# A Cellular Model for Configuration Spaces of Points on a Graph

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## Abstract

Consider $n$-robots moving on a terrain modelled by a topological space $X$. The robots are modelled as points in $X$. It is natural to assume that the robots move continuously and without collisions. A configuration of the robots can be thought of as an element of $X^n$, no two entries of which are equal. The set of all the configurations is called the configuration space. Therefore, asking whether the robots can move from one configuration to the other is equivalent to asking if there is a path between the corresponding elements of the configuration space. This scenario occurs frequently in industries which implement automated guided vehicles (AGVs) which move on the factory floor and carry load from one point to the other. AGVs which can move freely in any direction of the two dimensional floor are difficult to build and maintain. For this reason, many industries make use of 'line-following' AGVs which are constrained to move on a network of guidepath wires marked on the floor. Thus we can think of the AGVs as robots moving on a graph. Understanding the topology of the configuration spaces of graphs is therefore important for the motion-planning of the AGVs. The main aim of the thesis is to study the homotopy type of the configuration spaces of points on a graph.

One way to compute the homotopy type of a space is by constructing a finite cellular model for the space. A classical theorem states that the geometric realization of the face poset of a regular CW complex $X$ is homeomorphic to $X$. Thus if one can impart a regular CW complex structure on a space then the topology of the space can be combinatorially captured. Since the configuration spaces of $n$ points on a Hausdorff space is not compact whenever $n > 1$, this technique cannot be used to study configuration spaces of graphs because a finite CW complex structure cannot be given to such spaces.

Recently, by relaxing the definition of a CW complex, Dai Tamaki has come with the concept of totally normal cellular stratified spaces and proved an analogous result which combinatorially captures the homotopy type of a possibly non-compact space $X$. In this thesis we will discuss the main theorem of Tamaki and see the applications of his theorem in finding the homotopy type of configuration spaces of points of graphs.

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# Contents

# Introduction

In many industrial settings, the following situation arises. There are a number of automated guided vehicles (AGVs) moving on the factory floor which carry load from one point to the other. AGVs which enjoy two degrees of freedom are challenging to manufacture and are costly to build and maintain. Many industries therefore invest in AGVs which are constrained to move on a network of guidepath wires etched on the factory floor or hanging from the ceiling. An attempt to describe the above situation mathematically naturally leads to the notion of *configuration space* of points on a topological space. Each AGV can be though of as a point moving on a topological space which models the space accessible to the robots. The motion of $n$ AGVs on $X$ can be thought of as a path in $X^n$. Since we do not want the AGVs to collide at any point of time, we cannot let a path in $X^n$ pass through a point in which two coordinates are equal. The collection of all the forbidden points in $X^n$ is termed as the $n$-diagonal of $X$ and the collection of all the accessible points is called the configuration space of $n$ points on $X$.

As noted by R. Ghrist in [5], the problem of motion planning AGVs having a full two degree of freedom on a factory floor is a local problem, for two AGVs may avoid collision at the last moment. On the other hand, the situation when the AGVs are constrained to move on a set of guidepath wires calls for a global analysis. Therefore, using topological tools in order to study the configuration spaces of points on a graph is useful in motion planning such AGVs. We refer the interested reader to [5] for a survey of results concerning configuration spaces of trees and graphs in general. Many results about braid groups of trees can be found in [2] and in [8]. Investigation on general graphs have been done in [1].

In [3], Tamaki et al. have come up with a cellular model which combinatorially describes the homotopy type of configuration spaces of graphs. The authors start by first defining the concept of a *cellular stratified space*, which roughly is a CW complex in which we are allowed to have non-closed cells to act as domains of characteristic maps. By a non-closed cell we mean a subset of a closed unit disk in some Euclidean space which contains the open $n$-ball. Thus a non-closed cell is a closed unit disk with some parts of its boundary missing. Then, the authors define the notion of *total normality* for a cellular stratified space, by adding some conditions in the definition of the latter.

A classical precursor of the main theorem in [3] is the following. The geometric realization of the face poset of a regular CW complex $X$ is homeomorphic to $X$. This result is not true for non regular CW complexes. To see this, consider the minimal CW complex structure on $S^1$ which has a single 0-cell and a single 1-cell. The face poset is then just a total order on two elements and therefore the geometric realization is homeomorphic to the closed interval. The homotopy type of the geometric realization is therefore not the same as that of $S^1$. The reason for this is that the face poset fails to capture how many ways does the 0-cell sit inside the 1-cell. Roughly speaking, since the characteristic map of the 1-cell identifies the two endpoints of the closed interval, there are two ways in which the 0 cell can sit inside the 1-cell. This leads to the notion of a face category, which is a generalization of the face poset. What the authors have shown in [3] is that the geometric realization of the face category of a totally normal cellular stratified space embeds in $X$ as a strong deformation retract, and therefore has the same homotopy type as that of $X$. The authors then show that the configuration space of points on a graph always admits a totally normal cellular stratified structure, making their technique ideal for dealing with graphs.

## Chapter-wise Organization

The thesis is divided into four chapters.
**Chapter 1.** In this chapter we discuss some of the background material required for the remainder of the thesis. In section 1.1 we define posets, poset maps, products of posets, and the Alexandroff topology. In Section 1.2 we define abstract and geometric simplicial complexes, geometric realization of an abstract simplicial complex, and cones. Section 1.3 is dedicated to regular CW complexes. The main item here is the classical theorem in combinatorial algebraic topology which states that the geometric realization of the face poset of a regular CW complex $X$ is homeomorphic to $X$. Lastly, in Section 1.4 we discuss face categories and their geometric realizations.
**Chapter 2.** We define the main object of interest, namely the configuration space of points on a topological space, especially graphs. In Section 2.2 we give some examples of configuration spaces on graphs.
**Chapter 3.** This chapter discusses the central concept of totally normal cellular stratified spaces. In Section 3.1, we define the notion of a stratification on a topological space and give several examples of stratified spaces, including stratification of Euclidean spaces via hyperplane arrangements. Section 3.3 discusses the important concept of cellular stratified spaces which allow us to see non-compact spaces as finite cell complexes. The main concept of totally normal cellular stratified spaces is discussed in Section 3.4. The chapter ends with Section 3.5 in which we formally define face categories and state the main theorem of [3].
**Chapter 4** In this chapter we discuss two applications of the main theorem on configuration spaces of graphs.

# Chapter 1

# Preliminaries

## 1.1 Posets

**Definition 1.1.1.** A **poset** is a set $P$ along with a reflexive, transitive, and anti-symmetric relation '$\leq$'. In other words, we have

1. **Reflexivity.** $x \leq x$ for all $x \in P$.
2. **Anti-Symmetry.** For $x, y \in P$ with $x \neq y$, we do not have both $x \leq y$ and $y \leq x$.
3. **Tansitivity.** For $x, y, z \in P$, if $x \leq y$ and $y \leq z$ then $x \leq z$.

❖

**Definition 1.1.2.** Let $P$ be a poset. Declare a subset $D$ of $P$ *closed* if and only if $D$ satisfies the following condition

$$\left.\begin{array}{r} \lambda, \mu \in \Lambda,\ \lambda \in D \\ \mu \leq \lambda \end{array}\right\} \quad \Rightarrow \quad \mu \in D$$

It can be easily checked that this declaration gives a topology on $P$. This topology is referred to as the **Alexandroff topology** on $P$. ❖

**Definition 1.1.3.** Let $P$ and $Q$ be two posets. We say that $f : P \to Q$ is a **poset map** if whenever $x \leq y$ in $P$, we have $f(x) \leq f(y)$ in $Q$. Note that a poset map $f : P \to Q$ is continuous when $P$ and $Q$ are given the Alexandroff topology. ❖

Taking posets as objects and posets maps as morphisms, we get a category called the **category of posets**.

**Definition 1.1.4.** Let $P$ and $Q$ be two posets. For two elements $(a, b)$ and $(x, y)$ of $P \times Q$, write $(a, b) \leq (x, y)$ if and only if $a \leq x$ in $P$ and $b \leq y$ in $Q$. This gives a poset structure to $P \times Q$. Under this poset structure, we call $P \times Q$ as the **product** of $P$ and $Q$. ❖

It can be easily checked the product of two posets $P$ and $Q$ is the product in the category of posets.[1] Also, the Alexandroff topology on $P \times Q$ under the product poset structure is same as the product topology on $P \times Q$ when $P$ and $Q$ are given the Alexandroff topology.

[1] More precisely, if $P$ and $Q$ are two posets, then $(P \times Q, \pi_P : P \times Q \to P, \pi_Q : P \times Q \to Q)$ is a product of $P$ and $Q$ in the category of posets. Here $\pi_P : P \times Q \to P$ is the map which takes $(x, y)$ to $x$ for all $x \in P$ and $y \in Q$. Similarly for $\pi_Q$.

## 1.2 Simplicial Complexes

**Definition 1.2.1.** An **abstract simplicial complex** is a pair $(V, \mathcal{S})$, where $V$ is a finite set and $\mathcal{S}$ is a collection of subsets of $V$ such that whenever a subset $A$ of $V$ is in $\mathcal{S}$, then all subsets of $A$ are also in $\mathcal{S}$. The elements of $V$ are called **vertices** and the elements of $\mathcal{S}$ are called **simplices**. ❖

**Example 1.2.2.** Any simple graph can be thought of as an abstract simplicial complex. If $(V, \mathcal{E})$ is a simple graph, then we define a set $\mathcal{S}$ which contains $\mathcal{E}$ as well the end points of all the edges in $\mathcal{E}$ along with the empty set. This set $\mathcal{S}$ gives an abstract simplicial complex structure to the set of all the vertices $V$.

**Example 1.2.3.** Let $P$ be a poset. Let $A = (P, \mathcal{S})$ be an abstract simplicial complex defined as follows: The vertex set of $A$ is $P$, and a subset $S$ of $P$ is a simplex in $A$ if and only if $S$ is a chain in $P$. We call $A$ the **order complex** of $P$.

**Definition 1.2.4.** A **geometric** $k$**-simplex** in $\mathbb{R}^n$ is a subset $K$ of $\mathbb{R}^n$ such that there exist $k+1$ affinely independent points in $\mathbb{R}^n$ whose convex hull is $K$. ❖

**Theorem 1.2.5.** *Let $K$ be a geometric $n$-simplex in $\mathbb{R}^{n+1}$. Let $\mathbf{v}_1, \ldots, \mathbf{v}_{n+1}$ and $\mathbf{v}'_1, \ldots, \mathbf{v}'_{n+1}$ be two sets of $n+1$ points in $\mathbb{R}^{n+1}$ such that*

$$\operatorname{conv}(\mathbf{v}_1, \ldots, \mathbf{v}_{n+1}) = \operatorname{conv}(\mathbf{v}'_1, \ldots, \mathbf{v}'_{n+1}) = K$$

*where 'conv' stands for 'convex hull'. Then $\{\mathbf{v}_1, \ldots, \mathbf{v}_{n+1}\} = \{\mathbf{v}'_1, \ldots, \mathbf{v}'_{n+1}\}$.*

**Proof.** Since $K$ is affinely isomorphic to the standard $n$-simplex $\Delta^n$ in $\mathbb{R}^{n+1}$, we may assume, without loss of generality, that $K = \Delta^n$. It is clear that any vertex set of $\Delta^n$ has size exactly $n+1$. We prove the theorem by induction on $n$. The result clearly holds for $n = 1, 2$. Let $n > 2$ and assume that the theorem is true for all smaller values. Clearly, $E = \{\mathbf{e}_1, \ldots, \mathbf{e}_{n+1}\}$ is a vertex set of $\Delta^n$, where $\mathbf{e}_i$ is the $i$-th basis vector in $\mathbb{R}^{n+1}$. Suppose $V = \{\mathbf{v}_1, \ldots, \mathbf{v}_{n+1}\}$ is another vertex set of $\Delta^n$. Write $\Delta^n_k$ to denote $\operatorname{conv}(E \setminus \{\mathbf{e}_k\})$ and let $V_k$ be the set of all the members of $V$ which lie in $\Delta^n_k$.

We claim that $\operatorname{conv}(V_k) = \Delta^n_k$. It is obvious that $\operatorname{conv}(V_k) \subseteq \Delta^n_k$ simply because the latter is a convex set. For the reverse containment, pick any point $\mathbf{p} \in \Delta^n_k$. Since $\operatorname{conv}(V) = \Delta^n \supseteq \Delta^n_k$, we have

$$\mathbf{p} = a_1\mathbf{v}_1 + \cdots + a_{n+1}\mathbf{v}_{n+1} \tag{1}$$

for some $a_1, \ldots, a_{n+1} \geq 0$ with $a_1 + \cdots + a_{n+1} = 1$. Let $\pi_k : \mathbb{R}^{n+1} \to \mathbb{R}$ be the $k$-th coordinate projection map. Noting that $\pi_k(\Delta^n_k) = \{0\}$, we have from (1) that

$$0 \;=\; \sum_{i:\ \mathbf{v}_i \notin V_k} a_i \pi_k(\mathbf{v}_i)$$

Again, note that each $\pi_k(\mathbf{v}_i)$ above is strictly greater than 0. This forces $a_i = 0$ for each $i$ satisfying $\mathbf{v}_i \notin V_k$, whence from (1) it follows that $\mathbf{p} \in \operatorname{conv}(V_k)$. Now since $\Delta^n_k$ can be thought of as $\Delta^{n-1}$, by induction, and by possibly renumbering the $\mathbf{v}_i$'s, we have $V_k = E \setminus \{\mathbf{v}_k\}$. ■

The above theorem leads to the following definition.

