On the topology of the cyclopermutohedron and its quotient

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by
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Abstract

The emerging field of topological robotics lies on the crossroads of algebraic topology and engineering. One of the objects studied in this area is the set of all configurations of a given mechanical linkage. A planar mechanical linkage is a mechanism consisting of $n+1$ metal bars of fixed lengths $l_1, \ldots, l_{n+1}$ connected by revolving joints, that can rotate full $360°$, forming a closed polygonal chain.

Mechanical linkages are modelled by closed piecewise linear paths in $\mathbb{R}^2$ called planar polygons with specified side lengths. The configuration space of such a system is the set of its all possible states.

We begin by formally defining the configuration space.

Definition 0.1. Consider a length vector $L := (l_1, l_2, \ldots, l_{n+1}) \in \mathbb{R}_+^{n+1}$ that prescribes side lengths of planar $n+1$-gons. The moduli space of such polygons viewed up to the action of orientation-preserving isometries is denoted $\mathcal{M}_L$ and defined as:

$$\mathcal{M}_L = \{(u_1, u_2, \ldots, u_{n+1}) \in S^1 \times S^1 \times \cdots \times S^1; \sum_{i=1}^{n+1} l_i u_i = 0\}/\mathrm{SO}(2).$$

The moduli space of $n+1$-gons viewed up to the action of all isometries is denoted $\bar{\mathcal{M}}_L$ and defined as:

$$\bar{\mathcal{M}}_L = \{(u_1, u_2 \ldots u_{n+1}) \in S^1 \times S^1 \times \cdots \times S^1; \sum_{i=1}^{n+1} l_i u_i = 0\}/O(2).$$

Here, both the groups act diagonally.

Geometrically, the elements of $\mathcal{M}_L$ represent closed piecewise linear paths that differ either by a rotation or a translation (or both). Similarly, the elements of $\bar{\mathcal{M}}_L$ represent closed piecewise linear paths that differ in addition by a reflection.

Definition 0.2. A length vector $L \in \mathbb{R}_+^{n+1}$ is **generic** if $\forall I \subset [n+1]$

$$\sum_{i \in I} l_i \neq \sum_{j \notin I} l_j$$

Definition 0.3. A subset $I \subset [n+1]$ is called **short with respect to L** if

$$\sum_{i \in I} l_i < \sum_{i \notin I} l_i$$

Definition 0.4. A **cyclically ordered partition** of the set $[n+1] := \{1, \ldots, n+1\}$ is an equivalence class of ordered partitions with the relation that two ordered partitions equivalent if one can be obtained from the other by a cyclic permutation of its blocks.

Given a generic length vector L, the space $\mathscr{M}_L$ is a manifold and was given a natural regular cell structure using cyclically ordered partitions of $[n+1]$ in [9]. The k-cells of $\mathscr{M}_L$ are given by cyclically ordered partition of $[n+1]$ into $(k+3)$ blocks where each block of the partition is a short subset of $[n+1]$ and the attaching relations are given by the refinement of partitions.

Motivated by above, the cyclopermutohedron CP_{n+1} was introduced by G. Panina in [8]. It is an $(n-2)$-dimensional regular CW complex whose k-cells are labeled by cyclically ordered partitions of the set $[n+1]$ into $(n+1-k)$ non-empty parts, where $(n+1-k) > 2$. The boundary relations in the complex correspond to the refinement of partitions. The cyclopermutohedron is a "universal object" for moduli spaces of polygonal linkages i.e. given a generic length vector L, CP_{n+1} contains a subcomplex homeomorphic to $\mathscr{M}_L$.

The aim of this thesis is to understand the topological and combinatorial properties of CP_{n+1}. In [7] the authors showed that the homology groups of CP_{n+1} are torsion free and computed their Betti numbers. This was done using discrete Morse theory.

The moduli space $\mathscr{M}_L$ admits a natural free $\mathbb{Z}_2$ action, wherein each polygon is mapped to its reflection about the X-axis. The quotient under this action is precisely $\bar{\mathscr{M}}_L$. The space CP_{n+1} mimicking this action also admits a free $\mathbb{Z}_2$ action. The quotient space $\mathrm{CP}_{n+1}/\mathbb{Z}_2$ will be called *bicyclopermutohedron* and will be denoted by QP_{n+1}. This quotient is the universal object for the moduli spaces $\bar{\mathscr{M}}_L$ in the same sense as described above.

The main aim of this thesis is to compute homology of QP_{n+1}. This computation is more involved compared to that of $H_i(\mathrm{CP}_{n+1})$. Our main tool is discrete Morse theory and approach is similar to the one taken in [8]. However, in our case the boundary maps in the Morse complex do not vanish. As a result there is torsion in the homology. In Theorem 4.11 we compute the integer homology of QP_{n+1} and show that there is only 2-torsion. On the other hand the mod-2 homology of QP_{n+1} is relatively easy to compute; which is done in Theorem 4.8.

Chapter-wise organization

Chapter 1. The first chapter contains basic definitions from combinatorics and topology of posets, and discrete Morse theory needed to understand the results presented in this thesis. We also introduce here some of the foundational, well-known results of the area (taken mainly from [10], [11] and [6]).

Chapter 2. We discuss various topological and combinatorial properties of two posets in the second chapter, the poset of ordered partitions and the poset of unordered partitions. The main theorem of the chapter is the *Homotopy complementation formula for posets*. We discuss in detail the proof of the theorem. Topological aspects of the poset of ordered partitions is illuminated using permutohedron.

Chapter 3. With this chapter, the main content of the thesis begins. We start the chapter with the definition of cyclopermutahedron, following that, a combinatorial description for the boundary maps in the cellular chain complex is presented. Finally, we discuss the computation of cyclopermutohedron's homology as performed in [7]. The authors explicitly construct a perfect Morse function thereby reducing the complexity of the problem.

