# HW6

Throughout this HW, $X$ is a topological space, $Z$ a closed subset of $X$ and $i\colon Z \hookrightarrow X$ the inclusion map. Let $\mathscr{F} \in \mathrm{Sh}_X$. Recall that the pre-sheaf

$$U \mapsto \Gamma_{Z\cap U}(U,\, \mathscr{F}) \qquad (U \text{ open in } X)$$

is a sheaf and is denoted $\underline{\Gamma_Z}(X,\, \mathscr{F})$, or more often as simply $\underline{\Gamma_Z}\mathscr{F}$.

(1) Show that if $Y$ is a second closed subset of $X$ then $\Gamma_Y \circ \underline{\Gamma_Z} = \Gamma_{Y\cap Z}$.

(2) If $\mathscr{F} \in \mathrm{Sh}_X$ is flasque, show that so is $\underline{\Gamma_Z}\mathscr{F}$, for $Z$ a closed subset of $X$.

(3) Let $Z$ be a closed subset of $X$, $U$ the open set $U = X \setminus Z$, $j\colon U \to X$ the inclusion map.
   (a) Show that the sequence
$$0 \to \underline{\Gamma_Z}\mathscr{F} \to \mathscr{F} \to j_*j^{-1}\mathscr{F}$$
   is exact for every $\mathscr{F} \in \mathrm{Sh}_X$.
   (b) If $\mathscr{F}$ is *flasque* show that
$$0 \to \underline{\Gamma_Z}\mathscr{F} \to \mathscr{F} \to j_*j^{-1}\mathscr{F} \to 0$$
   is exact.

(4) Let $Y$ be a second closed subset of $X$ and let $V = X \setminus Y$.
   (a) Show that for a sheaf $\mathscr{F}$ on $X$ we have an exact sequence
$$0 \to \Gamma_{Y\cap Z}(X,\, \mathscr{F}) \to \Gamma_Z(X\, \mathscr{F}) \to \Gamma_{V\cap Z}(V,\, \mathscr{F}).$$
   (b) If $\mathscr{F}$ is *flasque* show that
$$0 \to \Gamma_{Y\cap Z}(X,\, \mathscr{F}) \to \Gamma_Z(X\, \mathscr{F}) \to \Gamma_{V\cap Z}(V,\, \mathscr{F}) \to 0$$
   is exact.

(5) Let $V$ be an open subset $X$ and $\mathscr{F}$ a sheaf on $X$.
   (a) Show that we have an exact sequence, the so-called *excision exact sequence*:
$$\cdots \to \mathrm{H}^p_{Z\setminus V}(X,\, \mathscr{F}) \to \mathrm{H}^p_Z(X,\, \mathscr{F}) \to H^p_{Z\cap V}(V,\, \mathscr{F}) \to H^{p+1}_{Z\setminus V}(X,\, \mathscr{F}) \to \dots.$$
   [Hint: Use the previous problem.]
   (b) Let $V$ contain $Z$. Deduce *excision isomorphism*
$$\mathrm{H}^p_Z(X,\, \mathscr{F}) \xrightarrow{\ \sim\ } \mathrm{H}^p_Z(V,\, \mathscr{F}) \qquad (p \in \mathbb{Z}).$$