**Definition 1.2.6.** Let $K$ be a geometric $k$-simplex in $\mathbb{R}^n$. The **vertex set** of $K$ is the set of $k+1$ points in $\mathbb{R}^n$ whose convex hull is $K$. A **face** of $K$ is the convex hull of a subset of the vertex set of $K$. ❖

**Definition 1.2.7.** Let $A_1 = (V_1, \mathcal{S}_1)$ and $A_2 = (V_2, \mathcal{S}_2)$ be abstract simplicial complexes. A map $f : V_1 \to V_2$ is said to be a **simplicial map** if $f(S_1) \in \mathcal{S}_2$ for all $S_1 \in \mathcal{S}_1$, that is, if $f$ maps simplices to simplices. We say that $f$ is an isomorphism if $f$ is a bijective simplicial map whose inverse is also simplicial. ❖

**Definition 1.2.8.** A **geometric simplicial complex** in $\mathbb{R}^n$ is a collection $\mathcal{K}$ of geometric simplices in $\mathbb{R}^n$ such that

1. If a geomeFtric simplex $K$ is in $\mathcal{K}$, then all faces of $K$ are also in $\mathcal{K}$.
2. If $K$ and $T$ are two geometric simplices in $\mathcal{K}$ which share a point in common, then $K \cap T$ is a common face of both $K$ and $T$.

❖

Given a geometric simplicial complex $\mathcal{K}$, we can construct an abstract simplicial complex $A = (V, \mathcal{S})$ in the following way. For each member $K$ of $\mathcal{K}$, let $S_K$ denote the set of all the vertices of $K$. Let $V = \bigcup_{K \in \mathcal{K}} S_K$ and $\mathcal{S} = \{S_K\}_{K \in \mathcal{K}} \cup \{\emptyset\}$. It can be easily seen that this indeed defines an abstract simplicial complex $A$. We refer to $A$ as the abstract simplicial complex **underlying** $\mathcal{K}$.

**Definition 1.2.9.** Let $A$ be an abstract simplicial complex. A **geometric realization** of $A$ is a geometric simplicial complex whose underlying abstract simplicial complex is isomorphic to $A$. We denote a geometric realization of $A$ by $|A|$. ❖

Every abstract simplicial complex $A = (V, \mathcal{S})$ admits a geometric realization. Let $n = |V| - 1$ and identify the points in $V$ to the vertex set of a geometric $n$-simplex $\Delta$. Then for each simplex $S$ in $A$ we define a face $F_S$ of $\Delta$ as the convex hull of the vertices of $\Delta$ corresponding to the points in $S$. The collection $\mathcal{K} = \{F_S\}_{S \in \mathcal{S}}$ gives a geometric simplicial complex whose underlying abstract simplicial complex is isomorphic to $A$.

**Definition 1.2.10.** Let $A = (V, \mathcal{S})$ be an abstract simplicial complex and '$x$' be a formal symbol. We define an abstract simplicial complex $x * A$, whose vertex set is $V \cup \{x\}$, and whose simplices are, in addition to all the simplices in $\mathcal{S}$, all sets of the form $\{x\} \cup S$ where $S \in \mathcal{S}$. One can check that $x * A$ is indeed an abstract simplicial complex and is called the **cone** on $A$. ❖

Recall that given a topological space $X$, we define the **(topological) cone** on $X$ as the space $X \times I / X \times \{0\}$ and denote it by $\mathsf{C}X$. It is straightforward to see that the geometric realization of the cone on an abstract simplicial complex $A$ is homeomorphic to the topological cone on the geometric realization of $A$.

## 1.3 Regular CW Complexes

We closely follow the material in Section 5.3 of [4].

**Definition 1.3.1.** A CW complex structure on a topological space $X$ is said to be **regular** if each cell in $X$ admits a characteristic map which is a homeomorphism. ❖

**Definition 1.3.2.** Given a subset $S$ of a CW complex $X$, we define the **carrier** of $S$ as the intersection of all the subcomplexes of $X$ which contain $S$. ❖

Since an arbitrary intersection of subcomplexes is again a subcomplex, the carrier of a subset $S$ of a CW complex $X$ is the smallest subcomplex of $X$ which contains $S$.

**Theorem 1.3.3** (Borsuk-Ulam Theorem). *Let $f : S^n \to \mathbb{R}^n$ be a continuous map. Then there exists a point $\mathbf{x} \in S^n$ such that $f(\mathbf{x}) = f(-\mathbf{x})$.*

**Proof.** See Corollary 2B.7 in [6]. ■

**Corollary 1.3.4.** *There is no embedding of $S^n$ in $\mathbb{R}^n$.*

**Proof.** By the Borsuk-Ulam theorem, any continuous map $S^n \to \mathbb{R}^n$ admits a pair of antipodal points in $S^n$ which have the same image. Thus there does not exist an injective continuous map $S^n \to \mathbb{R}^n$, proving the theorem. ■

**Theorem 1.3.5** (Invariance of Domain). *Let $f : U \to \mathbb{R}^n$ be an injective continuous map from an open subset $U$ of $\mathbb{R}^n$ into $\mathbb{R}^n$. Then $f(U)$ is open in $\mathbb{R}^n$ and $f$ maps $U$ homeomorphically onto $f(U)$.*

**Proof.** See Theorem 2B.3 in [6]. ■

**Definition 1.3.6.** Let $e_\alpha$ and $e_\beta$ be cells in a CW complex. We say that $e_\alpha$ is a **face** of $e_\beta$ if $e_\alpha \subseteq \bar{e}_\beta$. ❖

**Theorem 1.3.7.** *Let $e_\alpha^{n-1}$ and $e_\beta^n$ be $(n-1)$ and $n$-cells respectively in a regular CW complex. Assume that $e_\alpha^{n-1} \cap \partial e_\beta^n \neq \emptyset$. Then $e_\alpha^{n-1} \subseteq \partial e_\beta^n$. In other words, $e_\alpha^{n-1}$ is a face of $e_\beta^n$.*

**Proof.** Since $\partial e_\beta^n \subseteq X^{n-1}$ and $e_\alpha^{n-1}$ is open in $X^{n-1}$, we have $e_\alpha^{n-1} \cap \partial e_\beta^n$ is open in $\partial e_\beta^n$. Note that $e_\alpha^{n-1} \cap \partial e_\beta^n \neq \partial e_\beta^n$. For otherwise we would have an embedding of $S^{n-1} \cong \partial e_\beta^n$ into $\mathbb{R}^{n-1} \cong e_\alpha^{n-1}$, contrary to Theorem 1.3.4. This helps us invoke Theorem 1.3.5 to conclude that $e_\alpha^{n-1} \cap \partial e_\beta^n$ is open in $e_\alpha^{n-1}$. But since $\partial e_\beta^n$ is compact, we also know that $\partial e_\beta^n \cap e_\alpha^{n-1}$ is closed in $e_\alpha^{n-1}$. Therefore if $e_\alpha^{n-1} \cap \partial e_\beta^n \neq \emptyset$, the connectedness of $e_\alpha^{n-1}$ forces $e_\alpha^{n-1} \subseteq \partial e_\beta^n$. ■

**Theorem 1.3.8.** *Let $e$ be a cell in a regular CW complex $X$. Then the carrier of a cell $e$ in $X$ is the closure of $e$.*

**Proof.** We prove by induction on the dimension of $e$. The base case is $\dim e = 0$, in which case the proof is trivial. Let $n > 0$ and inductively assume that the theorem is true whenever $\dim e < n$. Let $e^n$ be an $n$ cell in a regular CW complex $X$. We need to show that $C(e^n) = \bar{e}^n$. Since $X$ is a regular CW complex, we can choose an attaching map $\varphi : S^{n-1} \to X^{n-1}$ for $e^n$ such that $\varphi$ is an embedding. Let $e_1^{n-1}, \ldots, e_k^{n-1}$ be all the $(n-1)$-cells in $X$ which intersect $\partial e^n$ and define $F = \bigcup_{i=1}^k \bar{e}_i^{n-1}$. Since each $\bar{e}_i^{n-1}$ is contained in $\partial e^n$, we have $F \subseteq \partial e^n$.

We claim that $F = \partial e^n$. Suppose not. Then $O := \partial e^n - F$ is open in $\partial e^n$ and is nonempty. Therefore $\varphi : \varphi^{-1}(O) \to X^{n-1}$ is an embedding of a nonempty open subset of $S^{n-1}$ into $X^{n-1}$, contradicting Theorem 1.3.5. ■

**Corollary 1.3.9.** *Let $X$ be a regular CW complex and $e^n_\beta$ be an $n$-cell in $X$. If $e^k_\alpha \cap \partial e^n_\beta \neq \emptyset$ for some $k$-cell $e^k_\alpha$ in $X$, where $k < n$, then $e^k_\alpha \subseteq \partial e^n_\beta$. In other words, $e^k_\alpha$ is a face of $e^n_\beta$.*

**Definition 1.3.10.** Let $X$ be a regular CW complex. We define a poset structure on the set of all the cells in $X$ by writing

$$e_\alpha \leq e_\beta$$

if and only if two cells $e_\alpha$ and $e_\beta$ in $X$ satisfy $e_\alpha \cap \bar{e}_\beta \neq \emptyset$. This poset is called the **face poset** of $X$ and is written as $\mathrm{FP}(X)$. ❖

Corollary 1.3.9 implies that the above definition makes sense, that is, writing $e_\alpha \leq e_\beta$ if and only if $e_\alpha$ intersects the boundary of $e_\beta$ actually gives a poset structure to the set of all the cells.

**Definition 1.3.11.** Let $X$ be a regular CW complex. The order complex of the face poset of $X$ is called the **barycentric subdivision** of $X$ and is written as $\mathrm{sd}(X)$. ❖

**Definition 1.3.12.** Given a convex subset $C$ of $\mathbb{R}^n$, we define the **relative interior** of $C$ as the topological interior of $C$ in the affine subspace spanned by $C$. We denote the relative interior of $C$ as $\mathrm{RelInt}(C)$. ❖

**Theorem 1.3.13.** *Let $X$ be a finite regular CW complex. Then the geometric realization of the barycentric subdivision of $X$ is homeomorphic to $X$. In fact there is a homeomorphism $h : |\mathrm{sd}(X)| \to X$ such that for each $k$-simplex $\{e_0, \ldots, e_k\}$ in $\mathrm{sd}(X)$ with $e_0 < \cdots < e_n$, we have $h(\mathrm{RelInt}(|\{e_0, \ldots, e_k\}|)) \subseteq e_k$.*

**Proof.** We prove by induction on the dimension of $X$, the base case being $\dim X = 0$, in which case the theorem clearly holds. Suppose $n > 0$ and we have a homeomorphism $h_{n-1} : |\mathrm{sd}(X^{n-1})| \to X^{n-1}$ satisfying the property in the theorem, where $X^k$ denotes the $k$-skeleton of $X$. We will extend this map to $X^n$.

Note that, by the property in the theorem, for each $n$-cell $e$ in $X$ we have $|\mathrm{sd}(\partial e)|$ is mapped homeomorphically onto $\partial e$ by $h_{n-1}$. Therefore, there is a homeomorphism $\mathrm{cone}(|\mathrm{sd}(\partial e)|) \to \mathrm{cone}(\partial e)$ induced by the restriction of $h_{n-1}$ on $|\mathrm{sd}(\partial e)|$. But $\mathrm{cone}(\mathrm{sd}(\partial e))$ is homeomorphic to $|e * \mathrm{sd}(\partial e)|$. Further, since $\partial e$ is homeomorphic to $S^{n-1}$, $\mathrm{cone}(\partial e)$ is homeomorphic to $D^n$, and hence to $\bar{e}$ too. So we get an homeomorphism $|e * \mathrm{sd}(\partial e)| \to \bar{e}$ which extends the restriction of $h_{n-1}$ on $\mathrm{sd}(\partial e)$. Doing this for all the $n$-cells in $X$, we get a map $h_n : |\mathrm{sd}\, X^n| \to X^n$ which extends $h_{n-1}$ and satisfies the property in the statement of the theorem. The only thing left to check is that $h_n$ is continuous, but that is obvious by using the pasting lemma (See Theorem 18.3 in [7]). ■

## 1.4 Acyclic Categories

**Definition 1.4.1.** A category $C$ is said to be an **acyclic category** if

1. For all objects $A$ in $C$, we have $\mathrm{End}(A) = \{\mathrm{id}_A\}$, and
2. For any two distinct objects $A$ and $B$ in $C$, we have either $\mathrm{Mor}(A, B)$ is empty or $\mathrm{Mor}(B, A)$ is empty. ❖

Every poset can be naturally thought of as an acyclic category. If $P$ is a poset, then we get a category $C$ whose objects are all the elements of $P$, and for two objects $x$ and $y$ in $\mathrm{Ob}(C)$, we have a morphism from $x$ to $y$ if and only if $x \leq y$.

**Definition 1.4.2.** Given a category $C$ and $n \geq 1$. An $n$-**chain** in $C$ is an $n$-tuple $(u_n, \dots, u_1)$ of morphisms $u_1, \dots, u_n$ in $C$ such that for each $1 \leq i < n$ the composition $u_{i+1} \circ u_i$ makes sense. A **non-degenerate** $n$-**chain** is an $n$-chain having no entry as the identity morphism. Also, a 0-chain in $C$ is same as an object in $C$. We denote the set of all the $n$-chains in $C$ by $N_n(C)$ and the set of all the non-degenerate $n$-chain in $C$ by $\bar{N}_n(C)$. ❖

**Definition 1.4.3.** Let $C$ be a category and $n > 1$. For each $0 \leq i \leq n$ we define a map $d_i : N_n(C) \to N_{n-1}(C)$ as

$$d_i(u_n, \dots, u_1) = \begin{cases} (u_n, \dots, u_2) & \text{if} \quad i = 0 \\ (u_n, \dots, u_{i+1} \circ u_i, \dots, u_1) & \text{if} \quad 1 \leq i \leq n_1 \\ (u_{n-1}, \dots, u_1) & \text{if} \quad i = n \end{cases}$$

Also, define maps $d_0, d_1 : N_1(C) \to N_0(C)$ by declaring, for each morphism $u$ in $C$, $d_0(u)$ as the source of the morphism $u$ and $d_1(u)$ as the target of the morphism $u$. The maps $d_i$ defined in this way are called the **face operators**. ❖

Note that if $C$ is an acyclic category, then we may restrict the face operators to get maps $d_i : \bar{N}_n(C) \to \bar{N}_{n-1}(C)$.