Chapter 4. The quotient space QP_{n+1} is the focus of this chapter and computing its $\mathbb{Z}$-homology is the main result regarding this space. As in Chapter 3, we construct a discrete Morse function and this is sufficient to compute the $\mathbb{Z}_2$-homology of this quotient space. Each equivalence class in QP_{n+1} contains two cells and the $\mathbb{Z}$-homology requires a delicate calculation of comparing the orientation induced by each of its representative. All results in this chapter are, to the best of our knowledge, new.

Acknowledgement

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Contents

Chapter 1

Introduction

In this Chapter we state the basics of ideas and techniques needed to understand this thesis. We have tried to present the material in a self contained manner without going into too many details.

1.1 Posets

This section introduces combinatorial and topological aspects of posets. The main reference is Stanley's book[10].

Definition 1.1. A *partially ordered set* P (or poset, for short) is a set together with a binary relation denoted $\leq$ satisfying the following three axioms:

- For all $t \in P, t \leq t$ (reflexivity).
- If $s \leq t$ and $t \leq s$, then $s = t$ (antisymmetry).
- If $s \leq t$ and $t \leq u$, then $s \leq u$ (transitivity).

We use the obvious notation $t \geq s$ to mean $s \leq t$, $s < t$ to mean $s \leq t$ and $s \neq t$, and $t > s$ to mean $s < t$. We say that two elements s and t of P are comparable if $s \leq t$ or $t \leq s$; otherwise s and t are incomparable, denoted $s||t$.

Example 1.1. Let $n \in \mathbb{N}, n \geq 0$. The set $[n]$ with its usual order forms an n-element poset with the special property that any two elements are comparable. This poset is denoted $[n]$.

Example 1.2 (Boolean Poset). Let $n \in \mathbb{N}, n \geq 0$. We can make the set $\mathcal{P}\text{ow}[n]$ of all subsets of $[n]$ into a poset B_n by defining $S \leq T$ in B_n if $S \subset T$ as sets. One says that B_n consists of the subsets of $[n]$ "ordered by inclusion."

Example 1.3 (Partition Lattice). Let $n \in \mathbb{N}, n \geq 0$. We can make the set Π_n of all partitions of $[n]$ into a poset by defining $\pi \leq \sigma$ in Π_n if every block of π is contained in a block of σ. We then say that π is a refinement of σ and that Π_n consists of the partitions of $[n]$ "ordered by refinement."

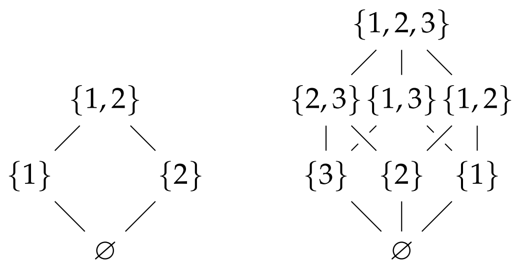
Two posets P and Q are *isomorphic*, denoted $P \cong Q$, if there exists an order-preserving bijection $\phi : P \to Q$ whose inverse is order-preserving; that is,

$$s \leq t \text{ in } P \iff \phi(s) \leq \phi(t) \text{ in } Q.$$

If $s, t \in P$, then we say that t covers s or s is covered by t, denoted $s \prec t$, if $s < t$ and no element $u \in P$ satisfies $s < u < t$. For $a \leq b$, the *closed interval* $[a, b]$ is the set of elements $x \in P$ satisfying $a \leq x \leq b$. If every interval of P is finite, then P is called a *locally finite poset*. A locally finite poset P is completely determined by its cover relations. The *Hasse diagram* of a finite poset P is the graph whose vertices are the elements of P, whose edges are the cover relations, and such that if $s < t$ then t is drawn "above" s. See Fig. 1.1 for example.

An element $\hat{1} \in P$ is called a *greatest element* if for every element $s \in P,\ s \leq \hat{1}$. An element $\hat{0} \in P$ is called a *least element* if for every element $t \in P,\ t \geq \hat{0}$. A poset can only have one greatest or least element.

A poset P is said to be *bounded* if it has a top element $\hat{1}$ and a bottom element $\hat{0}$. The proper part of a bounded poset P, for which $|P| > 1$, is defined to be $\bar{P} := P - \{\hat{0}, \hat{1}\}$. Given a poset P, we define the bounded extension $\hat{P} := P \cup \{\hat{0}, \hat{1}\}$, where new elements $\hat{0}$ and $\hat{1}$ are adjoined (even if P already has a bottom or top element).



B_2 B_2

Figure 1.1: Hasse diagram of B_2 and B_3

A *chain* in P is a subposet in which any two elements are comparable. A chain C in P is called *maximal* if it is not contained in a larger chain of P. The chain C of P is called *saturated (or unrefinable)* if there does not exist $u \in P - C$ such that $s < u < t$ for some $s, t \in C$ and such that $C \cup \{u\}$ is a chain. Thus maximal chains are saturated, but not conversely. The *length* $l(C)$ of a finite chain is defined by $l(C) = |C| - 1$. The length of a finite poset P is $l(P) := \max\{l(C) : C \text{ is a chain of } P\}$. The length of an interval $[s, t]$ is denoted $l(s, t)$. If every maximal chain of P has the same length n, then we say that P is *graded* of rank n. In this case there is a unique rank function $\rho : P \to [n]$ such that $\rho(s) = 0$ if s is a minimal element of P, and $\rho(t) = \rho(s) + 1$ if t covers s.

