HW5

Use the notes entitled *Čech complexes, tensor product of complexes,* $\mathrm{Hom}^\bullet(A^\bullet, B^\bullet)$*, and the Koszul complex.*

Definitions and Notations. Let A be a ring.[1] A *non-zero divisor* of M is an element $a \in A$ such that for $m \in M$ $am = 0$ only if $m = 0$. In other words a is a non-zero divisor of M (abbreviated to "NZD of M") if an only if the map $M \xrightarrow{a} M$ is injective. For any ideal I of A, we write IM for the submodule of M generated by elements of the form xm with $x \in I$ and $m \in M$. If $\boldsymbol{t} = (t_1, \dots, t_d)$ is a sequence of elements in A we sometimes write $\boldsymbol{t}M$ or $(t_1, \dots, t_d)M$ for IM where I is the ideal of A generated by the t_i, $i = 1, \dots, d$.

(1) Let A be a ring, $\boldsymbol{t} = (t_1, \dots, t_d)$ a sequence of elements in A. For each $p = 0, \dots, d$ let $K^{\boldsymbol{e}}_p(\boldsymbol{t})$ be the free A-module of rank $\binom{d}{p}$ on the free generators $\{e_{i_1\cdots i_p} \mid 1 \le i_1 < \cdots < i_p \le d\}$, i.e.,

$$K^{\boldsymbol{e}}_p(\boldsymbol{t}) := \bigoplus_{1\le i_1<\cdots<i_p\le d} Ae_{i_1\cdots i_p}$$

with the e's forming a basis. Define a differential $d\colon K^{\boldsymbol{e}}_p(\boldsymbol{t}) \to K^{\boldsymbol{e}}_{p-1}(\boldsymbol{t})$ by setting

$$d(e_{i_1\cdots i_p}) := \sum_{r=1}^{p} (-1)^{r-1} t_{i_r} e_{i_1\cdots \widehat{i_r} \cdots i_p}.$$

Show that the Koszul complex $K_\bullet(\boldsymbol{t}, A)$ is canonically isomorphic to the complex $K^{\boldsymbol{e}}_\bullet(\boldsymbol{t})$. (In fact the isomorphism is so canonical that for the rest of the course we identify the two.)

(2) Let A be a ring, $\boldsymbol{t} = (t_1, \dots, t_d)$ and $\boldsymbol{f} = (f_1, \dots, f_d)$ sequences of elements in A, with $f_i = a_i t_i$ for every i, for some elements $a_i \in A$. Show that we have a natural map of complexes $K_\bullet(\boldsymbol{t}, M) \to K_\bullet(\boldsymbol{f}, M)$ which on $e_{i_1\cdots i_p}$ is given by multiplication by a suitable minor of the diagonal matrix $T = \mathrm{diag}\,(a_1, \dots, a_d)$.[2]

(3) Let A be a commutative ring, $\boldsymbol{t} = (t_1, \dots, t_d)$ a sequence of elements in A, and M an A-module. Suppose t_1 is NZD of M and t_i is a NZD of $M/(t_1, \dots, t_{i-1})M$ for $i = 2, \dots, d$. Show that

$$\mathrm{H}_i(K_\bullet(\boldsymbol{t}, M)) = 0 \qquad (i = 1, \dots, d)$$

and

$$\mathrm{H}_0(K_\bullet(\boldsymbol{t}, M) = M/\boldsymbol{t}M.$$

[1] Commutative of course.

[2] There is a more general statement that can be made for $\boldsymbol{f}$ of the form $f_i = \sum_{i=1}^{d} a_{ij} t_j$. We leave that for another day.

[Hint: Use induction, and use the dual of problem 5 (b) of HW 3, with trivial modifications. Check that

$$K_\bullet((t_1, \ldots, t_i, M) = K_\bullet(t_1, \ldots, t_{i-1},\, M) \otimes_A K_\bullet(t_i,\, A)$$

and note that the right side is the total complex of a third quadrant cochain double complex (i.e. third quadrant cohomology complex), if we "raise indices" and convert our chain complexes into cochain complexes.]

Definition. Sequences $\boldsymbol{t}$ with the property vis-a-vis M as in Problem 3 are called *M-sequences*, and A-sequences are called *regular sequences*. (Such sequences have an important role to play in algebraic geometry and commutative algebra. Problem 3 shows that if $\boldsymbol{t}$ is an M-sequence, then $K_\bullet(\boldsymbol{t},\, M) \to M/\boldsymbol{t}M$ is a resolution of $M/\boldsymbol{t}M$.)

(4) Let $t \in A$, with A a ring. Consider the direct system $(M_n, \mu_{m,n})$ where $M_n = A$ for every n, and $\mu_{m,n}$ is the map $a \mapsto t^{n-m}a$ $(m \leq n)$. Show that

$$\varinjlim_n M_n = A_t$$

where, as usual, A_t is the localization of A at the multiplicative system $\{1,\, t,\, t^2, \ldots\}$.

Tensor products and direct limits. If A is a ring and $(N_\lambda)_{\lambda\in\Lambda}$ is a direct system of A-modules then it is well-known and easy to prove (using the universal property of tensor products and the universal property of direct limits) that for every A-module A, there is a functorial isomorphism

$$\varinjlim_\lambda (M \otimes_A N_\lambda) \xrightarrow{\ \sim\ } M \otimes_A \left(\varinjlim_\lambda N\right).$$

Feel free to use this in what follows. Supply a proof for yourself if you haven't seen this before but don't submit such a proof to me. We will treat such canonical functorial isomorphisms as identities. The other important thing to remember is this. According to one of the HW problems you did, $\varinjlim_\lambda$ is an exact functor. An easy consequence is:

$$\varinjlim_\lambda \mathrm{H}^n(C^\bullet_\lambda) \xrightarrow{\ \sim\ } \mathrm{H}^n(\varinjlim_\lambda C^\bullet_\lambda) \qquad (n \in \mathbb{Z})$$

for a direct sequence of complexes of A=modules $(C^\bullet_\lambda)_{\lambda\in\Lambda}$. Feel free to use this in what follows, and again, if you haven't seen such things before, provide for yourself the easy proof.