**Definition 1.4.4** (Lemma 2.10 in [3])**.** Let $C$ be a finite acyclic category. We define the **geometric realization** of $C$ as

$$\left( \coprod_{n=0}^{\infty} \bar{N}_n(C) \times \Delta^n \right) / \sim$$

where $\sim$ is an equivalence relation on $\coprod_{n=0}^{\infty} \bar{N}_n(C) \times \Delta^n$ defined by declaring

$$(d_i(u_n, \dots, u_1), (t_0, \dots, t_{n-1})) \sim ((u_n, \dots, u_1), (t_0, \dots, t_{i-1}, 0, t_i, \dots, t_{n-1}))$$

Here $\Delta^n$ is the standard $n$-simplex in $\mathbb{R}^{n+1}$. ❖

# Chapter 2

# Configuration Spaces of Points

## 2.1 Motivation

An automated guided vehicle (AGV) is a robot which can move on the floor and carry load from one point to other. Some AGVs are designed to move freely in any direction of the two dimensional floor. But such AGVs are costly to build and maintain. Therefore, many industries use 'line following' AGVs which are constrained to move on a network of guidepath wires etched on the factory floor or hanging from the ceiling.

Consider $n$ distinct robots moving on a terrain modelled by a topological space $X$ such that the robots move continuously and no two robots collide. The motion of $n$ robots on $X$ can be equivalently thought of as the motion of a single point on $X^n$. The $i$-th entry at any point of time gives the position of the $i$-th robot. Since we do not want the robots to collide, a particular subset of $X^n$ is forbidden. This forbidden subset, called the *$n$-diagonal* of $X$, is the set of all the points in $X^n$ which have at least two distinct entries equal. The complement of the $n$-diagonal is called the *configuration space of $n$ points on $X$*. This leads to the following definition.

**Definition 2.1.1.** Let $X$ be a topological space. The $n$-**diagonal** of $X$ is defined as

$$\mathrm{Diag}_n(X) = \{(x_1, \ldots, x_n) : \; x_i = x_j \text{ for some } i \neq j\}$$

The **configuration space** of $n$ points on $X$ is defined as

$$\mathrm{Conf}_n(X) = X^n - \mathrm{Diag}_n(X)$$

❖

A network of guidepath wires on a factory floor can be modelled by a graph. Therefore we are mainly interested in the case when $X$ is a graph.

## 2.2 Examples

**Example 2.2.1.** Let $X$ be the closed interval $I = [0, 1]$. Then $\mathrm{Conf}_2(X)$ can be pictured as follows:

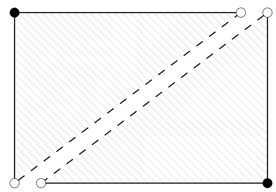


Figure 2.1: $\mathrm{Conf}_2(I)$

**Example 2.2.2.** Let $X = S^1$. The configuration space of two points on $X$ is a cylinder. To see this, first think of the torus $S_1 \times S_1$ as formed by identifying opposite edges of a rectangle. Then it can be seen that removing the four endpoints of the rectangle as well as the diagonal, and then identifying the opposite edges gives $\mathrm{Conf}_2(S^1)$. The rest is clear from the following diagram:

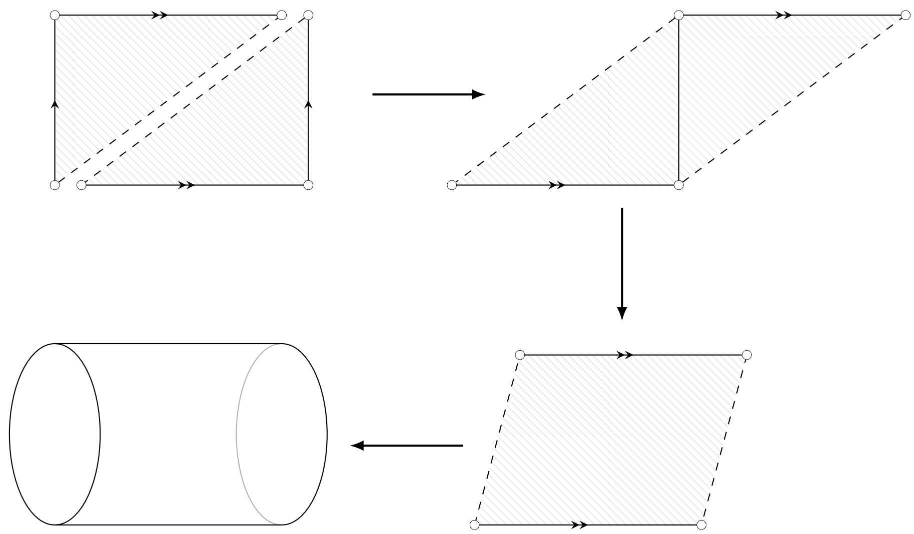


Figure 2.2: $\mathrm{Conf}_2(S^1)$

# Chapter 3

# Stratified Spaces

## 3.1 Stratified Spaces

**Definition 3.1.1** (Definition 2.23 in [3])**.** Let $X$ be a topological space and $\Lambda$ be a poset. A **stratification** of $X$ is a map $\pi : X \to \Lambda$ such that

1. For each $\lambda \in \mathrm{Im}(\pi)$, $\pi^{-1}(\lambda)$ is locally closed[1] and connected.
2. For $\lambda, \mu \in \mathrm{Im}(\pi)$, $\pi^{-1}(\lambda) \subseteq \overline{\pi^{-1}(\mu)}$ if and only if $\lambda \leq \mu$. This is same as demanding that $\pi$ is continuous when $\Lambda$ is given the Alexandroff topology.

For each $\lambda \in \Lambda$, we write $e_\lambda$ to denote $\pi^{-1}(\lambda)$ and call it a **stratum** indexed by $\lambda$. A **stratified space** is a pair $(X, \pi)$, where $\pi : X \to \Lambda$ is a stratification of $X$. The image of $\pi$ is called the **face poset** of $X$ and is denoted by $P(X, \pi)$ or simply by $P(X)$. ❖

A stratification of a topological space $X$ can be thought of as a partition $\mathcal{E}$ of $X$ such that if we write $e_\alpha \leq e_\beta$ for two members $e_\alpha$ and $e_\beta$ of $\mathcal{E}$ if and only if $e_\alpha$ intersects $\bar{e}_\beta$, then this makes $\mathcal{E}$ into a poset. We further insist that the elements of $\mathcal{E}$ be connected and locally closed.

**Example 3.1.2.** Any Hausdorff space admits a trivial stratification. For let $X$ be any Hausdorff space and $\Lambda$ be a poset which as a set is in bijection with $X$ and no two distinct elements in $\Lambda$ are comparable. Then any bijection $\pi : X \to \Lambda$ gives a stratification on $X$.

**Example 3.1.3.** Any normal CW complex[2] structure on a Hausdorff topological space $X$ induces a stratification on $X$ in the following way. Let $\{e_\lambda\}_{\lambda \in \Lambda}$ be all the cells in the CW complex. We give a poset structure to $\Lambda$ by declaring $\lambda \leq \mu$ for two elements $\lambda$ and $\mu$ in $\Lambda$ if and only if $e_\lambda \subseteq \bar{e}_\mu$. Define $\pi : X \to \Lambda$ as $\pi(e_\lambda) = \lambda$ for all $\lambda \in \Lambda$. Then $\pi$ is a stratification on $X$. This makes use of the fact that a characteristic map $\Phi : D^n \to \bar{e}_\lambda$ for an $n$-cell $e_\lambda$ is a quotient map which maps the interior of $D^n$ homeomorphically onto $e_\lambda$.

**Notation** Let $e$ be a stratum in a stratification of a space $X$. Then we write $\partial e$ to denote $\bar{e} - e$. Note that this may be different from the topological boundary of $e$ in $X$.

[1] A subset $A$ of a topological space $X$ is said to be **locally closed** if $A$ is open in $\bar{A}$.

[2] A CW complex is said to be **normal** if whenever a cell $e_\alpha$ intersects the closure of a cell $e_\beta$, then $e_\alpha \subseteq \bar{e}_\beta$. Note that all regular CW complexes are normal.

**Definition 3.1.4.** Let $\pi : X \to \Lambda$ be a stratification of a topological space $X$. A stratum $e_\mu$ is said to be **normal** if $e_\lambda \subseteq \bar{e}_\mu$ whenever $e_\lambda \cap \bar{e}_\mu$ is non-empty. This is equivalent to saying that $\partial e_\mu$ is a union of strata in $X$. We say that a stratification on $X$ is a **normal stratification** if each stratum is normal. ❖

**Example 3.1.5.** We give an example of a stratification which is not normal. Let $A$ and $B$ be subspaces of $\mathbb{R}^2$ defined as $A = \{(-1, y) : -1 < y < 1\}$ and $B = \{(x, y) : x^2 + y^2 < 1\}$. Let $X \subseteq \mathbb{R}^2$ be the union of $A$ and $B$.

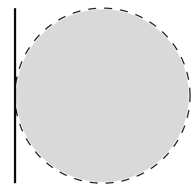


Figure 3.1: A non-normal stratification.

Let $P = \{a, b\}$ be the poset where neither $a < b$ nor $b < a$. Then $\pi : X \to P$ defined as $\pi(A) = \{a\}$ and $\pi(B) = \{b\}$ gives a stratification on $X$ which is not normal.

**Example 3.1.6.** Let $P = \{-1, 0, 1\}$ be a poset with $0 < \pm 1$. Define a map $\operatorname{sgn} : \mathbb{R} \to P$ which takes positive numbers to 1, 0 to 0, and negative numbers to $-1$. This defines a normal stratification on $\mathbb{R}$ which we call the **sign stratification of** $\mathbb{R}$.

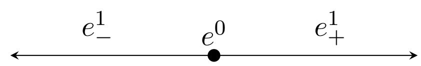



Figure 3.2: The sign stratification on $\mathbb{R}$.

**Lemma 3.1.7.** *Let $\rho : Y \to \Gamma$ be a stratification of a topological space $Y$ and let $f : X \to Y$ be an open continuous map which pulls back connected subspaces of $Y$ to connected subspaces of $X$. Then $\rho \circ f : X \to \Gamma$ gives a stratification of $X$. Further, if $(Y, \rho)$ is normal, then so is $(X, \rho \circ f)$.*

**Proof.** Write $\pi = \rho \circ f$. For each $\gamma \in \Gamma$, write $a_\gamma$ to denote $\pi^{-1}(\gamma)$ and $b_\gamma$ to denote $\rho^{-1}(\gamma)$. To settle the first assertion of the proposition we only need to check that if $\gamma \in \Gamma$ is in the image of $\pi$, then $a_\gamma$ is locally closed. Since $f$ is open, we have $f^{-1}(\bar{B}) = \overline{f^{-1}(B)}$ for all $B \subseteq Y$.[3] Now let $\gamma \in \Gamma$ be in the image of $\pi$. Since $b_\gamma$ is open in $\bar{b}_\gamma$, we have $f^{-1}(b_\gamma)$ open in $\overline{f^{-1}(b_\gamma)}$. Using $a_\gamma = f^{-1}(b_\gamma)$, we are done.

To prove the second part, let $\gamma, \theta \in \Gamma$ be such that $a_\gamma \cap \bar{a}_\theta \neq \emptyset$. We need to show that $a_\gamma \subseteq \bar{a}_\theta$. Let $x \in a_\gamma \cap \bar{a}_\theta$. Then since the openness of $f$ gives $f^{-1}(\bar{b}_\theta) = \overline{f^{-1}(b_\theta)}$, we have $f(x) \in b_\gamma \cap \bar{b}_\theta$. By normality of $(Y, \rho)$, we have $b_\gamma \subseteq \bar{b}_\theta$, giving $a_\gamma \subseteq \bar{a}_\theta$. ■

**Example 3.1.8.** Let $\ell : \mathbb{R}^n \to \mathbb{R}$ be a surjective affine map and let $H = \ell^{-1}(0)$. Then $H$ is a hyperplane in $\mathbb{R}^n$. Let $P = \{-1, 0, 1\}$ be the poset given by $0 < \pm 1$. Define $\pi : \mathbb{R}^n \to P$ as $\pi = \operatorname{sgn} \circ \ell$. Since $\ell$ is an open map, by Lemma 3.1.7 we know that $\pi$ gives a normal stratification on $\mathbb{R}^n$. This stratification on $\mathbb{R}^n$ is called the stratification induced by the hyperplane $H$.[4]

[3] This is not true if $f$ is not open. For consider the identity map $\mathrm{id} : \mathbb{R} \to \mathbb{R}$, where the domain $\mathbb{R}$ has the discrete topology and the target $\mathbb{R}$ has the cofinite topology. Then $\mathbb{Z}$ is dense in the target $\mathbb{R}$ but not in the domain $\mathbb{R}$. Therefore $\mathbb{R} = \mathrm{id}^{-1}(\bar{\mathbb{Z}}) \neq \overline{\mathrm{id}^{-1}(\mathbb{Z})} = \mathbb{Z}$. I am indebted to my graduate colleague Gautam Aishwarya for this example.

[4] Rather, we should call is the stratification induced by the affine map $\ell : \mathbb{R}^n \to \mathbb{R}$.

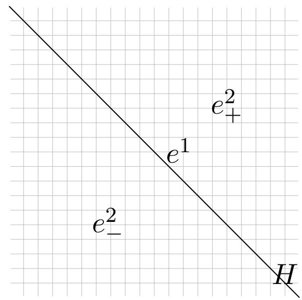



Figure 3.3: Stratification of $\mathbb{R}^2$ via a line.

