HW-I

As always, "map" is used for "morphism".

Direct Limits

Let $(\Lambda, \prec)$ be a directed set. If $(M_\lambda)_{\lambda\in\Lambda}$ and $(N_\lambda)_{\lambda\in\Lambda}$ is a direct system of abelian groups, then $\mathrm{Hom}_\Lambda((M_\lambda),(N_\lambda))$ is the group of maps between direct systems, i.e. an element of the above set is a collection of maps $(\varphi_\lambda\colon M_\lambda \to N_\lambda)$ compatible with the direct system structures on (M_λ) and (N_λ). Every abelian group T will be regarded as a direct system, namely as the constant direct system. Thus the symbol $\mathrm{Hom}_\Lambda((M_\lambda),T)$ makes sense. In particular, by definition of a direct limit

$$\mathrm{Hom}_\Lambda((M_\lambda),\,T) \xrightarrow{\sim} \mathrm{Hom}(\varinjlim_{\lambda\in\Lambda} M_\lambda,\,T)$$

for every abelian group T, where the Hom on the right is in the category of abelian groups.

A sub-directed set $\Gamma \subset \Lambda$ is said to be *cofinal* with respect to Λ, if given $\lambda \in \Lambda$ there exists a $\gamma \in \Gamma$ with $\lambda \prec \gamma$. In what follows we fix a directed set Λ and a cofinal subset Γ of Λ.

(1) Show that if (M_λ) is a direct system, then there is an isomorphism

$$\mathrm{Hom}_\Lambda((M_\lambda)_{\lambda\in\Lambda},T) \xrightarrow{\sim} \mathrm{Hom}_\Gamma((M_\gamma)_{\gamma\in\Gamma},T)$$

for every abelian group T.

(2) Show that

$$\varinjlim_{\lambda\in\Lambda} M_\lambda \xrightarrow{\sim} \varinjlim_{\gamma\in\Gamma} M_\gamma.$$

Category Theory

Read the definitions of an exact, additive, and abelian categories from the notes handed out in class (downloadable from the website).

Definition 1. *Let $\mathscr{C}$ be an exact category. An* exact sequence *in $\mathscr{C}$ is a sequence of maps in $\mathscr{C}$*

$$\dots C^{p-1} \xrightarrow{d^{p-1}} C^p \xrightarrow{d^p} C^{p+1} \xrightarrow{d^{p+1}} \dots$$

such that $\mathbf{im}\, d^{p-1} = \ker d^p$*,* $p \in \mathbb{Z}$.

Definition 2. *If $\mathscr{C}$ is any category, its* opposite category*, $\mathscr{C}^\circ$ is the category whose objects are the objects of $\mathscr{C}$, and a morphism $A \to B$ in $\mathscr{C}^\circ$ is a morphism $B \to A$ in $\mathscr{C}$. A* contravariant *functor on $\mathscr{C}$ is nothing but a functor on $\mathscr{C}^\circ$.*

(3) Suppose $\mathscr{C}$ is an exact category. Show that $\mathscr{C}^\circ$ is also an exact category, where, if $\alpha\colon A \to B$ is a map in $\mathscr{C}^\circ$ corresponding to $\check{\alpha}\colon B \to A$ in $\mathscr{C}$, then $\ker\alpha$ corresponds to $\mathrm{coker}\,\check{\alpha}$, etc., etc.

(4) Let $\mathscr{C}$ be an exact category. Consider the commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & P & \longrightarrow & Q & \longrightarrow & R \\ & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} \\ 0 & \longrightarrow & P' & \longrightarrow & Q' & \longrightarrow & R' \end{array}$$

with exact rows.

(a) Show that $\ker \alpha \to \ker \beta$ is the kernel for $\ker \beta \to R$.

(b) Show that $0 \to \ker \alpha \to \ker \beta \to \ker \gamma$ is exact.

(c) What would be the "dual" statement to (4b)? In other words consider the statement (4b) in the exact category $\mathscr{C}^\circ$ and translate that statement back to $\mathscr{C}$, and tell me what the new statement is.

(5) Let

$$\begin{array}{ccccccc} G & \longrightarrow & H & \longrightarrow & I & \longrightarrow & 0 \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle i} & & \\ G' & \longrightarrow & H' & \longrightarrow & I' & \longrightarrow & 0 \\ \downarrow & & & & & & \\ 0 & & & & & & \end{array}$$

be a commutative diagram in an exact category $\mathscr{C}$ such that the rows are exact, and the left column is exact.