If P and Q are posets, then the *direct product* of P and Q is the poset $P \times Q$ on the set $\{(s, t) : s \in P, t \in Q\}$ such that $(s, t) \leq (s_0, t_0)$ in $P \times Q$ if $s \leq s_0$ in P and $t \leq t_0$ in Q. If P and Q are posets on disjoint sets, then the *disjoint union* (or direct sum) of P and Q is the poset $P + Q$ on the union $P \cup Q$ such that $s \leq t$ in $P + Q$ if either (a) $s, t \in P$ and $s \leq t$ in

P, or (b) $s, t \in Q$ and $s \leq t$ in Q. It is clear from the definitions that $P \times Q \cong Q \times P$ and $P + Q \cong Q + P$.

The *join* $P * Q$ of posets P and Q is the poset whose underlying set is the disjoint union of P and Q and whose order relation is given by $x < y$ if either

- $x < y$ in P,
- $x < y$ in Q, or
- $x \in P$ and $y \in Q$.

Definition 1.2. The *Möbius funtion* μ of a poset P is defined recursively as follows

$$\mu(s,s) = 1, \text{ for all } s \in P.$$

$$\mu(s,u) = \begin{cases} -\sum\limits_{s \leq t < u} \mu(s,t), & s < u; \\ 0, & \text{otherwise.} \end{cases}$$

Proposition 1.1. *Let P and Q be posets. Then for $(p_1, q_1) \leq (p_2, q_2) \in P \times Q$,*

$$\mu_{P\times Q}((p_1,q_1),(p_2,q_2)) = \mu_P(p_1,p_2) \cdot \mu_Q(q_1,q_2).$$

1.1.1 Whitney Numbers and Stirling numbers

In what follows let P be a graded poset.

Definition 1.3 (Whitney numbers of the first kind)**.** The characteristic polynomial $\chi(x)$ of a poset P is defined by

$$\begin{aligned} \chi(x) &= \sum_t \mu(\hat{0},t) x^{n-rnk(t)} \\ &= \sum_{k=0}^{n} w_k x^{n-k}. \end{aligned}$$

The coefficients w_k are called the Whitney number of the first kind.

Definition 1.4 (Whitney numbers of the second kind)**.** The number of elements of P of rank k is denoted W_k and is called the k^{th} Whitney number of P of the second kind. Thus the rank-generating function $F(P,x)$ of P is given by

$$\begin{aligned} F(P,x) &= \sum_{t\in P} x^{\rho(t)} \\ &= \sum_{k=0}^{n} W_k x^k. \end{aligned}$$

Definition 1.5 (Stirling numbers of the second kind)**.** The Stirling numbers of the second kind, written $S(n,k)$ or $\left\{ \begin{matrix} n \\ k \end{matrix} \right\}$, count the number of ways to partition a set of $[n]$ into k nonempty subsets. For $n \geq 1$, $S(n,k) = 0$ if $k > n$, $S(n,0) = 0$, $S(n,1) = 1$, $S(n,2) = 2^{n-1} - 1$, $S(n,n) = 1$, $S(n,n-1) = \binom{n}{2}$.

The Stirling numbers of the second kind satisfy the following basic recurrence:

$$S(n,k) = k \cdot S(n-1,k) + S(n-1,k-1).$$

The Stirling numbers of the second kind are also given by the explicit formula:

$$S(n,k) = \sum_{j=1}^{k} (-1)^{k-j} \frac{j^{n-1}}{(j-1)!(k-j)!} = \frac{1}{k!} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} j^n.$$

1.2 Order Complexes and Face posets

This section primarily follows Wachs's lecture notes [11] and focuses on the topology and shellability of a poset. By the topology of a poset, we mean the topology of a certain simplicial complex associated to the poset, called the order complex of the poset. Shellability is a combinatorial property of simplicial and more general cell complexes, with strong topological and algebraic consequences.

1.2.1 Simplicial Complex

Definition 1.6. An *abstract simplicial complex* Δ on finite vertex set V is a nonempty collection of subsets of V such that

- $v \in \Delta$ for all $v \in V$.
- if $G \in \Delta$ and $F \subset G$ then $F \in \Delta$.

The elements of Δ are called *faces (or simplices)* of Δ and the maximal faces are called *facets*. We say that a face F has dimension d and write $\dim(F) = d$ if $d = |F| - 1$. Faces of dimension d are referred to as d-faces. The *dimension* $\dim(\Delta)$ of Δ is defined to be $\max_{F \in \Delta} \dim(F)$. We also allow the (-1)-dimensional complex $\{\varnothing\}$, which we refer to as the empty simplicial complex. A *pure simplicial k-complex* Δ is a simplicial complex where every simplex of dimension less than k is a face of some simplex $\sigma \in \Delta$ of dimension exactly k i.e., all facets have dimension k.

Definition 1.7. A d-dimensional *geometric simplex* in $\mathbb{R}^n$ is defined to be the convex hull of $d+1$ affinely independent points in $\mathbb{R}^n$ called vertices. The convex hull of any subset of the vertices is called a face of the geometric simplex. A *geometric simplicial complex* K in $\mathbb{R}^n$ is a nonempty collection of geometric simplices in $\mathbb{R}^n$ such that

- Every face of a simplex in K is in K.
- The intersection of any two simplices of K is a face of both of them.

From a geometric simplicial complex K, one gets an abstract simplicial complex $\Delta(K)$ by letting the faces of $\Delta(K)$ be the vertex sets of the simplices of K. Every abstract simplicial complex Δ can be obtained in this way, i.e., there is a geometric simplicial complex K such that $\Delta(K) = \Delta$. We refer to this space as the *geometric realization* of Δ and denote it by $|\Delta|$. A simplicial map is a map between simplicial complexes with the property that the images of the vertices of a simplex always span a simplex.