**Example 3.1.9** (Example 2.11 in [9]). Let $\Delta^n = \{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} : x_0+\cdots+x_n = 1, x_i \geq 0 \text{ for all } i\}$ be the standard $n$-simplex in $\mathbb{R}^{n+1}$. Let $P$ be the partial order given by inclusion on the power set of $\{0, \ldots, n\}$. Define a map $\pi_n : \Delta^n \to P$ as

$$\pi_n(x_0, \ldots, x_n) = \{i : \ x_i \neq 0\}$$

Then $\pi$ defines a normal stratification on $\Delta^n$ whose strata are the faces $\Delta^n$.

**Example 3.1.10** (Lemma 2.14 in [9]). Let $\pi : X \to \Lambda$ and $\rho : Y \to \Gamma$ be stratifications of $X$ and $Y$. We claim that $\pi \times \rho : X \times Y \to \Lambda \times \Gamma$ is a stratification of $X \times Y$, called the **product stratification** on $X \times Y$. Let $\lambda \in P(X)$ and $\gamma \in P(Y)$. We check that $(\pi \times \rho)^{-1}(\lambda, \gamma) = e_\lambda \times e_\gamma$ is connected and locally closed in $X \times Y$. The connectedness is clear since product of two connected spaces is again connected. The local closure follows from observing that $e_\lambda \times e_\gamma$ is open in $\overline{e_\lambda \times e_\gamma} = \bar{e}_\lambda \times \bar{e}_\gamma$ since $e_\lambda$ is open in $\bar{e}_\lambda$ and $e_\gamma$ is open in $\bar{e}_\gamma$.

To check the second condition we need only check that $\pi \times \rho$ is continuous when $\Lambda \times \Gamma$ has the Alexandroff topology. But this is clear since the product topology on $\Lambda \times \Gamma$ is same as the Alexandroff topology on the product poset $\Lambda \times \Gamma$.

It is easily checked that the product of two normal stratified spaces is again a normal stratified space.

**Lemma 3.1.11.** *Let $C$ be a convex subset of $\mathbb{R}^n$. Let $\mathbf{x}_0$ be a point in the relative interior of $C$ and $\mathbf{x}$ be a point in $\bar{C}$. Then the relative interior of the line segment joining $\mathbf{x}$ and $\mathbf{x}_0$ is contained in $C$.*

**Proof.** Let $\mathbf{y}$ be a point in the relative interior of the line segment joining $\mathbf{x}$ and $\mathbf{x}_0$. We need to show that $\mathbf{y} \in C$. Let $(\mathbf{x}_n)$ be a sequence of points in $C$ which converge to $\mathbf{x}$. Without loss of generality we may assume that the affine span of $C$ is whole of $\mathbb{R}^n$, for otherwise we may pass to the affine span of $C$. So $\mathbf{x} \in \mathrm{Int}(C)$, and we can choose an open ball $B$ around $\mathbf{x}$ such that $B$ is contained in $C$. For sufficiently large $N$, the line passing through $\mathbf{x}_N$ and $\mathbf{y}$ also passes through a point $\mathbf{b}$ in $B$. Since both $\mathbf{x}_N$ and $\mathbf{b}$ lie in $C$, the convexity of $C$ forces that $\mathbf{y} \in C$ and we are done. ■

**Lemma 3.1.12.** *Let $L = \{\ell_1, \ldots, \ell_k\}$ be a set of surjective affine maps $\mathbb{R}^n \to \mathbb{R}$. For each $1 \leq i \leq k$, let $S_i$ be one of the three subsets $\ell_i^{-1}((-\infty, 0))$, $\ell_i^{-1}(0)$, and $\ell_i^{-1}((0, \infty))$ of $\mathbb{R}^n$. Assume that $S_1 \cap \cdots \cap S_k \neq \emptyset$. Then we have*

$$\bar{S}_1 \cap \cdots \cap \bar{S}_k = \overline{S_1 \cap \cdots \cap S_k}$$

**Proof.** Write $S = S_1 \cap \cdots \cap S_k$. Since $\bar{S}_1 \cap \cdots \cap \bar{S}_k$ is a closed subset of $\mathbb{R}^n$ which contains $S$, we have

$\bar{S} \subseteq \bar{S}_1 \cap \cdots \cap \bar{S}_k$. So we need to show that reverse containment. Since $S$ is non-empty, we can choose a point $\mathbf{x}_0$ in $S$. Now let $\mathbf{x}$ be chosen arbitrarily in $\bar{S}_1 \cap \cdots \cap \bar{S}_k$. We will show that $\mathbf{x} \in \bar{S}$.

Note that the relative interior of each $S_i$ is all of $S_i$. Let $(\mathbf{x}_0, \mathbf{x})$ denote the set of all the points in the relative interior of the line segment joining $\mathbf{x}_0$ and $\mathbf{x}$. By Lemma 3.1.11, $(\mathbf{x}_0, \mathbf{x})$ is contained in each $S_i$. Therefore we have $(\mathbf{x}_0, \mathbf{x}) \subseteq S$. Now since $\mathbf{x}$ is in the closure of $(\mathbf{x}_0, \mathbf{x})$, we have $\mathbf{x} \in \bar{S}$ and we are done. ■

**Example 3.1.13** (Example 2.10 in [9]). Let $L = \{\ell_1, \dots, \ell_k\}$ be a set of surjective affine maps $\mathbb{R}^n \to \mathbb{R}$ and $H_i$ be the hyperplane in $\mathbb{R}^n$ defined as $H_i = \ell_i^{-1}(0)$. The set $\mathcal{A} := \{H_1, \dots, H_k\}$ is called the **hyperplane arrangement** determined by the affine maps in $L$. Let $P = \{-1, 0, 1\}$ be the poset given by $0 < \pm 1$. Define maps $\pi_i : \mathbb{R}^n \to P$ as $\pi_i = \operatorname{sgn} \circ \ell_i$ and define $\pi : \mathbb{R}^n \to P^k$ as $\pi(\mathbf{x}) = (\pi_1(\mathbf{x}), \dots, \pi_k(\mathbf{x}))$. We claim that $\pi$ is a stratification of $\mathbb{R}^n$, where $P^k$ has the product poset structure.

Define $H_i^+ = \pi_i^{-1}(1)$ and $H_i^- = \pi_i^{-1}(-1)$. Note that each fibre of $\pi$ is of the form $S_1 \cap \cdots \cap S_k$, where for each $i$, $S_i$ is one of $H_i^-$, $H_i$, or $H_i^+$. Let $A$ and $A'$ be two fibres of $\pi$, and say $A = S_1 \cap \cdots \cap S_k$ and $A' = S_1' \cap \cdots \cap S_k'$. Suppose $A \cap \bar{A}'$ is non-empty. We proceed to show that $A \subseteq \bar{A}'$. To see this, we use Lemma 3.1.12 to write $\bar{A}' = \bar{S}_1' \cap \cdots \cap \bar{S}_k'$. Thus from $A \cap \bar{A}' \neq \emptyset$ we get for each $i$ that $S_i \cap \bar{S}_i' \neq \emptyset$. But it is clear that if $S_i \cap \bar{S}_i' \neq \emptyset$, then $S_i \subseteq \bar{S}_i'$. So we have $S_i \subseteq \bar{S}_i'$ for each i, and thus

$$S_1 \cap \cdots \cap S_k \subseteq \bar{S}_1' \cap \cdots \cap \bar{S}_k'$$

giving $A \subseteq \bar{A}'$. This shows that if for two members $\lambda, \mu \in P^k$, we write $\pi^{-1}(\lambda) \leq \pi^{-1}(\mu)$ if and only if $\pi^{-1}(\lambda) \cap \overline{\pi^{-1}(\mu)} \neq \emptyset$, then we get a poset structure on the set of all the fibres of $\pi$. Also, each fibre of $\pi$ is an intersection of open half-spaces and hyperplanes. Therefore, the fibres of $\pi$ are all convex sets which are open in their affine hulls, and are hence connected and locally closed in $\mathbb{R}^n$.

Now all that remains is to show that $\pi$ is continuous when $P^k$ is given the Alexandroff topology. For each $i$, choose a point $\mathbf{p}_i \in H_i$ and a unit vector $\mathbf{v}_i \in \mathbb{R}^n$ perpendicular to $H_i$ such that $\mathbf{p}_i + \mathbf{v}_i \in H_i^+$. Let $K_i$ denote the affine subspace $\{\mathbf{p}_i + t\mathbf{v}_i : \; t \in \mathbb{R}\}$. Let $\rho_i : \mathbb{R}^n \to K_i$ be the projection on $K_i$ with respect to the hyperplane $H_i$. Just like the sign stratification on $\mathbb{R}$, the map $\tau_i : K_i \to P$ defined as $\tau_i(\mathbf{p}_i + t\mathbf{v}_i) = \operatorname{sgn}(t)$ defines a stratification of $K_i$. Therefore, the map $\tau = \tau_1 \times \cdots \times \tau_k : K_1 \times \cdots \times K_k \to P^k$ gives a stratification of $K := K_1 \times \cdots \times K_k$. Now consider the following diagram

$$\begin{tikzcd}
\mathbb{R}^n \arrow[r, "\rho = \rho_1 \times \cdots \times \rho_k"] \arrow[dr, "\pi"'] & K_1 \times \cdots \times K_k \arrow[d, "\tau = \tau_1 \times \cdots \times \tau_k"] \\
& P^k
\end{tikzcd}$$

This shows that $\pi$ is continuous when $P^k$ has the Alexandroff topology since $\pi = \tau \circ \rho$, finishing the proof that $\pi$ is a stratification of $\mathbb{R}^n$. The stratification of $\mathbb{R}^n$ so obtained is called the stratification of $\mathbb{R}^n$ determined by $\mathcal{A}$. From the work done above it is clear that this stratification is normal.

**Definition 3.1.14.** Let $n$ be a positive integer and for each $1 \leq i, j \leq n$ define $H_{i,j} = \{(x_1, \dots, x_n) \in \mathbb{R}^n : \; x_i = x_j\}$. The hyperplane arrangement $\{H_{i,j} : \; 1 \leq i, j \leq n\}$ is called the **braid arrangement** of rank $n-1$ and is denoted by $\mathcal{A}_{n-1}$. The stratification on $\mathbb{R}^n$ obtained by the braid arrangement is called the **braid stratification**. ❖

**Example 3.1.15.** The following figure depicts a normal stratification of the punctured torus.

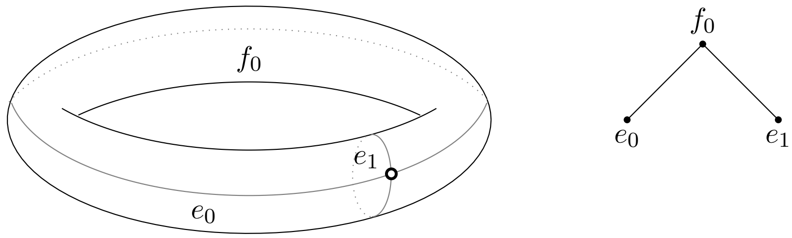



Figure 3.4: The Punctured Torus and its Face Poset.

**Definition 3.1.16** (Definition 2.15 in [9])**.** Let $\pi : X \to \Lambda$ and $\rho : Y \to \Gamma$ be stratifications of topological spaces $X$ and $Y$ indexed by the posets $\Lambda$ and $\Gamma$. A **morphism of stratified spaces** is a pair $(F, f)$ of a continuous map $F : X \to Y$ and a poset map $f : P(X) \to P(Y)$ making the following diagram commute:

$$\begin{tikzcd} X \arrow[r, "F"] \arrow[d, "\pi"] & Y \arrow[d, "\rho"] \\ P(X) \arrow[r, "f"] & P(Y) \end{tikzcd}$$

We say that $(F, f)$ is a **strict morphism** if $F(e_\lambda) = e_{f(\lambda)}$ for all $\lambda \in P(X)$. ❖

It can be seen that the identity map $(\mathrm{id}, \mathrm{id})$ from a stratified space to itself is a morphism and that composition of two morphisms of stratified spaces is again a morphism. The associativity of the compositions of morphisms is clear. Thus we have a category of stratified spaces.

**Example 3.1.17.** Following gives an example of a morphism of stratified spaces. We have a map $f : S^1 \to S^1$ which sends $z$ to $z^2$, when $S^1$ is thought of as a subset of the complex plane. The stratifications on the domain and the target $S^1$'s are depicted in the diagram. The map $f$ sends $+1$ and $-1$ to 1 and $+0$ and $-0$ to 0.

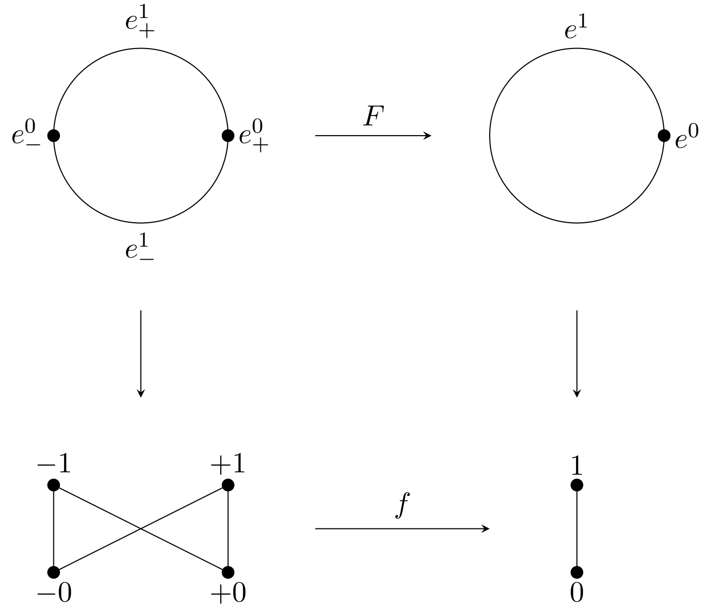



Figure 3.5: A strict morphism between two stratification of $S^1$.