(a) Show that $\operatorname{coker} h \to \operatorname{coker} i$ is an isomorphism. (Use the last part of the previous problem is necessary.)

(b) Show that $G' \to \mathbf{im}\, h \to \mathbf{im}\, i$ is exact. [Hint: Replace P, Q, R, P', Q', R' in 4 by $\ker(H' \to I')$, H', I', 0, $\operatorname{coker} h$, and $\operatorname{coker} i$ respectively, and use the result from (4b).]

(c) Use the above to show that $\ker h \to \ker i$ is surjective.

(6) Let $\mathscr{C}$ be an exact category. Suppose we have a commutative diagram with exact rows, with the column on the extreme left also exact:

$$\begin{array}{ccccccc} P & \longrightarrow & Q & \longrightarrow & R & \longrightarrow & S \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \delta} \\ P' & \longrightarrow & Q' & \longrightarrow & R' & \longrightarrow & S' \\ \downarrow & & & & & & \\ 0 & & & & & & \end{array}$$

Show that the induced sequence

$$\ker \beta \to \ker \gamma \to \ker \delta$$

is also exact.

Using the above we will prove the **snake lemma** in class. This lemma says that if we have an exact commutative diagram in an exact category:

$$
\begin{array}{ccccccccc}
 & & & & & & & & 0 \\
 & & & & & & & & \downarrow \\
A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & D & \longrightarrow & E \\
\downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle \epsilon} \\
A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & D' & \longrightarrow & E' \\
\downarrow & & & & & & & & \\
0 & & & & & & & &
\end{array}
$$

then there is an exact sequence:

$$\ker \beta \to \ker \gamma \to \ker \delta \to \operatorname{coker} \beta \to \operatorname{coker} \gamma \to \operatorname{coker} \delta$$

(7) Assume the snake lemma. Show that if we have a commutative diagram

$$
\begin{array}{ccccccccc}
A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & D & \longrightarrow & E \\
\downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle \epsilon} \\
A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & D' & \longrightarrow & E'
\end{array}
$$

with exact rows in an exact category such that α, β, δ, and ϵ are isomorphisms, then so is γ.

Sheaves

For a presheaf F on a topological space, we will use the notations we used in class. Thus $\mathscr{E}(F)$ is the topological space associated with F, $\pi\colon \mathscr{E}(F) \to X$ the natural map, F^+ the sheafification of F etc.

In what follows, X is a topological space, and F a presheaf on X.

(8) Show that $\pi\colon \mathscr{E}(F) \to X$ is a local homeomorphism.

(9) Show that if U is open in X and $s \in F(U)$, and for $x \in U$, s_x the germ of s at x,[1] then the map

$$\sigma_s\colon X \to \mathscr{E}(F) = \coprod_{x \in X} F_x$$

given by $x \mapsto s_x$, $x \in U$, is a continuous map.

Recall that the natural map $\theta(= \theta_F)\colon F \to F^+$ is the map defined on every open set U of X by $s \mapsto \sigma_s$ with the notation as above.

(10) If F is a sheaf, show that θ_F is an isomorphism.

[1] In other words, s_x is the image of $s \in F(U)$ in the stalk F_x under the natural map $F(U) \to F_x$ arising from the definition of a direct limit.

(11) Let E be a topological space, $p\colon E \to X$ a local homeomorphism such that for every $x \in X$, $p^{-1}(x)$ is an abelian group. Define $E \times_X E$ to be the subspace of $E \times E$ consisting of pairs (e, e') with $p(e) = p(e')$. Suppose the two maps $E \times_X E \to E$, $(e, e') \mapsto e + e'$ and $E \to E$, $e \mapsto -e$ are continuous. Let $\mathscr{F} = \mathscr{F}_E$ be the sheaf of sections of $p\colon E \to X$, i.e., for an open subset U of X, $\mathscr{F}(U)$ is the abelian group of continuous maps from $\sigma\colon U \to E$ such that $p \circ \sigma = 1_U$. Show the following.

(a) For $x \in X$, there is a natural isomorphism of abelian $\psi_x\colon F_x \xrightarrow{\sim} p^{-1}(x)$.

(b) There is a natural isomorphism $\psi\colon \mathscr{E}(F) \xrightarrow{\sim} E$ such that $p \circ \psi = \pi$.