Design and Analysis of Algorithms
End-semester Examination

April 27, 2012

Maximum marks: 40

Instructions: Write correctness proofs for algorithms. Try to design as efficient algorithms as possible. There are 7 questions.

1. **(4 marks)** State whether the following are true or false. Justify. Here $d[u]$ and $f[u]$ are the discovery and finish time of vertex u.

   (a) If there is a path from u to v in a directed graph G, and if $d[u] < d[v]$ in a DFS of G, then v is a descendant of u in the depth-first forest produced.

   (b) If there is a path from u to v in a directed graph G, then any DFS must result in $d[v] \leq f[u]$.

2. **(4 marks)** Let X and Y be two arrays, each containing n elements already in sorted order. Give an $O(\log n)$ time algorithm to find the median (n^{th} smallest) of all the $2n$ elements in arrays X and Y.

3. **(4 marks)** We are given a directed graph $G = (V, E)$ on which each edge $e = (u, v)$ has an associated value $r(u, v)$, which is a real number in the range $0 \leq r(u, v) \leq 1$ that represents the reliability of a communication channel from vertex u to vertex v. Thus $r(u, v)$ is the probability that the channel from u to v will not fail. Assuming that these probabilities are independent, give an efficient algorithm to find the most reliable path between two given vertices s and t.

   (Hint: Reliability of a path is the product of reliabilities of edges on the path.)

4. **(6 marks)** The *edge connectivity* of an undirected graph is the minimum number of edges that must be removed to disconnect the graph. For example, the edge connectivity of a tree is 1, and the edge connectivity of a cycle is 2.

   Show how the edge connectivity of an undirected graph $G = (V, E)$ can be determined by running a max-flow algorithm on at most $|V|$ flow networks, each with $O(V)$ vertices and $O(E)$ edges.

5. **(8 marks)** A *perfect matching* is a matching in which every vertex is matched. Let $G = (V, E)$ be a bipartite graph with vertex partition $L \cup R$, and $|L| = |R| = n$. For any $X \subseteq V$, define the *neighbourhood* of X as

   $$N(X) = \{y \in V : (x, y) \in E \text{ for some } x \in X\}$$

   That is, the set of vertices adjacent to some member of X. Prove that there exists a perfect matching in G if and only if $|A| \leq |N(A)|$ for every $A \subseteq L$.

   (Hint: Construct a flow network. Show that it has a cut of strictly less than n edges if there is $A \subseteq L$ such that $|A| > |N(A)|$. Prove the other direction similarly.)

6. **(10 marks)** In the *balanced partition problem*, a set S of numbers is given as input. The goal is to find an $A \subseteq S$ such that $|\sum_{x \in A} - \sum_{x \in S \setminus A}|$ is minimized. Give an algorithm for this problem and analyze its time complexity.

   Consider the special case of the corresponding decision problem where the goal is to check whether there is an $A \subseteq S$ such that $\sum_{x \in A} = \sum_{x \in S \setminus A}$. Show that this problem is NP-complete.

7. **(4 marks)** In the subset sum problem, input is a set S of integers and a target number t. The goal is to determine if there exists $A \subseteq S$ such that $\sum_{x \in A} = t$. We have seen that this problem is NP-complete. Prove that the problem can be solved in polynomial time if t is given in unary.