**Definition 3.1.18** (Definition 2.18 in [9])**.** Let $\pi : X \to \Lambda$ be a stratification of a space $X$. Let $A$ be a subspace of $X$ and the restriction $\pi|_A : A \to \Lambda$ be a stratification of $A$. Then we say that $(A, \pi|_A)$ is a **stratified subspace** of $(X, \pi)$. Note that the pair $(i, \mathrm{id})$ is a morphism from $(A, \pi|_A)$ into $(X, \pi)$, where $i : A \hookrightarrow X$ is the inclusion map. When $(i, \mathrm{id})$ is a strict morphism, we say that $(A, \pi|_A)$ is a **strict stratified subspace** of $(X, \pi)$. ❖

**Theorem 3.1.19.** *Let $\pi : X \to \Lambda$ be a stratification on a topological space $X$ and $A$ be a subspace of $X$. Then $A$ is a strict stratified subspace of $X$ if and only if $A$ is a union of strata.*

**Proof.** First suppose that $A$ is a strict stratified subspace of $X$. Then $\pi|_A : A \to \Lambda$ is a stratification on $A$. Let $x \in A$ be arbitrary and $e_\lambda$ be the strata of $X$ which contains $x$. Since $(i, \mathrm{id})$ is a strict morphism between $(A, \pi|_A)$ and $(X, \pi)$, we have $i(\pi|_A^{-1}(\lambda)) = e_\lambda$, showing that $A \cap e_\lambda = e_\lambda$. Therefore $e_\lambda \subseteq A$ and we see that $A$ is a union of strata of $X$.

Now suppose that $A$ is a union of strata of $X$. Since $\pi : X \to \Lambda$ is continuous when $\Lambda$ is given the Alexandroff topology, so is $\pi|_A : A \to \Lambda$. Also, if $\lambda \in \operatorname{Im} \pi_A$, then $\pi|_A^{-1}(\lambda) = \pi^{-1}(\lambda)$, and therefore $\pi|_A^{-1}(\lambda)$ is connected. So we only need to show that for $\lambda \in \operatorname{Im} \pi|_A$, $e_\lambda = \pi^{-1}(\lambda)$ is open in the closure of $e_\lambda$ in $A$. To this end, let $x \in e_\lambda$ be arbitrary. Since $e_\lambda$ is open in $\bar{e}_\lambda$, we know that there is a neighborhood $U$ of $x$ in $X$ such that $U \cap \bar{e}_\lambda \subseteq e_\lambda$. Therefore $U \cap \mathrm{cl}_A(e_\lambda) \subseteq e_\lambda$ and we are done. ■

## 3.2 CW Stratifications

**Definition 3.2.1** (Definition 2.19 in [9])**.** A stratification $\pi : X \to \Lambda$ on a space $X$ is said to be a **CW stratification** if the following conditions are satisfied:

**(CF)** For each $\lambda \in P(X)$, $\bar{e}_\lambda$ is covered by a finite number of strata.

**(WT)** $X$ has the weak topology with respect to the covering $\{\bar{e}_\lambda\}_{\lambda \in P(X)}$, that is, the topology on $X$ is final with respect to the collection of maps $\{i_\lambda : \bar{e}_\lambda \hookrightarrow X\}_{\lambda \in P(X)}$. ❖

**Definition 3.2.2.** A stratification of a topological space $X$ is said to be finite if the face poset of the stratification is finite. ❖

**Theorem 3.2.3** (Lemma 2.20 in [9])**.** *A finite stratification is always CW.*

**Proof.** Let $\pi : X \to \Lambda$ be a finite stratification of a topological space $X$. The closure finiteness is obvious. We show that the (WT) condition holds. Let $A \subseteq X$ be such that $A \cap \bar{e}_\lambda$ is closed in $\bar{e}_\lambda$ for all $\lambda \in \Lambda$. We need to show that $A$ is closed in $X$. Let $e_1, \dots, e_k$ be all the strata in $X$ and $x \in X \setminus A$. For each $i$ we can find a neighborhood $U_i$ of $x$ such that $U_i \cap (A \cap \bar{e}_i) = \emptyset$. Thus $U := U_1 \cap \cdots \cap U_k$ is a neighborhood of $x$ which does not intersect $A$, showing that $A$ is closed and we are done. ■

## 3.3 Cellular Stratifications

**Definition 3.3.1.** A **globular $n$-cell** is a subset $D$ of the closed $n$-ball $D^n$ such that $\mathrm{Int}(D^n)$ is contained in $D$. The **boundary** of a globular $n$-cell $D$ is defined as $D \cap \partial D^n$ and is denoted by $\partial D$. The **interior** of $D$ is simply defined as $\mathrm{Int}(D^n)$ and is denoted by $\mathrm{Int}(D)$. ❖

**Definition 3.3.2** (Definition 2.24 in [3])**.** Let $X$ be a topological space and $e$ be a subspace of $X$. An **$n$-cell structure** on $e$ is a pair $(D, \varphi)$ of a globular $n$-cell $D$ and a continuous map $\varphi : D \to X$ satisfying the following two conditions:

1. $\varphi|_{\mathrm{Int}(D)} : \mathrm{Int}(D) \to e$ is a homeomorphism.
2. $\varphi : D \to \bar{e}$ is a quotient map.

The number $n$ is called the **dimension** of $e$. The map $\varphi$ will be called the **cell structure map**. An $n$-cell structure $(D, \varphi)$ on $e$ is said to be **closed** if $D = D^n$ and **regular** if $\varphi : D \to \bar{e}$ is a homeomorphism. ❖

**Notation** For a stratum $e_\lambda$ in a stratified space $(X, \pi)$, we use $\partial e_\lambda$ to denote $\bar{e}_\lambda - e_\lambda$.

A *cellular stratified space* is crudely a CW complex in when we have globular cells as domains of characteristic maps. The following definition makes this idea precise.

**Definition 3.3.3** (Definition 2.24 in [3])**.** Let $X$ be a Hausdorff space. A **cellular stratification** of $X$ is a pair $(\pi, \Phi)$ of a stratification $\pi : X \to \Lambda$ and a collection $\Phi = \{\varphi_\lambda : D_\lambda \to X\}_{\lambda \in \pi(X)}$ of cell structures on the strata $e_\lambda$ of $X$ such that for each $n$-cell $e_\lambda$, $\partial e_\lambda$ is covered by cells of dimension no more than $n-1$. A **cellular stratified space** is a triple $(X, \pi, \Phi)$, where $(\pi, \Phi)$ is a cellular stratification on $X$. ❖

Note that a cell structure map separates the boundary and interior of the globular cell. More precisely, if $\varphi_\lambda : D_\lambda \to \bar{e}_\lambda$ is a cell structure map for a stratum $e_\lambda$ of a stratified space, then $\varphi_\lambda(\mathrm{Int}(D_\lambda)) \cap \varphi_\lambda(\partial D_\lambda) = \emptyset$. Therefore $\varphi_\lambda(\partial D_\lambda) = \bar{e}_\lambda - e_\lambda$. This uses the fact that each stratum is locally closed.

An important difference between a CW complex and a cellular stratified space is that a CW complex does not come with the data of characteristic maps. We only insist that each cell of a CW complex 'admits' a characteristic map. On the other hand, a cellular stratified space comes with the information of cell structure maps.

**Theorem 3.3.4.** *Let $(X, \pi, \Phi)$ be a finite cellular stratified space. Then top dimensional*[5] *cells in $X$ are open in $X$.*

**Proof.** Let $n$ be the highest dimension of any stratum in $X$ and let $e^n_\alpha$ be an $n$-cell in $X$. By definition of a stratification, we know that $e^n_\alpha$ is open in $\bar{e}^n_\alpha$. If $e^k_\beta$ is any other stratum, then $\bar{e}^k_\beta$ does not intersect $e^n_\alpha$ because $\partial e^k_\beta$ is covered by cells of dimension no more that $k-1$. Since a finite stratification is automatically CW, we know that the topology on $X$ is finial with respect to the closure of all the cells in $X$. As observed above, $e^n_\alpha$ is open in the closure of each cell in $X$ and is hence open in $X$. ■

**Definition 3.3.5.** A cellular stratified space $(X, \pi, \Phi)$ is said to be **regular** if each cell structure map in $\Phi$ is regular and is said to be **closed** if each cell structure map in $\Phi$ is closed. ❖

As Theorem 1.3.8 shows, each regular CW complex is a normally stratified space. But it not true that each regular cellular stratified space is normal since the stratified space in Example 3.1.5 is not normal but admits a regular cellular stratification.

**Example 3.3.6.** A finite closed cellular stratification on a topological space $X$ is nothing but a finite normal CW complex structure on $X$ for each cell of which we have chosen a cell structure map.

**Example 3.3.7** (Example 2.42 in [9])**.** Not all stratified spaces admit a cellular stratification. Let $S = \{(x, \sin(1/x)) : 0 < x \leq 1\}$ and $X = \bar{S}$. The space $X$ is known as the topologist's sine curve. Consider the poset $P$ given by the following Hasse diagram:

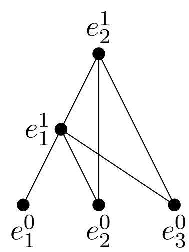



Figure 3.6

Let $E^0_1 = \{(0, 1)\}$, $E^0_2 = \{(0, -1)\}$, $E^0_2 = \{(1, \sin(1))\}$, $E^1_1 = \{(0, t) : -1 < t < 1\}$, and $E^1_2 = \{(x, \sin(1/x)) : 0 < x < 1\}$. Define a map $\pi : X \to P$ which maps all points in $E^i_j$ to $e^i_j$. Then $\pi$ is a normal stratification on $X$. We show that $(X, \pi)$ does not admit a cellular stratification. The reason for this is that the stratum $E^1_2$ does not admit a cell structure map. For there is no way to extend a homeomorphism $\mathrm{Int}(D^1) = (-1, 1) \to E^1_2$ to a continuous map $[-1, 1] \to X$.

**Definition 3.3.8.** Let $(X, \pi, \Phi)$ be a cellular stratified space. Let $\Phi = \{\varphi_\lambda : D_\lambda \to \bar{e}_\lambda\}_{\lambda \in P(X)}$. We say that a strict stratified subspace $A$ of $X$ is a **strict cellular stratified subspace** if for each cell $e_\lambda$ contained in $A$, the restriction $\varphi_\lambda|_{D^A_\lambda} : D^A_\lambda = \varphi^{-1}_\lambda(\bar{e}_\lambda \cap A) \to \bar{e}_\lambda \cap A = \mathrm{cl}_A(e_\lambda)$ is a quotient map. ❖

**Example 3.3.9.** Let $X$ be a topological space and $\mathcal{E}$ be a cell decomposition on $X$ which makes $X$ into a CW complex. By choosing a characteristic map $\varphi_\lambda : D_\lambda \to \bar{e}_\lambda$ for each cell $e_\lambda$ in $X$, we can give a cellular stratification on $X$. We show that under such a chosen cellular stratification of $X$, each (strict)

[5] By a top-dimensional cell we mean a cell of the largest dimension.

stratified subspace of $X$ is also a (strict) cellular stratified subspace of $X$. This is clear from the definition and the following lemma.

**Lemma 3.3.10.** *Let $f : X \to Y$ be a surjective continuous map from a compact space to a Hausdorff space. Then $f|_A : A \to f(A)$ is a quotient map whenever $A$ is a saturated subspace of $X$.*[6]

**Proof.** Let $A$ be a saturated subspace of $X$ and write $B = f(A)$. We need to show that $f|_A : A \to B$ is a quotient map. To this end, let $V \subseteq B$ be such that $f|_A^{-1}(V) = f^{-1}(V) \cap A$ be open in $A$. Since $A$ is saturated, $f^{-1}(V)$ is contained in $A$, and therefore $f|_A^{-1}(V) = f^{-1}(V)$.

We need to prove that $V$ is open in $B$. Now there is $U$ open in $X$ such that $U \cap A = f^{-1}(V)$. Since $X$ is compact, $X - U$ is a compact space. Therefore $f(X - U)$ is a closed subspace of $Y$. Note that since $A$ is saturated, $f(X - U)$ does not intersect $V$. Further, $f(X - U)$ contains $Y - V$. Therefore $Y - V$ is closed in $Y$, proving that $V$ is open in $B$. ∎

**Example 3.3.11** (Example 2.35 in [9]). Let $\mathcal{A} = \{H_1, \ldots, H_m\}$ be a collection of hyperplanes in $\mathbb{R}^n$. We will show that the stratification of $\mathbb{R}^n$ by $\mathcal{A}$ admits a regular cellular stratification. Without loss of generality we may assume that $\mathcal{A}$ is essential[7]. So we can choose a closed ball $B$ which contains all the bounded strata. Each stratum in $\mathbb{R}^n$ is a convex set and hence has an affine dimension. Suppose $e$ is a stratum with affine dimension $k$. Then we can choose a homeomorphism $\varphi : D^k \to \overline{e \cap B}$ such that $\varphi$ restricted to $\mathrm{Int}(D^k)$ is a homeomorphism onto $e \cap \mathrm{Int}(B)$. Define $D = \varphi^{-1}(\bar{e} \cap \mathrm{Int}(B))$. Further, we can also find a homeomorphism $\psi : \bar{e} \to \bar{e} \cap \mathrm{Int}(B)$. Now the composition $\psi^{-1} \circ \varphi|_D : D \to \bar{e}$ gives a regular cell structure on $e$.

**Example 3.3.12.** The following figure illustrates that the stratification of the punctured torus discussed in Example 3.1.15 admits a cellular stratification.

[6]We say that $A \subseteq X$ is a saturated subspace of $X$ if there is $B \subseteq Y$ such that $A = f^{-1}(B)$.

[7]We say that a collection of hyperplanes in $\mathbb{R}^n$ is essential if the normal vectors to the hyperplanes span the whole of $\mathbb{R}^n$.

