Linear Programming and Combinatorial Optimization
Tutorial 1

January 4, 2019

1. Find the rank of the following matrix by Gaussian elimination:

$$\begin{bmatrix} 1 & 2 & 1 & 0 \\ 2 & 1 & 3 & 1 \\ 4 & 5 & 5 & 1 \end{bmatrix}$$

2. Show that row operations (and hence Gaussian elimination) preserve the solutions to $Ax = b$.

3. Show that the set of solutions to $Ax = 0$ is a vector space.

4. Let x_1 and x_2 be solutions to $Ax = b$, $b \neq 0$. If $\alpha x_1 + \beta x_2$ is also a solution, what is the relation between α and β?

   Note: The relation derived above is called an *affine combination*.

5. The set of solutions to $Ax = b$ in the above question form an *affine subspace*. Informally, an affine subspace is a translation of a subspace. Let A be an affine subspace of a vector space V. Show that, if A is a translation of a d-dimensional vector space, then there are $d + 1$ points in A that span whole of A. Here the span is considered as all affine combinations of the $d + 1$ points.

6. Write an LP for the following problems:

   (a) **Matching:** Given a graph $G = (V, E)$, a matching M is a subset of E such that no two edges in M share an end-point. The goal is to find a largest possible set M in G.

   (b) **Network flow:** The input is a directed graph $G = (V, E)$ with a source s and a sink t, along with a non-negative capacity c_e on each edge e. A flow in G is an assignment of a non-negative value f_e to each edge e such that

   - f_e does not exceed c_e,
   - for each $v \in V \setminus \{s, t\}$, the sum of flows along incoming edges of v equals the sum of flows along outgoing edges of v. The goal is to maximize the total flow along incoming edges of t.

   (c) **Shortest path:** Given a directed graph with positive integer weights on edges, and two designated vertices s, t, find a minimum weight path from s to t.

   (d) **Vertex cover:** Given a graph $G = (V, E)$, find a smallest possible set $S \subset V$ such that each edge has at least one end-point in S.