1.2.2 Order Complex

To every poset P, one can associate an abstract simplicial complex $\Delta(P)$ called the *order complex* of P. The vertices of $\Delta(P)$ are the elements of P and the faces of $\Delta(P)$ are the chains of P. This association is functorial i.e., Δ is a functor from the category of posets to the category of simplicial complexes,

$$\begin{array}{ccc} \{\text{Posets}\} & \xrightarrow{\Delta} & \{\text{Simplicial Complexes}\} \\ \{\text{Order Preserving maps}\} & \longrightarrow & \{\text{Simplicial Maps}\}. \end{array}$$

To every simplicial complex Δ, one can associate a poset $P(\Delta)$ called the *face poset* of Δ, which is defined to be the poset of nonempty faces ordered by inclusion. The face lattice $L(\Delta)$ is $P(\Delta)$ with a smallest element $\hat{0}$ and a largest element $\hat{1}$ attached.

If we start with a simplicial complex Δ, take its face poset $P(\Delta)$, and then take the order complex $\Delta(P(\Delta))$, we get a simplicial complex known as the *barycentric subdivision* of Δ. The geometric realizations are always homeomorphic, $\Delta \cong \Delta(P(\Delta))$.

Theorem 1.2 (Philip Hall). *For any poset P*

$$\widetilde{\chi}(\Delta(P)) = \mu(\hat{P}).$$

Define the *join* of two simplicial complexes Δ and Γ on disjoint vertex sets to be the simplicial complex given by

$$\Delta * \Gamma := \{A \cup B : A \in \Delta, B \in \Gamma\}$$

Clearly $\Delta(P * Q) = \Delta(P) * \Delta(Q)$.

1.2.3 Shellable Simplicial Complexes

For each face F of a simplicial complex Δ, let (F) denote the subcomplex generated by F, i.e., $(F) = \{G : G \subset F\}$. All simplicial complexes that we consider are pure.

Definition 1.8. A simplicial complex Δ is said to be *shellable* if its facets can be arranged in linear order $F_1, F_2, \ldots, F_t$ in such a way that the subcomplex $(\bigcup_{i=1}^{k-1}(F_i)) \cap (F_k)$ is pure and $(\dim \Delta - 1)$-dimensional for all $k = 2, 3, \ldots, t$. Such an ordering of facets is called a *shelling*.

Shellability does have strong topological consequences, however, as is shown by the following result.

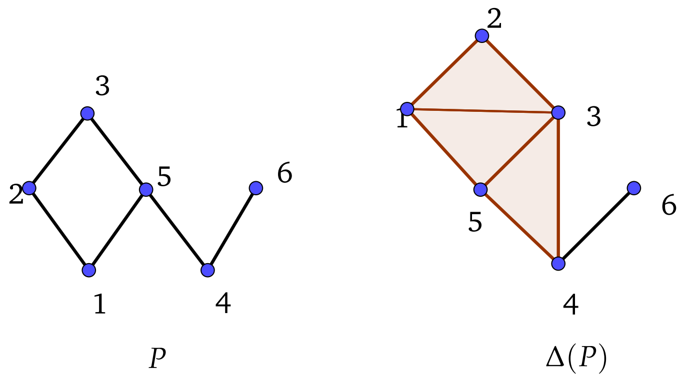


Figure 1.2: Order Complex of a Poset

Theorem 1.3. *[4] A shellable simplicial complex has the homotopy type of a wedge of spheres, where the number of spheres is the number of facets whose entire boundary is contained in the union of the earlier facets. Such facets are usually called homology facets.*

Corollary 1.4. *If Δ is shellable then*

$$\widetilde{H}_i(\Delta, \mathbb{Z}) \cong \begin{cases} \mathbb{Z}^r, & i = \dim(\Delta); \\ 0, & \text{otherwise.} \end{cases}$$

where r is the number of homology facets of Δ.

1.2.4 Shellable Posets

Definition 1.9. A poset P is said to be *shellable* if the maximal chains $\mathfrak{M}$ of P has a shelling, that is, a linear order Ω such that if $k <^{\Omega} m$ for $k, m \in \mathfrak{M}$ then there is an $h \in \mathfrak{M}$ with $h <^{\Omega} m$ such that $(k \cap m) \subset (h \cap m)$ and $|h \cap m| = |m| - 1$.

For any finite poset P we let $C(P)$ denote its covering relation, $C(P) = \{(x, y) \in P \times P : x \prec y\}$. An edge-labeling of P is a map $\lambda : C(P) \to \Lambda$, where Λ is some poset. An edge-labeling therefore corresponds to an assignment of elements of Λ to the edges of the Hasse diagram of P. An unrefinable chain $x_0 \prec x_1 \prec \cdots \prec x_n$ in a poset with an edge-labeling λ will be called *rising* if $\lambda(x_0, x_1) < \lambda(x_1, x_2) < \cdots < \lambda(x_{n-1}, x_n)$.

Definition 1.10 (EL-shellable posets)**.** Let $\lambda : C(P) \to \Lambda$ be an edge-labeling of a graded poset P. λ is said to be an *R-labeling* if in every interval $[x, y]$ of P there is a unique rising unrefinable chain $x = x_0 \prec x_1 \prec \cdots \prec x_n = y$. λ is said to be an *EL-labeling* or simply *L-labeling* in case
(i) λ is an R-labeling;
(ii) for every interval $[x, y]$ of P if $x = x_0 \prec x_1 \prec \cdots \prec x_n = y$ is the unique rising unrefinable chain and $x \prec z \leq y, z \neq x$, then $\lambda(x, x_1) < \lambda(x, z)$.

Definition 1.11. A poset is *lexicographically shellable* (or L-shellable) if it is graded and admits an L-labeling.