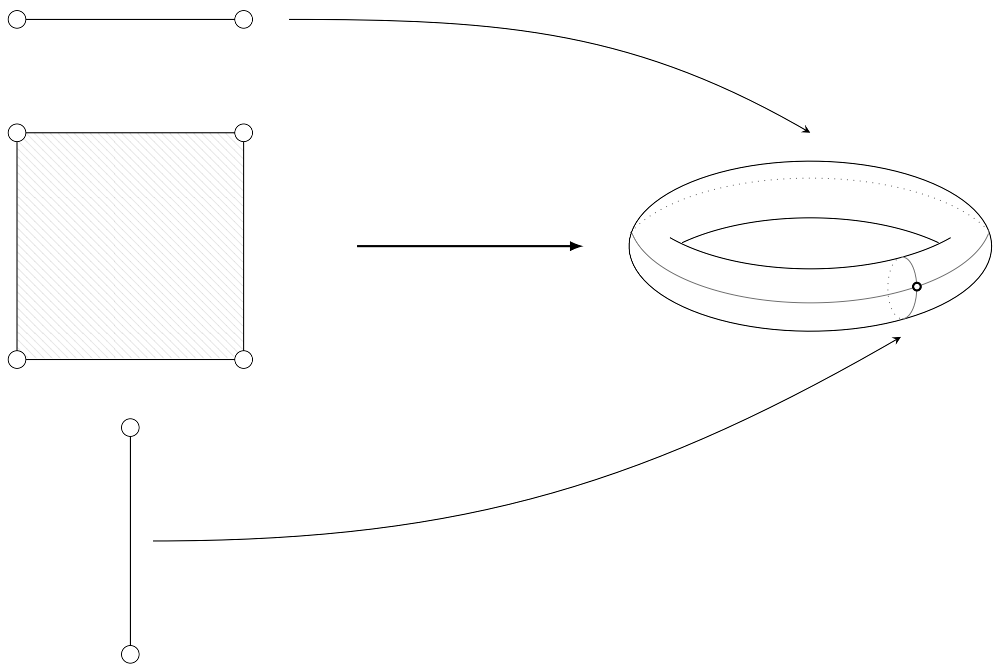


Figure 3.7: The punctured torus admits a cellular stratification.

**Definition 3.3.13.** A **graph** is a 1-dimensional cellular stratified space. ❖

**Theorem 3.3.14.** *All graphs are normal.*

**Proof.** Let $(X, \pi, \Phi)$ be a graph. Let $e_\lambda$ and $e_\mu$ be cells in $X$ such that $e_\mu \cap \bar{e}_\lambda \neq \emptyset$. It is clear that $\dim e_\lambda = 1$. We claim that $\dim e_\mu = 0$. Assume on the contrary that $\dim e_\mu = 1$ and let $p \in \partial e_\lambda$ be in $e_\mu$. By Theorem 3.3.4, we know that $e_\mu$ is open in $X$. Since $p$ is in the closure of $e_\lambda$, we must have $e_\mu \cap e_\lambda \neq \emptyset$, which is a contradiction. Therefore $\dim e_\mu = 0$. This means $e_\mu$ is a singleton so we have $e_\mu \subseteq \bar{e}_\lambda$ and we are done. ■

**Lemma 3.3.15.** *Let $X$ be a locally compact*[8] *Hausdorff space. Let $x$ be a point in $X$ and $U$ be a neighborhood of $x$. Then there is a compact subset $C$ of $X$ contained in $U$ which contains a neighborhood of $x$.*

**Proof.** Since $X$ is locally compact, there is a compact subset $K$ of $X$ which contains a neighborhood $V$ of $X$. Now $K \cap (X - U) = K - U$ is a closed subspace of $K$, and therefore it is compact. Using the fact that $X$, is Hausdorff, we can find an neighborhood $W$ of $x$ which is disjoint from an open set $O$ containing $K - U$. Now $C := K - O$ is a closed, and hence compact, subspace of $K$, which contains the neighborhood $W$ of $x$. Since $C$ is contained in $U$, we are done. ■

**Lemma 3.3.16.** *Let $C$ be a compact space and $X$ be any topological space. Then the projection map $X \times C \to X$ is a closed map.*

**Proof.** Let $p : X \times C \to X$ denote the projection map. Let $A$ be a closed subset of $X \times C$ and $x$ be

[8]We say that a topological space $X$ is locally compact if for all points $x \in X$, there is a compact subspace of $X$ which contains a neighborhood of $x$.

point in $X$ not in $p(A)$. We need to show that there exists a neighborhood $x$ in $X$ disjoint from $p(A)$. Since $x$ is not in $p(A)$, the set $x \times C$ is disjoint with $A$. For each $y \in C$, we can choose neighborhoods $U_y$ of $x$ in $X$ and $V_y$ of $y$ in $C$ such that $U_y \times V_y$ is disjoint with $A$. Now $\{V_y\}_{y \in C}$ is an open cover of $C$ and therefore it admits a finite subcover. Let $y_1, \ldots, y_n$ be in $C$ such that $V_{y_1}, \ldots, V_{y_n}$ cover $C$. Then it is easily seen that $\bigcap_{i=1}^{n} U_{x_i}$ is a neighborhood of $x$ in $X$ disjoint from $p(A)$ and we are done. ■

**Lemma 3.3.17.** *Let $f : X \to Y$ be a quotient map and $Z$ be a locally compact Hausdorff space. Then $f \times \mathrm{id}_Z : X \times Z \to Y \times Z$ is also a quotient map.*

**Proof.** Write $g = f \times \mathrm{id}_Z$. Let $B$ be a subset of $Y \times Z$ such that $A := g^{-1}(B)$ is open in $X \times Z$. We need to show that $B$ is open in $Y \times Z$. Let $(y_0, z_0)$ be arbitrary in $B$ and let $x_0 \in X$ be such that $f(x_0) = y_0$. By Lemma 3.3.15, we can find a compact subset $C$ of $Z$ which contains a neighborhood of $z$ such that $x_0 \times C \subseteq A$. Since $A$ is saturated, we can in fact write $f^{-1}(y_0) \times C \subseteq A$. Define $V = \{y \in Y : \ f^{-1}(y) \times C \subseteq A\}$. We show that $V$ is open in $Y$. By Lemma 3.3.16, the projection map $p : X \times C \to X$ is a closed map. Now

$$\begin{aligned} f^{-1}(V) &= \{x \in X : \ f(x) \in V\} \\ &= \{x \in X : \ f^{-1}(f(x)) \times C \subseteq A\} \\ &= \{x \in X : \ x \times C \subseteq A\} \\ &= X - p(X \times C - A) \end{aligned}$$

which shows that $f^{-1}(V)$ is open in $X$, and thus $V$ is open in $Y$. Noting that $(y_0, z_0) \in V \times C \subseteq B$ so we conclude that $B$ is open in $Y \times Z$ and we are done. ■

**Theorem 3.3.18** (Whitehead, See Lemma 4 in [10])**.** *Let $f : A \to B$ and $g : C \to D$ be quotient maps. Suppose $B$ and $C$ are locally compact Hausdorff. Then the map $f \times g : A \times C \to B \times D$ is also a quotient map.*

**Proof.** Consider the following diagram

$$\begin{tikzcd} A \times C \arrow[r, "f \times \mathrm{id}_C"] \arrow[dr, "f \times g"'] & B \times C \arrow[d, "\mathrm{id}_B \times g"] \\ & B \times D \end{tikzcd}$$

Applying Lemma 3.3.17 twice immediately leads to the desired result. ■

**Theorem 3.3.19** (Lemma 2.29 in [3])**.** *Let $(X, \pi_X, \Phi_X)$ and $(Y, \pi_Y, \Phi_Y)$ be cellular stratified spaces and consider the product stratification $\pi_X \times \pi_Y : X \times Y \to P(X) \times P(Y)$. For each pair of strata $e_\lambda$ and $e_\mu$ in $X$ and $Y$ respectively, define a map*

$$\varphi_{\lambda,\mu} : D_{\lambda,\mu} \cong D_\lambda \times D_\mu \xrightarrow{\varphi_\lambda \times \varphi_\mu} \bar{e}_\lambda \times \bar{e}_\mu = \overline{e_\lambda \times e_\mu} \subseteq X \times Y^{9}$$

[9] Here $D_{\lambda,\mu}$ is the subspace of $D^{\dim e_\lambda + \dim e_\mu}$ defined by pulling back $D_\lambda \times D_\mu$ via the homeomorphism $D^{\dim e_\lambda + \dim e_\mu} \cong D^{\dim e_\lambda} \times D^{\dim e_\mu}$. In fact, there is no loss in thinking of $D_{\lambda,\mu}$ as $D_\lambda \times D_\mu$ itself and this will be particulalry useful if both $X$ and $Y$ are 1-dimensional cellular stratified spaces.

*If $\varphi_{\lambda,\mu}$ is a quotient map for each $\lambda \in P(X)$ an $\mu \in P(Y)$, then $\Phi := \{\varphi_{\lambda,\mu}\}_{\lambda\in P(X),\mu\in P(Y)}$ gives a cellular stratification on $X \times Y$ and is called the* ***product cellular stratification*** *on $X \times Y$.*

**Proof.** Immediate from the definitions. ■

**Corollary 3.3.20.** *Let $\Gamma_1, \dots, \Gamma_k$ be graphs. Then the space $\Gamma = \Gamma_1 \times \cdots \times \Gamma_k$ is a normal cellular stratified space under the product cellular stratification.*

**Proof.** The domain globular cell of each stratum as well as the closure of each strata in each $\Gamma_i$ is locally compact Hausdorff, and therefore, by Theorem 3.3.18, $\Gamma$ is a cellular stratified space under the product stratification. The normality of $\Gamma$ follows from the fact that the product of two normally stratified spaces is also a normally stratified space. ■

## 3.4 Totally Normal Cellular Stratified Spaces

The following definition is a stronger version of Definition 2.35 in [3].

**Definition 3.4.1.** Let $(X, \pi, \Phi)$ be a normal cellular stratified space. We say that $X$ is **totally normal** if for each $n$-cell $e_\lambda$ in $X$ there exists a structure of a regular CW complex on $S^{n-1}$ which contains $\partial D_\lambda$ as a strict stratified subspace of $S^{n-1}$ such that for each cell $e$ in the stratification of $\partial D_\lambda$, there exists a cell $e_\mu$ in $X$, and a homeomorphism $b : D_\mu \to \bar{e}$ with $b(\mathrm{Int}(D_\mu)) = e$ and $\varphi_\lambda \circ b = \varphi_\mu$.

$$\begin{tikzcd}
e \arrow[r, hook] & \partial D_\lambda \arrow[r, "\varphi_\lambda|_{\partial D_\lambda}"] & X \\
\mathrm{Int}(D_\mu) \arrow[u, two heads, "b|_{\mathrm{Int}(D_\mu)}"] \arrow[r, hook] & D_\mu \arrow[u, "b"'] \arrow[ur, "\varphi_\mu"] &
\end{tikzcd}$$

Such a map $b$ is called a **lift** from $D_\mu$ to $D_\lambda$. The partition induced on $\partial D_\lambda$ by a regular CW complex structure on $S^{n-1}$ such that the above holds is called a **lifting structure** on $\partial D_\lambda$. ❖

**Example 3.4.2.** All closed finite regular cellular stratifications are totally normal. In other words, every finite regular CW complex can be thought of as a totally normal cellular stratified space once we choose a regular characteristic map for each cell.

**Example 3.4.3** (Example 3.19 in [9])**.** *A regular normal cellular stratified space may not be totally normal.* Let $X = \mathbb{R} \times \mathbb{R}_{\geq 0}$. For each $n \in \mathbb{Z}$, let $e_n^0 = \{(n, 0)\}$, and $e_n^1 = (n, n+1) \times \{0\}$. Also define $e^2 = \mathbb{R} \times \mathbb{R}_{>0}$. Then $e_n^0$'s, $e_n^1$'s and $e^2$ form a partition of $X$ and determine a stratification of $X$. This admits a cellular stratification. The cell structure maps for the $e_n^0$'s and $e_n^1$'s are trivial. The cell structure map for $e^2$ is given by extending the stereographic projection $S^1 - \{(0, 1)\} \to \mathbb{R}$. The domain of this cell structure map is $D = D^2 - \{(0, 1)\}$. This gives a regular cellular stratification of $X$. But this is not totally normal since the lifting structure on $D$ would then have infinitely many cells in it. This is not possible since a CW complex structure on $S^1$ cannot have infinitely many cells because $S^1$ is compact.

**Example 3.4.4.** *The minimal CW complex structure on $S^1$ is totally normal.* Consider the stratification $\pi : S^1 \to \{0, 1\}$ of $S^1$ which sends the point $(1, 0)$ to 0 and all other points to 1. Write $e^0 = \pi^{-1}(0)$ and

$e^1 = \pi^{-1}(1)$. The poset structure on $\{0,1\}$ is given by $0 < 1$. We can choose cell structure maps $\varphi_0 : D^0 \to \bar{e}^0$ and $\varphi_1 : D^1 \to \bar{e}^1 = S^1$. This clearly gives a normal cellular stratification on $(X, \pi)$. The following diagram shows that this stratification is in fact totally normal, where $D^0$ admits two different lifts into $\partial D^1$.

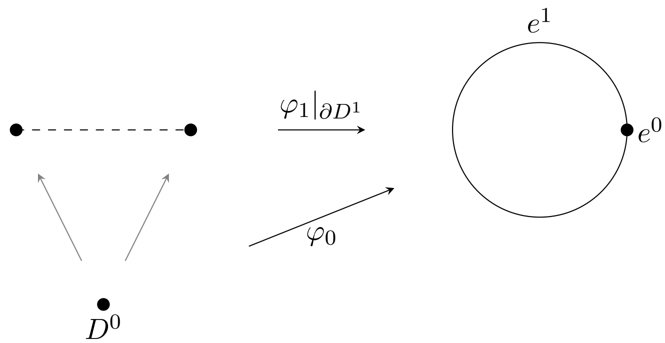



Figure 3.8: A totally normal cellular stratification of $S^1$.