Stable Koszul complex. Let A be a ring, $\boldsymbol{t} = (t_1, \ldots, t_d)$ a sequence of elements in A, and M an A-module. The *stable Koszul complex* of M with respect to $\boldsymbol{t}$ is

$$K^\bullet_\infty(\boldsymbol{t},\, M) := \varinjlim_{\boldsymbol{n}} K^\bullet(\boldsymbol{t^n},\, M).$$

Here the index $\boldsymbol{n} = (n_1, \ldots, n_d)$ varies in the set of d-tuples of non-negative integers, and $\boldsymbol{n} \leq \boldsymbol{m}$ if $n_i \leq m_i$ for $i = 1, \ldots, d$. The maps of the direct system of complexes are given by Problem 2. In more explicit terms, if $(n_1, \ldots, n_d) \leq (m_1, \ldots, m_d)$, then set $a_i = t_i^{m_i - n_i}$ and note that $t_i^{m_i} = a_i t_i^{n_i}$. By Problem 2 we therefore have a chain map

$$\varphi_\bullet(\boldsymbol{n},\, \boldsymbol{m})\colon K_\bullet(\boldsymbol{t^n},\, A) \to K_\bullet(\boldsymbol{t^m},\, A),$$

and these maps give us the required direct system.

Other ad-hoc conventions. For a complex $C^\bullet$ and an integer n, let $C^\bullet\{n\}$ denote the complex $C^\bullet$ shifted to the "left" by n-units[3], without the intervention of signs for the coboundary maps.[4] For m an integer, $C^\bullet_{\geq m}$ is the brutally truncated complex which is zero in degrees less than m and equals C^p for $p \geq m$, with the coboundaries being the ones in $C^\bullet$ for $p \geq m$.

(5) Let A be a ring, $\boldsymbol{t} = (t_1, \ldots, t_d)$ a sequence of elements in A and M an A module. Let $I = \boldsymbol{t}A$, the ideal of A generated by the $\boldsymbol{t}$. Let $X = \operatorname{Spec} A$, $Z = V(I)$ the closed subset of X consisting of prime ideals containing I, and $U_i = D(t_i)$, $i = 1, \ldots, d$, the basic open sets in $\operatorname{Spec} A$ associated to t_i in the standard basis for the topology on $\operatorname{Spec} A$. In other words for a fixed i, U_i is the set of prime ideals which do not contain t_i. Let U be the complement of Z in X.

$$U := X \setminus Z.$$

Note that $\mathfrak{U} = (U_i)_{i=1}^d$ is an open cover of U. Let $\mathscr{F} = \widetilde{M}$ be the sheaf on X corresponding to the A-module M. Show that

$$[K^\bullet_\infty(\boldsymbol{t},\, M)]_{\geq 1}\{1\} = C^\bullet(\mathfrak{U},\, \mathscr{F}|_U)$$

where the complex on the right is the Čech complex of $\mathscr{F}|_U$ with respect to the covering $\mathfrak{U}$.[5]

(6) Let $S = R[X_1, \ldots, X_n]$ be the polynomial ring in n-variables over a ring R. Let U be the punctured spectrum of S punctured at the zero locus of the X's. In other words, $U = \operatorname{Spec} S \setminus V(X_1, \ldots, X_n)$. (If R is a field, then U is obtained by puncturing the affine n-space $\mathbb{A}^n_R$ at the origin.) Let $U_i = D(X_i)$ and let $\mathfrak{U} = (U_i)$ be the resulting open cover of U. Show that the Čech cohomologies of $\mathscr{O}_U$ with respect to $\mathfrak{U}$ are zero for $i \notin \{0,\, n-1\}$.

(7) In the above situation with $n \geq 2$ and with $W = \operatorname{Spec} S$, show that the restriction map $\Gamma(W,\, \mathscr{O}_W) \to \Gamma(U, \mathscr{O}_U)$ is an isomorphism. This means that every global function on U can be extended past the puncture in a unque way to give a global function on W. (Think about what goes wrong for $n = 1$).

(8) With $n \geq 2$ as above, show that $\mathrm{H}^{n-1}(\mathfrak{U},\, \mathscr{O}_U)$ is canonically isomorphic to the S-module E of *inverse polynomials*

$$\sum_{\mu_1 \geq 1, \ldots, \mu_n \geq 1} a_{\mu_1, \ldots, \mu_n} X_1^{-\mu_1} \cdots X_n^{-\mu_n}$$

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[3]Which means that if n is negative, we shift to the right by $-n$ units.

[4]Thus $C^\bullet\{n\}$ is not equal to $C^\bullet[n]$ unless n is even, but is isomorphic to it. This isomorphism, when n is odd is given by $C^p \xrightarrow{(-1)^p} C^p$.

[5]Čech complex in the sense of the notes I posted to go with this HW, i.e., the Hartshorne Čech complex.

with $a_\mu \in R$. These are formal expression. The S-module structure is given by (with μ's positive integers and ν's non-negative integers)

$$(X_1^{\nu_1} \cdots X_n^{\nu_n}) \cdot X_1^{-\mu_1} \cdots X_n^{-\mu_n} = \begin{cases} X_1^{-\mu_1+\nu_1} \cdots X_n^{-\mu_n+\nu_n} & \text{if } \nu_i < \mu_i \ \forall \ i \\ 0 & \text{otherwise.} \end{cases}$$