Example 1.4. The Boolean poset B_n is Lexicographically shellable.
Given $x \prec y$ in B_n, there exists $k \in [n]$ such that $y - x = \{k\}$. The label λ we associate with $(x,y) \in C(B_n)$ is k *i.e.,* $\lambda(x,y) = k$. It is clear that the labelling λ is a L-labelling on B_n.

Theorem 1.5. *Let P be a lexicographically shellable poset. Then P is shellable.*

Theorem 1.6. *[4], [2] Suppose P is a bounded poset with an L-labeling. Then the lexicographic order of the maximal chains of P is a shelling of $\Delta(P)$. Moreover, the corresponding order of the maximal chains of $\bar{P}$ is a shelling of $\Delta(\bar{P})$.*

1.3 Discrete Morse Theory

Discrete Morse theory is a technique for analyzing the topology of a regular cell complex by defining a special type of function on it, called a discrete Morse function. It is similar to smooth Morse theory. Several notions of classical Morse theory like critical points and gradient paths are modified to the category of CW complexes and in turn used to state discrete versions of the Morse inequalities. This section closely follows R. Forman's treatment in [6].

1.3.1 Discrete Morse Function

Let $\mathcal{K}$ be any finite regular CW complex, K the cells of $\mathcal{K}$, and K_p the cells of dimension p. We say a function $f : K \to \mathbb{R}$ a *discrete Morse function* if $\forall \sigma^p \in K_p$

$$\begin{aligned} \#\{\tau^{p+1} > \sigma^p : f(\tau) \leq f(\sigma)\} \leq 1, \\ \#\{\nu^{p-1} < \sigma^p : f(\nu) \geq f(\sigma)\} \leq 1. \end{aligned} \tag{1.1}$$

We say σ^p is critical (with index p) if

$$\begin{aligned} \#\{\tau^{p+1} > \sigma^p : f(\tau) \leq f(\sigma)\} = 0, \\ \#\{\nu^{p-1} < \sigma^p : f(\nu) \geq f(\sigma)\} = 0. \end{aligned} \tag{1.2}$$

Given $c \in \mathbb{R}$, we define a level subcomplex $M(c)$ by

$$M(c) = \bigcup_{f(\tau) \leq c} \bigcup_{\sigma < \tau} \sigma. \tag{1.3}$$

Lemma 1.7. *Suppose $p > 0$. For each cell α^p of $\mathcal{K}$, at least one of the inequalities in Eq.* (1.1) *must be strict.*

Proof. Suppose there is some α for which neither inequality is strict. That is, there exists a $\beta > \alpha$ and a $\gamma < \alpha$ such that $f(\beta) \leq f(\alpha)$ and $f(\gamma) \geq f(\alpha)$. But that means

$$f(\gamma) \geq f(\beta). \tag{1.4}$$

Now, since we are in a regular complex, there exists some p-cell α' such that $\gamma < \alpha' < \beta$. Applying condition 1 of the definition to γ, we see that $f(\alpha') > f(\gamma)$. Applying condition 2 to β, we see that $f(\alpha') < f(\beta)$. But these inequalities contradict Eq. (1.4). □

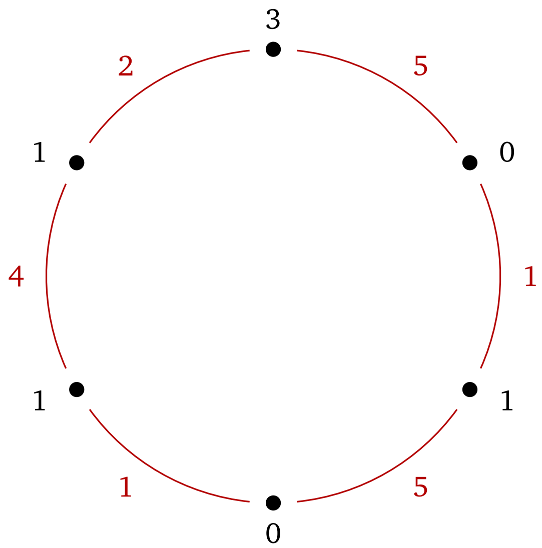


Figure 1.3: The discrete Morse function f

Theorem 1.8. *Suppose the interval $[a, b]$ contains no critical values of f. Then $M(a)$ is a deformation retract of $M(b)$. Moreover, $M(b)$ simplicially collapses onto $M(a)$.*

Theorem 1.9. *Suppose σ_p is a critical cell with $f(\sigma) \in [a, b]$, and there are no other critical cells with values in $[a, b]$. Then $M(b)$ is homotopy equivalent to*

$$M(a) \bigcup e^p$$

where e^p is a p-cell, and it is glued to $M(a)$ along its boundary.

Corollary 1.10. *Suppose M is a regular cell complex with a discrete Morse function. Then M is homotopy equivalent to a CW-complex with exactly one cell of dimension p for each critical simplex of dimension p.*

1.3.2 Gradient Flow

Definition 1.12. A *Discrete Vector Field* V on $\mathcal{K}$ is a collection of pair (σ^p, τ^{p+1}) where $\sigma < \tau$, such that each cell is in at most one pair of V.

Suppose there is a discrete Morse function $f : K \to \mathbb{R}$. The conditions listed above ensure that if we pair the cells $\sigma^p < \tau^{p+1}$ whenever $f(\sigma) \geq f(\tau)$, the resulting collection of pairs is a discrete vector field, called the gradient vector field of f.

Definition 1.13. Given a discrete vector field V on K, a V-path is a sequence of cells

$$\sigma_1^p, \tau_1^{p+1}, \ldots, \sigma_t^p, \tau_t^{p+1}, \sigma_{t+1}^p$$

such that, for each $i \in [t]$,

- $\tau_i > \sigma_{i+1}$,

- (σ_i, τ_i) is a pair in V,
- $\sigma_{i+1} \neq \sigma_i$.