Note that there is no regular CW complex structure on $S^1$ with just two cells.

**Example 3.4.5.** The following figure illustrates that the cellular stratification of the punctured torus discussed in Example 3.3.12 is totally normal.

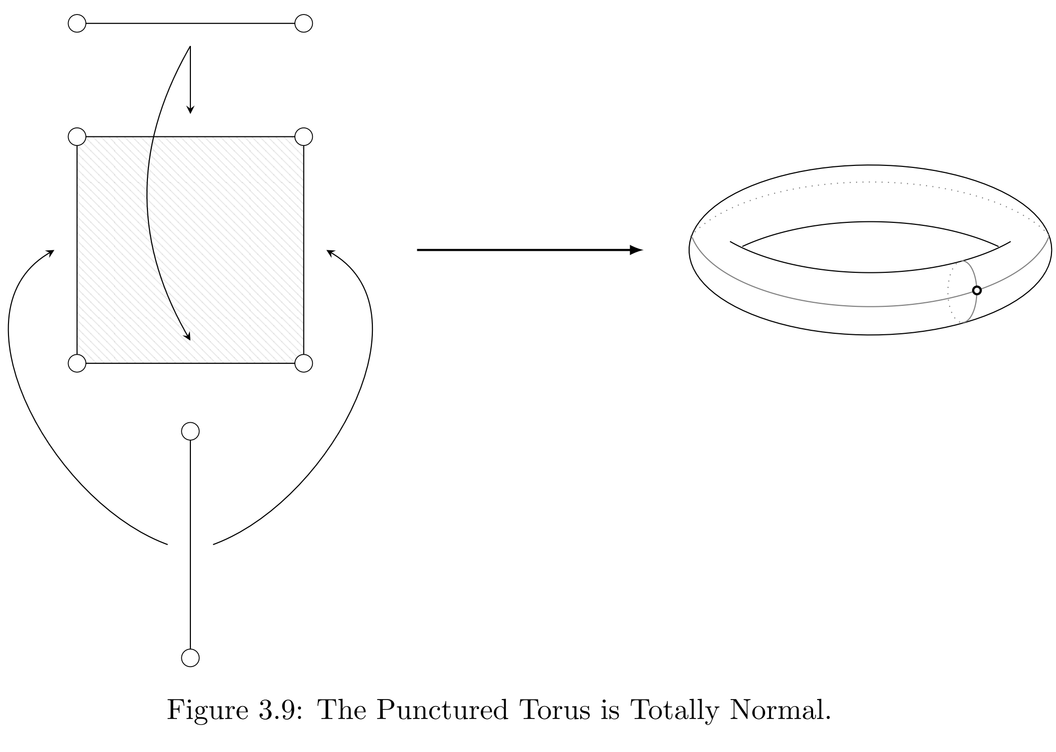


Figure 3.9: The Punctured Torus is Totally Normal.

**Theorem 3.4.6.** *All graphs are totally normal.*

**Proof.** Let $(X, \pi, \Phi)$ be a graph. By Theorem 3.3.14 we know that $X$ is normal. A one dimensional globular cell is one of the four intervals $[0,1]$, $[0,1)$, $(0,1]$ and $(0,1)$. Therefore the boundary is a strict stratified subspace of the (only) regular CW complex structure on $S^0$. So it remains to only show the lifting property. Let us show that the lifting condition holds for the globular cell $[0,1]$. The proof for other globular cells is similar. Let $\varphi : [0,1] \to \bar{e}$ be a cell structure map for a stratum $e$ in $X$. By normality of $X$, we must have $\varphi(0)$ is a 0-dimensional cell say $v$ in $X$. Therefore, we get a lift from the characteristic of $v$ which satisfies the lifting condition. Similarly for $\varphi(1)$. ■

**Theorem 3.4.7** (Remark 2.44 in [3]). *Let $\Gamma_1, \ldots, \Gamma_k$ be graphs. Then the product $\Gamma_1 \times \cdots \times \Gamma_k$ is a totally normal cellular stratified space.*

**Proof.** We follow the proof of Lemma 2.43 in [3] to prove the statement for the case $k = 2$. The general case is similar. So let $\Gamma_1$ and $\Gamma_2$ be two graphs. We have seen that the product $\Gamma_1 \times \Gamma_2$ is cellular stratified by the product cellular stratification. We show that this cellular stratification is totally normal. The possible domains for the cell structure maps for 1-cells and 2-cells are shown in the following diagrams



Figure 3.10

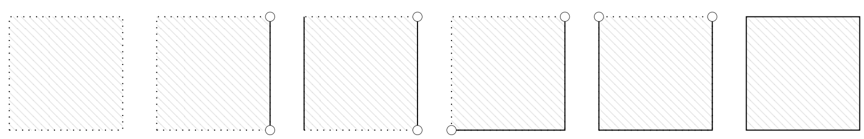


Figure 3.11

Here we think of globular cells not as discs but as rectangles. From the diagram above it is clear that the boundary of each globular rectangle is a strict stratified subspace of the regular CW complex structure on the rectangle in which each edge is a 1-cell and each vertex is a 0-cell. The lifting property of the cell structure maps is also clearly satisfied. ■

**Theorem 3.4.8.** *Let $X$ be a totally normal cellular stratified space. Then there is a lift from a cell $\varphi_\mu : D_\mu \to \bar{e}_\mu$ to a cell $\varphi_\lambda : D_\lambda \to \bar{e}_\lambda$ if and only if $e_\mu \subseteq \bar{e}_\lambda$*

**Proof.** It is clear that if there is a lift from a cell $e_\mu$ to a cell $e_\lambda$, then $e_\mu \subseteq \bar{e}_\lambda$. Conversely, assume that $e_\mu \subseteq \bar{e}_\lambda$. In case $e_\lambda = e_\mu$, the theorem is trivial. So assume $e_\mu \neq e_\lambda$. So we must have $e_\mu \subseteq \partial e_\lambda$. Let $e$ be a cell in the regular CW complex structure on $D_\lambda$ whose image under $\varphi_\lambda$ intersects $e_\mu$. By definition of a totally normal cellular stratified space, there is a stratum $e_\theta$ in $X$ and a continuous map $b : D_\theta \to \partial D_\lambda$ such that $b(\mathrm{Int}(D_\theta)) = e$ and $\varphi_\lambda \circ e = \varphi_\theta$. From this we see that $e_\theta \cap e_\mu \neq \emptyset$ and thus we must in fact have $e_\mu = e_\theta$. We have the following diagram

$$\begin{tikzcd}
e \arrow[r, hook] & \partial D_\lambda \arrow[r, "\varphi_\lambda|_{\partial D_\lambda}"] & X \\
\mathrm{Int}(D_\mu) \arrow[u, two heads, "b|_{\mathrm{Int}(D_\mu)}"] \arrow[r, hook] & D_\mu \arrow[u, "b"] \arrow[ur, "\varphi_\mu"] &
\end{tikzcd}$$

The diagram shows that $\varphi_\lambda(e) = e_\mu$ and we see that $b$ is a required lift. ■

**Lemma 3.4.9.** *Let $(X, \pi, \Phi)$ be a totally normal cellular stratified space and $\varphi_\lambda : D_\lambda \to \bar{e}_\lambda$ be an $n$-cell structure map for a stratum $e_\lambda$ in $X$. Let $e$ be a cell in the lifting structure of $\partial D_\lambda$. Then $\varphi_\lambda(e)$ is a stratum in $X$ and $\varphi_\lambda|_e : e \to \varphi_\lambda(e)$ is a homeomorphism.*

**Proof.** By total normality, there is a stratum $e_\mu$ in $X$ along with a homeomorphism $b : D_\mu \to \bar{e}$ such that $b(\mathrm{Int}(D_\mu)) = e$ and $\varphi_\lambda \circ b = \varphi_\mu$, whence the desired result follows. ■

**Lemma 3.4.10.** *Let $(X, \pi, \Phi)$ be a totally normal cellular stratified space. Let $\lambda, \mu \in P(X)$ be such that $e_\mu \subseteq \bar{e}_\lambda$. Then there are only finitely many lifts of $\varphi_\mu : D_\mu \to \bar{e}_\mu$ to $\varphi_\lambda : D_\lambda \to \bar{e}_\lambda$.*

**Proof.** Let $e$ be a cell in the lifting structure of $\partial D_\lambda$ such that there is a lift $D_\mu \to D_\lambda$ whose image is $\bar{e}$. Now let $b_1, b_2 : D_\mu \to D_\lambda$ be two such lifts. It is clear that $b_1$ and $b_2$ agree on $\mathrm{Int}(D_\mu)$. Let $x \in \partial D_\mu$ and $(x_n)$ be a sequence in $\mathrm{Int}(D_\mu)$ such that $x_n \to x$. Since $b_i(x_n) \to b_i(x)$, $i = 1, 2$, we see that $b_1(x) = b_2(x)$. So $b_1$ and $b_2$ agree on $D_\mu$. Now since there are only finitely many cells $e$ in $\partial D_\lambda$, and there is at most one lift corresponding to any given cell in $\partial D_\lambda$, we deduce that there are only finitely many lifts of $\varphi_\mu : D_\mu \to \bar{e}_\mu$ to $\varphi_\lambda : D_\lambda \to \bar{e}_\lambda$. ■

**Example 3.4.11.** *A normal cellular stratification may not be totally normal.* Let $\pi : S^2 \to \{0, 1\}$ be the map which sends $(1, 0, 0)$ to 0 and all other points of $S^2$ to 1. This gives a stratification of $S^2$ with strata $e^0 = \pi^{-1}(0)$ and $e^1 = \pi^{-1}(1)$, where the poset structure of $\{0, 1\}$ is given by $0 < 1$. Choose cell structure maps $\varphi_0 : D^0 \to \bar{e}^0$ and $\varphi_2 : D^2 \to \bar{e}^2 = S^2$. This gives a normal cellular stratification. We show that this cellular stratification, however, is not totally normal. For suppose it were. Then there exists a lifting structure on $\partial D^2 = S^1$. But any regular CW complex structure on $S^1$ is bound to contain a 1-cell. As seen from Lemma 3.4.9, $\varphi_2$ must be injective on this 1-cell. But this is not the case, for $\varphi_2$ maps all points of $\partial D^2$ to $(1, 0, 0)$. In fact, the only cell structure map available to provide a lift for this 1-cell in $\varphi_0 : D^0 \to \bar{e}^0$. This of course cannot work because $\dim D^0 < 1$.

A cellular stratified space is totally normal if each domain globular cell admits a lifting structure. Thus, given a cellular stratified space, it is natural to ask if there are two different ways in which it can be totally normal, that is, if some domain globular cells possess multiple lifting structures. We now proceed to show that the lifting structures on the domain globular cells of a totally normal cellular stratified space are unique.

**Lemma 3.4.12.** *Let $X$ be a topological space and $\mathcal{E}$ and $\mathcal{E}'$ are two finite CW complex structures on $X$. Let $\mathcal{L} \subseteq \mathcal{E}$ and $\mathcal{L}' \subseteq \mathcal{E}'$ be such that*

$$\bigcup_{e \in \mathcal{L}} e = \bigcup_{e' \in \mathcal{L}'} e' = A \text{ (say)}$$

*Assume that whenever two cells $e \in \mathcal{L}$ and $e' \in \mathcal{L}'$ intersect, we have $\dim e = \dim e'$. Then $\mathcal{L} = \mathcal{L}'$.*

**Proof.** It is clear by hypothesis that $\dim \mathcal{L} = \dim \mathcal{L}'$.[10] Say this dimension is $n$. We prove the result by induction on $n$. The base case is $n = 0$, whence the proof is trivial. So let $n \geq 1$ and assume that the theorem holds for all smaller values. Let $\mathcal{L}_n$ and $\mathcal{L}'_n$ be the collection of all the $n$-dimensional cells in $\mathcal{L}$ and $\mathcal{L}'$ respectively. We show that $\mathcal{L}_n = \mathcal{L}'_n$. Let $e \in \mathcal{L}_n$ and $e' \in \mathcal{L}'_n$ be such that $e \cap e' \neq \emptyset$. Note that

[10]By dimension of a cell decomposition we mean the largest dimension of all the cells.

$e$ and $e'$ are top dimensional cells in the $n$-skeleton of $(X, \mathcal{E})$ and $(X, \mathcal{E}')$ respectively, and therefore are open in the respective $n$-skeletons. Thus $e$ and $e'$ are open in $A$ since $A$ is contained in the $n$-skeleton of both $(X, \mathcal{E})$ and $(X, \mathcal{E}')$. We want to show that $e = e'$, for which by symmetry it suffices to establish that $e \subseteq e'$. Assume on the contrary that $e$ is not contained in $e'$.

<u>*Case 1:*</u> $e \cap (\mathrm{cl}_X(e') - e') \neq \emptyset$.

Let $x \in e \cap (\mathrm{cl}_X(e') - e')$. Let $u'$ be a cell in $\mathcal{E}'$ which contains $x$. Then $u' \in \mathcal{L}'$ and $\dim u' < n$. By hypothesis, then, $\dim e = \dim u' < \dim e' = \dim e$, which is absurd. So this case is not possible.

<u>*Case 2:*</u> $e \cap (\mathrm{cl}_X(e') - e') = \emptyset$.

Since we have assumed that $e$ is not contained in $e'$, we are forced to have $e \cap (X - \mathrm{cl}_X(e')) \neq \emptyset$. Write $U = e \cap e'$ and $V = e \cap (X - \mathrm{cl}_X(e'))$. Since $\mathrm{cl}_X(e')$ is compact in $X$, it is in particular a closed subset of $X$, giving $X - \mathrm{cl}_X(e')$ is open in $X$. Thus $V$ is open in $e$. As noted earlier, $e$ and $e'$ are both open in $A$, and therefore $U$ too is open in $e$. So we have $e = U \cup V$, where $U$ and $V$ are disjoint open subsets of $e$, implying $e$ is not connected, giving a contradiction.

