HW4

Throughout this course, rings will be assumed to have units, modules (left or right) will be unital, i.e., $1 \cdot m = m$ or $m \cdot 1 = m$, as the case may be. Ring homomorphisms will map unity to unity. For commutative rings, we do not distinguish between left and right modules. Two complexes $A^\bullet$ and $B^\bullet$ are (in this course) said to have *isomorphic cohomologies* if $\mathrm{H}^i(A^\bullet) \simeq \mathrm{H}^i(B^\bullet)$ for every $i \in \mathbb{Z}$. By a *resolution* $M \to R^\bullet$ (respectively $S^\bullet \to M$) of M we mean a quasi isomorphism with $R^n = 0$ for $n < 0$ (resp. $S^n = 0$ for $n > 0$). Note that this is the same as giving an exact sequence:

$$0 \to M \xrightarrow{\epsilon} R^0 \to R^1 \to \cdots \to R^n \to \ldots$$

(resp. an exact sequence

$$\cdots \to P^{-n} \to \cdots \to P^{-1} \to P^0 \xrightarrow{\epsilon} M \to 0).$$

The two cases are sometimes distinguished by the phrases "right resolution" and "left resolution". A resolution $M \to R^\bullet$ is an injective resolution if all the R^n are injective. Similarly one can talk about flat resolutions and projective resolutions. The latter are usually left resolutions.

(1) Let A be a commutative ring, M, N A-modules, $P^\bullet \to M$ a projective resolution of M and $N \to E^\bullet$ an injective resolution of N. Show that the complexes $\mathrm{Hom}^\bullet_A(P^\bullet, N)$ and $\mathrm{Hom}^\bullet_A(M, E^\bullet)$ have isomorphic cohomologies.

(2) Let A be as above, and suppose

$$0 \to F' \to F \to F'' \to 0$$

is an exact sequence of flat A-modules. Show that the induced sequence

$$0 \to F' \otimes_A M \to F \otimes_A M \to F'' \otimes_A M \to 0$$

is exact for every A-module M. [Hint: Take a flat resolution of $Q^\bullet \to M$ and consider the resulting short exact sequence of complexes

$$0 \to F' \otimes Q^\bullet \to F \otimes Q^\bullet \to F'' \otimes Q^\bullet \to 0.]$$

(3) Let A, M, and N be as above. Suppose and $P^\bullet \to M$ and $Q^\bullet \to N$ are *flat resolutions* of M and N respectively. Show that the complexes $P^\bullet \otimes_A N$ and $M \otimes_A Q^\bullet$ have isomorphic cohomologies.

Let X be a topological space, and $\mathfrak{U} = \{U_\alpha\}$ an open cover of X. For any open set V of X, set $\mathfrak{U} \cap V := \{U_\alpha \cap V\}$. Fix $p \in \{0, 1, 2, \ldots, n, \ldots\}$. If $C^\bullet(\mathfrak{U}, \mathscr{F})$ denotes the Cech complex of a sheaf of $\mathscr{F}$, let $\mathscr{C}^p(\mathfrak{U}, \mathscr{F})$ be the presheaf given by $V \mapsto C^p(\mathfrak{U} \cap V, \mathscr{F}|_V)$, V open in X. It is easy to check that $\mathscr{C}^\bullet(\mathfrak{U}, \mathscr{F})$ is a sheaf and that the coboundaries in the Cech complex restrict well to open subsets, and hence we have a complex, the so called *sheaf Cech complex*, $\mathscr{C}^\bullet(\mathfrak{U}, \mathscr{F})$ as well as a map $\mathscr{F} \to \mathscr{C}^p(\mathfrak{U}, \mathscr{F})$. As in Mumford's unpublished book, we also have the alternating Cech complex $C^\bullet_{\mathrm{alt}}(\mathfrak{U}, \mathscr{F})$, and a corresponding *alternating sheaf Cech complex* $\mathscr{C}^\bullet_{\mathrm{alt}}(\mathfrak{U}, \mathscr{F})$.

(4) Show that the natural map $C^\bullet_{\mathrm{alt}}(\mathfrak{U}, \mathscr{F}) \to C^\bullet(\mathfrak{U}, \mathscr{F})$ is a quasi-isomorphism.

(5) Show that the natural map $\mathscr{F} \to \mathscr{C}^\bullet_{\mathrm{alt}}(\mathfrak{U}, \mathscr{F})$ is a quasi-isomorphism. [Hint: Find a homotopy between the zero map and the identity map on the augumented complex

$$0 \to \mathscr{F} \to \mathscr{C}^0_{\mathrm{alt}}(\mathfrak{U}, \mathscr{F}) \to \mathscr{C}^1_{\mathrm{alt}}(\mathfrak{U}, \mathscr{F}) \to \cdots \to \mathscr{C}^n_{\mathrm{alt}}(\mathfrak{U}, \mathscr{F}) \to \dots.]$$