A path is called *closed* if $\sigma_1 = \sigma_{t+1}$.

Remark 1.1. If $\sigma_1, \tau_1, \ldots, \sigma_t, \tau_t, \sigma_{t+1}$ is a V-path, then $f(\sigma_1) \geq f(\tau_1) \geq \cdots \geq f(\tau_t) \geq f(\sigma_{t+1})$.

Theorem 1.11. *A discrete vector field is the gradient vector field of a discrete Morse function if and only if there are no closed V-paths.*

1.3.3 Morse Complex

For $\sigma^p < \tau^{p+1}$ in $\mathcal{K}$, the *incidence number* $[\tau : \sigma]$ is the degree of the attaching homeomorphism. Consider two distinct p-cells σ_1, σ_2 and a $(p+1)$-cell τ such that $\sigma_1 < \tau$ and $\sigma_2 < \tau$. Fixed orientations on σ_1 and τ, induce an orientation on σ_2 so that $[\tau : \sigma_1] \cdot [\tau : \sigma_2] = -1$. Let V be a discrete Morse function on $\mathcal{K}$ and let $C := \sigma_1, \tau_1, \ldots, \sigma_t, \tau_t, \sigma_{t+1}$ be a gradient path. An orientation on σ_1 induces an orientation on each σ_i in turn, and, in particular, on σ_{t+1}. Define $w(C) = 1$ if the fixed orientation on σ_1 induces the fixed orientation on a_{t+1}, and $w(C) = -1$ otherwise.

A cell σ is *critical* for a discrete Morse function V, if it is not paired in V. Let $\mathcal{M}_p$ denote the free abelian group generated by the critical p-cells. The Morse complex $\mathcal{M}_\bullet$ on $\mathcal{K}$ is defined as follows,

$$\mathcal{M}_\bullet : 0 \longrightarrow \ldots \xrightarrow{\tilde{\partial}} \mathcal{M}_{p+1} \xrightarrow{\tilde{\partial}} \mathcal{M}_p \xrightarrow{\tilde{\partial}} \ldots \xrightarrow{\tilde{\partial}} \mathcal{M}_0 \to 0$$

The boundary homomorphism is given by

$$\tilde{\partial}\tau = \sum_{\sigma^p \in \tau} \langle \tilde{\partial}\tau, \sigma \rangle \sigma, \ \forall \tau \in \mathcal{M}_{p+1},$$

$$\langle \tilde{\partial}\tau, \sigma \rangle := \sum_{\tilde{\sigma}^p < \tau} [\tau : \tilde{\sigma}] \sum_{c \in \Gamma(\tilde{\sigma}, \sigma)} w(c),$$

where, $\Gamma(\tilde{\sigma}, \tau)$ denotes the set of all gradient paths from $\tilde{\sigma}$ to τ.

Chapter 2

Posets of Partitions

Posets of partitions are those posets whose elements are certain partitions of a finite set. They play an important role in combinatorics. They serve not only as test cases for several theorems but also interesting in their own right. Several topological properties of these posets have purely combinatorial description. In this chapter we will discuss the topology of the poset of ordered and unordered partitions.

2.1 Lattices

Let P be a poset and $s, t \in P$, then an upper bound of s and t is an element $u \in P$ satisfying $u \geq s$ and $u \geq t$. A *least upper bound (or join or supremum)* of s and t is an upper bound u of s and t such that every upper bound v of s and t satisfies $v \geq u$. If a least upper bound of s and t exists, then it is clearly unique and is denoted $s \vee t$. Dually one can define the *greatest lower bound (or meet or infimum)* $s \wedge t$, when it exists. A *lattice* is a poset L for which every pair of elements has a least upper bound and greatest lower bound. Clearly,

- The operations $\vee$ and $\wedge$ are associative, commutative, and idempotent (i.e., $t \vee t = t \wedge t = t$);
- $s \wedge (s \vee t) = s = s \vee (s \wedge t)$ (absorption laws);
- $s \wedge t = s \iff s \vee t = t \iff s \leq t$.

Clearly all finite lattices have a $\hat{0}$ and $\hat{1}$. If every pair of elements of a poset P has a meet (respectively, join), then we say that P is a meet-semilattice (respectively, join-semilattice).

Lemma 2.1. *Let P be a finite meet-semilattice with $\hat{1}$. Then P is a lattice. (Dually a finite join-semilattice with $\hat{0}$ is a lattice.)*

Theorem 2.2. *[10] Let L be a finite lattice with at least two elements, and let $\hat{1} \neq a \in L$. Then*

$$\sum_{t : t \wedge a = \hat{0}} \mu(t, \hat{1}) = 0.$$

Proposition 2.3. *Let L be a finite lattice. The following two conditions are equivalent.*

1. *L is graded, and the rank function ρ of L satisfies $\rho(s) + \rho(t) \leq \rho(s \wedge t) + \rho(s \vee t) \forall s, t \in L$.*

2. *If s and t both cover $s \wedge t$, then $s \vee t$ covers both s and t.*

Definition 2.1. A finite lattice L satisfying either of the (equivalent) conditions of the previous proposition is called a *finite semimodular lattice*.

2.2 Homotopy Complementation formula for Posets

The main result of this section is a homotopy complementation formula and its application to semimodular lattices proved by Björner in [5].