Since the above two are the only cases possible, and in both cases we arrive at a contradiction, we deduce that our assumption that $e$ is not contained in $e'$ must be wrong. So we conclude that $e \subseteq e'$. Similarly, $e' \subseteq e$ and we have $e = e'$. Therefore, whenever a cell $e \in \mathcal{L}_n$ intersects with a cell $e' \in \mathcal{L}'_n$, we have $e = e'$. By the dimension condition in the hypothesis, a cell in $\mathcal{L}_n$ cannot intersect a cell in $\mathcal{L}' - \mathcal{L}'_n$. So we conclude that $\mathcal{L}_n = \mathcal{L}'_n$. Now by induction it follows that $\mathcal{L} - \mathcal{L}_n = \mathcal{L}' - \mathcal{L}'_n$. Therefore $\mathcal{L} = \mathcal{L}'$ and we are done. ■

**Theorem 3.4.13.** *Let $(X, \pi, \Phi)$ be a totally normal cellular stratified space and $\varphi_\lambda : D_\lambda \to \bar{e}_\lambda$ be a $n$-cell structure map. Then there is a unique lifting structure on $\partial D_\lambda$.*

**Proof.** Let $\mathcal{E}$ and $\mathcal{E}'$ be two regular CW complex structures on $S^{n-1}$ and $\mathcal{L} \subseteq \mathcal{E}$ and $\mathcal{L}' \subseteq \mathcal{E}'$ be two lifting structures on $\partial D_\lambda$. Let $e \in \mathcal{L}$ and $e' \in \mathcal{L}'$ be such that $e \cap e' \neq \emptyset$. By Lemma 3.4.9 we know that both $\varphi_\lambda(e)$ and $\varphi_\lambda(e')$ are strata in $X$. Since two strata intersect if and only if they are same, we deduce that $\varphi_\lambda(e) = \varphi_\lambda(e')$. This forces by Lemma 3.4.9 that $\dim e = \dim e'$. So what we have shown is that if $e \in \mathcal{L}$ intersects a cell $e' \in \mathcal{L}'$ then $\dim e = \dim e'$. By Lemma 3.4.12 we see that $\mathcal{L} = \mathcal{L}'$ and we are done. ■

**Theorem 3.4.14.** *Let $(X, \pi, \Phi)$ be a totally normal cellular stratified space. Let $\varphi_\lambda : D_\lambda \to \bar{e}_\lambda$ be an $n$-cell structure map, $e$ be a cell in the lifting structure of $\partial D_\lambda$, and $b : D_\mu \to \bar{e}$ be a lift of $\varphi_\mu : D_\mu \to \bar{e}_\mu$. Then the cellular stratification induced on $\partial D_\mu$ by pulling back the cellular stratification on $\partial e$ via $b$ is the lifting stratification on $\partial D_\mu$.*

**Proof.** Let $\mathcal{E}$ be the lifting structure on $\partial D_\mu$. Let $v$ be an arbitrary cell in the lifting structure of $\partial D_\lambda$ contained in $\partial e$, and define $u' = b^{-1}(v)$. Let $u$ be a cell in $\mathcal{E}$ which intersects $u'$.

We show that $\dim u = \dim u'$. Since $b$ maps $u'$ homeomorphically onto $v$ and, by Lemma 3.4.9, $\varphi_\lambda$ maps $v$ homeomorphically onto the stratum $\varphi_\lambda(v)$, we see that $b$ maps $u'$ homeomorphically onto $\varphi_\lambda(v)$. Therefore $\dim u' = \dim \varphi_\lambda(v)$. Again, by Lemma 3.4.9 we have $\varphi_\mu(u)$ is also a stratum and that $\varphi_\mu|_u : u \to \varphi_\mu(u)$ is a homeomorphism. This gives $\dim u = \dim \varphi_\mu(u)$. Since $u$ and $u'$ intersect, so do $\varphi_\mu(u)$ and $\varphi_\mu(u')$. This yields $\varphi_\mu(u) = \varphi_\mu(u')$ and therefore $\dim u' = \dim u$. So we conclude that if $u'$ intersects a cell in $\mathcal{E}$ then the dimension of $u'$ is same as that of the cell in $\mathcal{E}$.

Now we show that $u' \subseteq u$. If $u'$ is not contained in $u$, then $u'$ intersects a cell in the boundary of $u$, as argued in the proof of Lemma 3.4.12. Since $\partial u$ is a union of cells having dimension strictly lower than that of $u$, this means that the dimension of $u'$ is smaller than that of $u$, giving a contradiction. So we must have $u' \subseteq u$. The containment cannot be proper since $\varphi_\mu$ is injective on $u$ and both $\varphi_\mu(u')$ and $\varphi_\mu(u)$ are strata in $X$. We must have $u = u'$.

So we have shown that for each cell $v$ in the lifting structure of $\partial D_\lambda$ contained in $\partial e$, $b^{-1}(v)$ is in $\mathcal{E}$. But $\partial D_\mu$ is the union of inverse images of cells in the lifting structure of $\partial D_\lambda$ contained in $\partial e$ and therefore we have our result. $\blacksquare$

## 3.5 Face Categories

**Definition 3.5.1.** Let $X$ be a totally normal cellular stratified space. We define a category $C(X)$ whose objects are all the cells in $X$. A morphism from a cell $\varphi_\mu : D_\mu \to \bar{e}_\mu$ to a cell $\varphi_\lambda : D_\lambda \to \bar{e}_\lambda$ is the identity map if $e_\mu = e_\lambda$, and is a lift (as described in Definition 3.4.1) $b : D_\mu \to \partial D_\lambda$ if $\dim(e_\mu) < \dim(e_\lambda)$.

$$\begin{tikzcd}
\partial D_\lambda \arrow[r, "\varphi_\lambda"] & \bar{e}_\lambda \arrow[r, hook] & X \\
D_\mu \arrow[u, "b"] \arrow[r, "\varphi_\mu"] & \bar{e}_\mu \arrow[ur, hook] &
\end{tikzcd}$$

The composition of two morphisms is simply the composition of maps. The category thus obtained is called the **face category** of $X$. ❖

By Lemma 3.4.10 we know that there are only finitely many morphisms between any two objects in the face category of a totally normal cellular stratified space. Thus the face category of a finite totally normal cellular stratified space is finite. Also note that Theorem 1.3.5 implies that the face category of a totally normal cellular stratified space is acyclic. Lastly, note that if $X$ is a totally normal cellular stratified space, then the underlying poset of $C(X)$ is isomorphic to the face poset of $X$.

**Example 3.5.2.** The face category of a regular CW complex is same as the face poset, where of course, we think of the face poset as a category.

Following is the main theorem that we will use to find the homotopy type of configuration spaces of graphs.

**Theorem 3.5.3** (Theorem 2.50 in [3]). *Let $X$ be a totally normal cellular stratified space. Then the geometric realization of the face category of $X$ embeds in $X$ as a strong deformation retract.*

**Example 3.5.4.** The following figures shows that face category and the geometric realization of the totally normal cellular stratification of the punctured torus discussed in Example 3.3.12 and 3.4.5.

Figure 3.12: Face Category and its Geometric Realization

This shows, by using Theorem 3.5.3, that the homotopy type of the punctured torus is same as that of the wedge sum of two circles.

# Chapter 4

# Applications to Configuration Spaces of Points on a Graph

## 4.1 Braid Stratification of Product of Graphs

In this section we discuss how configuration spaces of graphs are naturally totally normal cellular stratified. Given a graph $\Gamma$, we know that $\Gamma^n$ is a totally normal cellular stratified space under the product cellular stratification. What we want to achieve is a finer cellular stratification of $\Gamma^n$ which has the diagonal as a strict stratified subspace. This is achieved by transferring the braid stratification on Euclidean spaces, allowing us to remove the diagonal from $\Gamma^n$ and be left with a totally normal cellular stratified space.

**Definition 4.1.1.** Let $X$ be a finite graph. Let $\{e^0_\lambda\}_{\lambda\in\Lambda_0}$ and $\{e^1_\lambda\}_{\lambda\in\Lambda_1}$. Fix a homeomorphism $\psi : \mathbb{R} \to \mathrm{Int}(D^1)$. Choose total orders in $\Lambda_0$ and $\Lambda_1$. For a cell $e^{\varepsilon_1}_{\lambda_1} \times \cdots \times e^{\varepsilon_k}_{\lambda_k}$, choose a permutation $\sigma \in S_k$[1] such that

$$\sigma(e^{\varepsilon_1}_{\lambda_1} \times \cdots \times e^{\varepsilon_k}_{\lambda_k}) = (\text{a product of 0-cells}) \times (e^1_{\mu_1})^{m_1} \times \cdots \times (e^1_{\mu_\ell})^{m_\ell}$$

where $\mu_1 < \cdots < \mu_\ell$. Now using the cell structure map $\varphi_{\mu_j} : \mathrm{Int}(D^1) \to e^1_{\mu_j}$, we get a homeomorphism

$$\begin{tikzcd} \mathbb{R}^{m_j} \arrow[r, "\psi^{m_j}"] \arrow[rr, bend left, "\cong"] & \mathrm{Int}(D^1)^{m_j} \arrow[r, "\varphi_{\mu_j}^{m_j}"] & (e^1_{\mu_j})^{m_j} \end{tikzcd}$$

Using this homeomorphism, transfer the braid stratification of $\mathbb{R}^{m_k}$ to stratify $(e^1_{\mu_j})^{m_j}$. The refined stratification on $X^k$ achieved by this procedure is called the **braid stratification** on $X^k$. ❖

**Theorem 4.1.2** (Prop. 3.7 in [3])**.** *Let $X$ be a finite graph. Then, the braid stratification on $X^k$ is totally normal and contains $\mathrm{Diag}_k(X)$ as a strict stratified subspace. Therefore, $\mathrm{Conf}_k(X)$ is also a totally normal cellular stratified space.*

[1] For a permutation $\sigma$ in $S_k$, we get a map $X^k \to X^k$ which maps $(x_1, \ldots, x_k)$ to $(x_{\sigma(1)}, \ldots, x_{\sigma(k)})$.

## 4.2 Examples

**Example 4.2.1.** We use Theorem 3.5.3 to find the homotopy type of $\mathrm{Conf}_2(S^1)$.

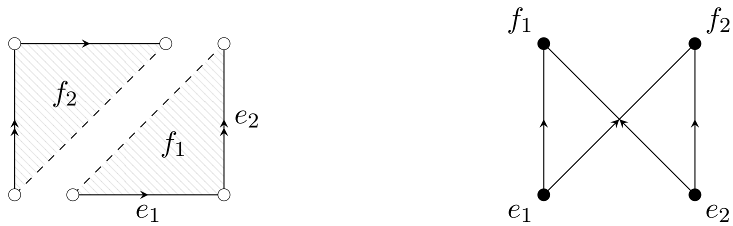
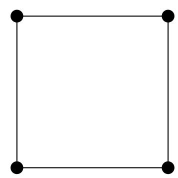



Figure 4.1: $\mathrm{Conf}_2(S^1)$ and its face category.

The geometric realization of the face category shown above is



Therefore the homotopy type of $\mathrm{Conf}_2(S^1)$ is same as that of $S^1$. We already saw this in Example 2.2.

**Example 4.2.2.** Let $X = S^1 \vee S^1$. We find the homotopy type of $\mathrm{Conf}_2(X)$.

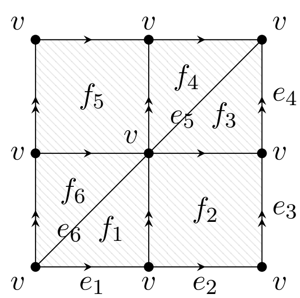



Figure 4.2: Braid Stratification of $(S^1 \vee S^1) \times (S^1 \vee S^1)$.

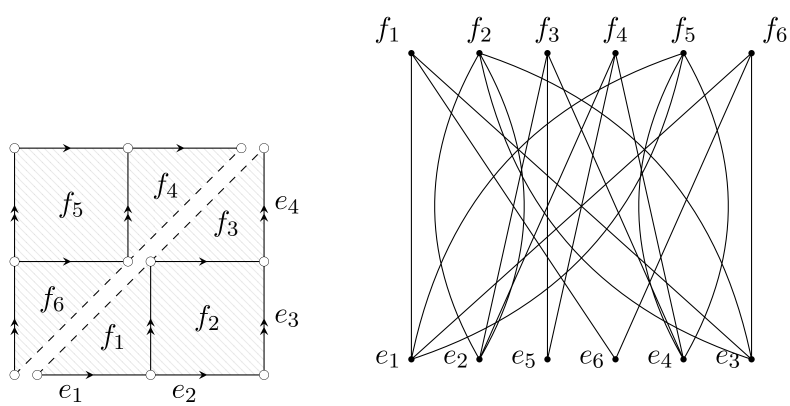



Figure 4.3: $\mathrm{Conf}_2(S^1 \vee S^1)$ and its Face Category.

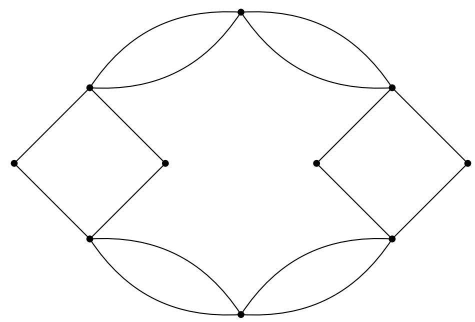


Figure 4.4: Geometric Realization

From the geometric realization of the face category, it can be seen that the homotopy type of $\mathrm{Conf}_2(S^1 \vee S^1)$ is same as that of the wedge sum of seven circles.

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