TALG 2025 problem set

January-April 2025

1 Instructions

You need to submit 10 problems during the course. Problems marked with $*$ are compulsory. I will continue to add more problems. *Do not* submit the problems you have solved/seen in one of my previous courses. The solutions might be available online. But to develop your own problem solving skills, you are strongly advised not to look at the solutions or to discuss with others (especially if they have already solved the problem!). If you are stuck, you are welcome to meet me and get a hint on how to proceed. In any case, for each problem, you are required to declare all the resources/persons that might have helped you in solving it. If you give a hint/entire solution to others, that needs to be declared as well.

2 Problems

1. * For the maximum spanning tree problem, consider the following LP:

$$\begin{aligned} \max \sum_{e \in E} w_e x_e & \\ \text{s.t.} \sum_{e \in E} x_e &\leq n-1 \\ \sum_{e: e \text{ crosses } S} x_e &\geq 1 \quad \forall S \subset V, S \neq \emptyset \\ 0 \leq x_e &\leq 1 \quad \forall e \in E \end{aligned}$$

   Show that there is a fractional solution to this LP which is larger than any integral solution.

2. For the maximum spanning tree problem, write a suitable LP and design a primal-dual algorithm.

3. The primal-dual algorithm designed in class corresponds to Kruskal's algorithm. Can you write another LP and design a primal-dual algorithm that would correspond to Prim's algorithm?

4. Write an LP for the vertex cover problem. Write its dual and execute a primal-dual algorithm. Show that whenever the algorithm terminates, it gives a 2-approximation to the vertex cover. Can you identify the combinatorial problem that the dual LP represents? Show that the dual LP can have a fractional optimum solution of value smaller than that of any integral optimum.

5. * Consider a solution $\tilde{y}$ to the tree packing LP. Define the support of $\tilde{y}$ to be those trees $T \in \mathcal{T}$ for which $\tilde{y}_T > 0$. Show that, for any mincut $S, V \setminus S$, there is a tree T in the support of $\tilde{y}$ such that T has only one edge crossing the cut.

6. *In class, we saw that, given an exact fractional tree packing $\{y_T\}_{T \in \mathcal{T}}$, a tree T picked at random with probability $\frac{y_T}{\sum_{T \in \mathcal{T}} y_T}$ uses at most two edges of a fixed mincut S with probability $\geq \frac{1}{2}$. What is the probability that such a tree uses at most one edge of S? You need to give a non-zero lower bound. How many repetitions are needed to boost it to $1 - \frac{1}{n}$?

7. Show an approximation ratio of $2\left(1 - \frac{1}{k}\right)$ for the following algorithm for the multiway cut problem: Find a minimum cut that separates a pair of terminals. Recursively find cuts in the two partitions that separate a pair of terminals from each of them.

8. Problems 2 and 3 from this set

Some problems for practicing LP and dual construction (not for submission):

1. Express the following problems as linear programs and write their duals: max-flow (between two vertices s and t), shortest path from s to t in a directed graph, maximum matching, Steiner trees.

2. Design a primal-dual algorithm for the shortest path problem.

More practice problems: Problems 1 to 5 from this homework. They are not directly based on the course material, but can be solved with your background.

3 Presentation Topics

Choose your presentation topic soon. You can do a presentation alone or in pairs, but not in a larger group. You can approach me with a topic or can choose one of the topics suggested below. Let me know as soon as you choose a topic. You are free to use any resources available online, apart from the ones suggested here. Before selecting a topic, do spend some time reading the paper and make sure that you find it readable. As the course progresses, I will add more topics:

1. MST verification in linear time (taken by Qusai and Shubh) - Paper by Komlos.

2. Cactus representation of mincuts Paper (Taken by Aditi)

3. Kaiser's proof of Tutte-Nash-Williams theorem Paper (Taken by Debmalya and Suneet)

4. A combinatorial algorithm for finding a largest set of disjoint spanning trees. [1]

5. The two-commodity flow problem Paper (Taken by Romit, Aryan)

6. Multicommodity max-flow min-cut theorems Paper

7. A separator theorem for graphs with excluded minors (Taken by Srijan) Paper

8. A Polynomial Algorithm for the k-Cut Problem for Fixed k Paper

9. Additive spanners and (α, β)-spanners Paper

10. Oblivious routing Notes

[1] Schrijver's book (Combinatorial Optimization: Polyhedra and Efficiency) has it in terms of matroids. Either find a source for the direct one or work it out yourself. The presentation should not use matroids as the class is not familiar with them.