Theorem 2.4. *Let L be a semimodular lattice of finite length r, $r \geq 2$. Then the geometric realization of $\Delta(L)$ has the homotopy type of a wedge of $(r-2)$-spheres. Furthermore, if L is finite the number of spheres in the wedge is $|\mu(\hat{0}, \hat{1})|$.*

The topology needed to prove Theorem 2.4 is condensed into the following two lemmas,

Lemma 2.5 (Contraction Carrier Lemma)**.** *Let Δ be a simplicial complex, X a topological space, and $f, g : |\Delta| \to X$ two continuous maps. Assume that to each simplex σ of Δ we can associate a subspace $C(\sigma)$ of X in such a way that*

1. *$C(\sigma)$ is contractible,*

2. *$\sigma \subset \tau \implies C(\sigma) \subset C(\tau)$*

3. *$f(|\sigma|) \cup g(|\sigma|) \subset C(\sigma)$.*

Then f and g are homotopic.

Lemma 2.6 (Contractible Subcomplex Lemma)**.** *If Δ is a simplicial complex and A a contractible subcomplex then the quotient map $\pi : |\Delta| \to |\Delta|/|A|$ is a homotopy equivalence.*

Now we will present some combinatorial conditions for homotopies of order-preserving maps.

Proposition 2.7. *Let $f, g : P \to Q$ be order-preserving maps of posets. If $f(x)$ and $g(x)$ are comparable for all $x \in P$, then f and g are homotopic.*

Proof. For each finite chain σ of P, let $C(\sigma) = f(\sigma) \cup g(\sigma)$. Since the subposet $C(\sigma)$ has a least element, it is a cone, hence contractible. The result follows by Lemma 2.5. □

A poset will be called *join-contractible* (via s) if there is some element s such that every element has a join with s. We define *meet-contractible* dually. By Proposition 2.7, a join-contractible poset is contractible: the identity map is homotopic to the map $x \mapsto s \vee x$, which in turn is homotopic to the constant map s.

Proposition 2.8. *Let P be a poset having an element s such that*

1. *$s \vee x$ or $s \wedge x$ exists for all $x \in P$, and*

2. *if $x < y$, $s \wedge x$ does not exist, but $s \wedge y$ does exist, then $(s \wedge y) \vee x$ exists.*

Then P is contractible.

Proof. Let $M = \{y \in P : s \wedge y \text{ exists}\}$ and $M^c = P - M$. For each finite non empty chain σ of P, define the subposet

$$C(\sigma) = \sigma \bigcup \{s\} \bigcup \{s \vee x : x \in \sigma \cap M^c\} \bigcup \{s \wedge y : y \in \sigma \cap M\}$$

$$\bigcup \{(s \wedge y) \vee x : x < y, x \in \sigma \cap M^c, y \in \sigma \cap M\}.$$

Let z be the least element of σ. It is easy to check that $C(\sigma)$ is join-contractible via z if $z \in M^c$, and $C(\sigma)$ is meet-contractible via z if $z \in M$. In fact, one only has to check for s and $\{s \wedge y : y \in \sigma \cap M\}$, since everything else in $C(\sigma)$ is above z. Clearly $\tau \subset \sigma$ implies $C(\tau) \subset C(\sigma)$. Since C carries both the identity map and the constant map s, it follows by Lemma 2.5 that these maps are homotopic, hence P is contractible. □

We will say that two elements x, y of a poset P are *complements* (in symbols, $x \perp y$) if the set $\{x, y\}$ has no upper or lower bound in P. For s in P we also define $\perp(s) = \{x \in P : x \perp s\}$. Now Proposition 2.8 can be rephrased as saying that if the poset P has an element s such that

1. for all $x \in P$, either $s \vee x$ or $s \wedge x$ exists, or else $x \in \perp(s)$, and

2. if $x < y, y \notin \perp(s)$ and $s \wedge x$ does not exist, but $s \wedge y$ does exist, then $(s \wedge y) \vee x$ exists.

Then $P - \perp(s)$ is contractible.

In the following, "Σ" denotes suspension, " $\cong$ " denotes homotopy equivalence. If $x \in P$ we write $P_{<x} = \{y \in P : y < x\}$ and $P_{>x}$ is defined similarly. A poset is said to be an antichain if no two distinct elements are comparable.

Theorem 2.9 (Homotopy Complementation formula). *Let P be a poset, and suppose that $s \in P$ satisfies the above conditions and that $\perp(s)$ is an antichain. Then*

$$|P| \cong \bigvee \Sigma(P_{<x} * P_{>x})$$

Proof. When $\perp(s)$ is an antichain the space $|P|$ can be obtained from $|P - \perp(s)|$ by attaching a cone over $(P_{<x} * P_{>x})$ for each $x \in \perp(s)$. Hence, the quotient space $|P|/|P - \perp(s)|$ is homeomorphic to $\bigvee_{x \perp (s)} (P_{<x} * P_{>x})$, where the wedge point is the image of $|P - \perp(s)|$. By Lemma 2.5 we know that $P - \perp(s)$ is contractible. Therefore, by Lemma 2.6, $|P| \cong |P|/|P - \perp(s)|$. □

We will also make use of two more facts of homotopy theory, namely:

1. the suspension of a wedge of d-spheres is homotopy equivalent to a wedge of $(d+1)$-spheres and

2. if $X_\alpha = Y_\alpha$ for all α in some indexing set, and these spaces are triangulable and connected, then $\bigvee_\alpha X_\alpha = \bigvee_\alpha Y_\alpha$.

