HW-II

(1) Show that if

$$0 \to (M'_\lambda) \to (M_\lambda) \to (M''_\lambda) \to 0$$

is an exact sequence direct systems, i.e. if at each level λ the corresponding sequence of abelian groups is exact, then the induced sequence

$$0 \to \varinjlim_\lambda M'_\lambda \to \varinjlim_\lambda M_\lambda \to \varinjlim_\lambda M''_\lambda \to 0$$

is an exact sequence of abelian groups. In other words, show that $\varinjlim_\lambda$ is an exact functor.

(2) Let X be a topological space, $\mathscr{F}$ a sheaf on X, U an open subset of X, and $\mathfrak{U} = \{U_\alpha\}$ an open cover of U. For every α and β set $U_{\alpha\beta} := U_\alpha \cap U_\beta$. Show that the sequence of abelian groups

$$0 \to \mathscr{F}(U) \xrightarrow{\epsilon} \prod_\alpha \mathscr{F}(U_\alpha) \xrightarrow{d^0} \prod_{\alpha,\beta} \mathscr{F}(U_{\alpha\beta})$$

is exact, where ϵ is the "diagonal" map $s \mapsto (s|_{U_\alpha})_\alpha$ and the map d^0 is defined by $d^0((s_\alpha)_\alpha) = (\sigma_{\alpha\beta})_{\alpha,\beta}$ where $\sigma_{\alpha\beta} = s_\alpha|_{U_{\alpha\beta}} - s_\beta|_{U_{\alpha\beta}}$.

For the remaining problems consider the following. Let X be a topological space, $\mathscr{B}$ a basis for the topology on X with the extra condition that if B_1 and B_2 are in $\mathscr{B}$ then so is $B_1 \cap B_2$ (e.g. the standard basis for the topology on Spec(A), where A is a commutative ring). Let F be a $\mathscr{B}$-sheaf (defined in class). For U an open set of X set

$$\mathscr{F}(U) := \ker \left[\prod_\alpha F(U_\alpha) \xrightarrow{d^0} \prod_{\alpha,\beta} F(U_{\alpha\beta}) \right] \qquad (*)$$

where (U_α) is an open cover of U with $U_\alpha \in \mathscr{B}$ for every α and d^0 is as in (2).

(3) Show that $\mathscr{F}(U)$ does not depend on the open cover (U_α) of U, i.e. any two covers by members of $\mathscr{B}$ give rise to isomorphic kernels as in $(*)$.

(4) Show that the assignment $U \mapsto \mathscr{F}(U)$ gives us a sheaf, which we will denote $\mathscr{F}$.

(5) Show that we have an isomorphism of $\mathscr{B}$-sheaves $\mathscr{F}|_{\mathscr{B}} \xrightarrow{\sim} F$.

(6) If G is a $\mathscr{B}$-sheaf and $\varphi\colon F \to G$ a map of $\mathscr{B}$-sheaves and if $\mathscr{G}$ is the sheaf on X arising from G via the process outlined in (4) then show that there

is a map $\tilde{\varphi}\colon \mathscr{F} \to \mathscr{G}$ such that the diagram

$$\begin{array}{ccc} \mathscr{F}|_{\mathscr{B}} & \xrightarrow{\sim} & F \\ {\scriptstyle\tilde{\varphi}}\downarrow & & \downarrow{\scriptstyle\varphi} \\ \mathscr{G}|_{\mathscr{B}} & \xrightarrow{\sim} & G \end{array}$$

commutes, where the horizontal isomorphisms are as in (5).

(7) Show that

$$\mathscr{F}(U) \xrightarrow{\sim} \varprojlim_{B} F(B)$$

where the inverse limit is taken over B such that $B \in \mathscr{B}$ and $B \subset U$.