Proof of Theorem 2.4. The proper part of any graded lattice of length 2 has the homotopy type of a wedge of 0-spheres. We continue by induction on r. If s is an atom and $x \perp s$, then $\rho(x) = r - 1$, as can be seen from the semimodular inequality. So for every $x \in \perp (s)$ the interval $[\hat{0}, x]$ is a semimodular lattice of length $r - 1$. By the induction assumption $(\hat{0}, x)$ has the homotopy type of a wedge of $(r-3)$-spheres, so using Theorem 2.9 and the initially cited facts (1) and (2), we conclude that L is of the required homotopy type. □

2.3 Poset of Unordered Partition

Let Π_n denote the set of all partitions of the finite set $[n]$ ordered by partition as in Example 1.3. Clearly Π_n is semimodular, so from Theorem 2.4 we conclude that Π_n has the homotopy of wedge of spheres of $(n-3)$-spheres. The number of spheres in the wedge is $|\mu(\hat{0}, \hat{1})|$.

2.3.1 The Möbius function of the partition poset

It is easy to check that Π_n is graded of rank $n - 1$. The rank $\rho(\pi)$ of $\pi \in \Pi_n$ is equal to n-(number of blocks of π) $= n - \#\pi$. Hence the rank-generating function of Π_n is given by

$$f(\Pi_n, x) = \sum_{k=0}^{n-1} S(n, n-k)x^k$$

where $S(n, n-k)$ is a Stirling number of the second kind. If $\pi, \sigma \in \Pi_n$ then $\pi \wedge \sigma$ has as blocks the nonempty sets $B \cap C$, where $B \in \pi$ and $C \in \sigma$. Hence Π_n is a meet-semilattice. Since the partition of $[n]$ with a single block is a $\hat{1}$ for Π_n, it follows from Lemma 2.1 that Π_n is a lattice. Suppose that $\pi = \{B_1, \ldots, B_k\}$. Then the interval $[\pi, \hat{1}]$ is isomorphic in an obvious way to Π_π, the lattice of partitions of the set $\{B_1, \ldots, B_k\}$. Hence $[\pi, \hat{1}] \cong \Pi_k$. Let us now consider the structure of any interval $[\sigma, \pi]$. Suppose that $\pi = \{B_1, \ldots, B_k\}$ and that B_i is partitioned into λ_i blocks in σ. An argument similar to above shows that

$$[\sigma, \pi] \cong \Pi_{\lambda_1} \times \Pi_{\lambda_2} \ldots \Pi_{\lambda_k}.$$

In particular, $[\hat{0}, \pi] \cong \Pi_1^{a_1} \times \ldots \Pi_n^{a_n}$ if π has a_i blocks of size i.

Now set $\mu_n = \mu(\hat{0}, \hat{1})$, where μ_n is the Möbius function of Π_n. If $[\sigma, \pi] = \Pi_{\lambda_1} \times \cdots \times \Pi_{\lambda_k}$, then we have $\mu(\sigma, \pi) = \mu_{\lambda_1} \times \cdots \times \mu_{\lambda_k}$. Hence to determine μ completely, it suffices to compute μ_n. Pick a to be the partition with the two blocks $\{1, 2, \ldots, n-1\}$ and $\{n\}$. An element t of Π_n satisfies $t \wedge a = \hat{0}$ if and only if $t = \hat{0}$ or t is an atom whose unique two-element block has the form $\{i, n\}$ for some $i \in [n-1]$. The interval $[t, \hat{1}]$ is isomorphic to Π_{n-1}, so from Theorem 2.2 we have $\mu_n = -(n-1)\mu_{n-1}$. Since $\mu_0 = 1$, we conclude that $\mu_n = (-1)^{n-1}(n-1)!$.

2.4 Poset of Ordered Partitions

Definition 2.2. Let $n \in \mathbb{N}, n \geq 0$. An *ordered partition* $\alpha = (I_1 \dots I_k)$ of $[n]$ is a partition in which the order of the I_j's matters. The subsets I_j's are called *blocks* of α.

We can make the set Ω_n of all ordered partitions of $[n]$ into a poset by defining $\pi \leq \sigma$ in Ω_n if every block of π is contained in a block of σ in an order preserving way. We then say that π is a order preserving refinement of σ and that Ω_n consists of the ordered partitions of $[n]$ "ordered by refinement."

Example 2.1. If $n = 9$ and if π has blocks $(137, 46, 2, 58, 9)$ and σ has blocks $(13467, 2589)$ then $\pi \leq \sigma$. We then say that π is a order preserving refinement of σ. If $\pi^{'} = (137, 2, 46, 58, 9)$ then $\pi^{'} \not\leq \sigma$ in Ω_n.

2.4.1 Permutohedron

The permutohedron of order n, denoted $\mathfrak{P}_n$, is an $(n-1)$-dimensional polytope embedded in an n-dimensional space. It is defined as the convex hull of all points in $\mathbb{R}^n$ that are obtained by permuting the coordinates of the point $(1, 2, \dots, n)$. It has the following properties:

1. The k-faces of $\mathfrak{P}_n$ are labeled by ordered partitions of the set $[n]$ into $(n-k)$ non-empty parts.

2. A face F of $\mathfrak{P}_n$ is contained in a face F' iff the label of F is a refinement of the label of F'. Here we mean the order-preserving refinement.

3. In particular, the vertices are labeled by the elements of the symmetry group S_n. For each vertex, the label is the inverse permutation of the coordinates of the vertex. Two vertices are joined by an edge whenever their labels differ by a permutation of two neighbor entries.

4. Each face of $\mathfrak{P}_n$ equals the Cartesian product of standard permutohedra of smaller dimensions.

5. The permutohedra $\mathfrak{P}_1$, $\mathfrak{P}_2$, and $\mathfrak{P}_3$ are a one-point polytope, a segment, and a regular hexagon respectively.

Refer [12], for more detailed analysis of permutohedron.

Theorem 2.10. *The face poset of the permutohedron $\mathfrak{P}_n$ is isomorphic to the poset of ordered partition Ω_n.*

Thus, the problem of understanding the topology of the poset Ω_n reduces to understanding the polytope $\mathfrak{P}_n$.

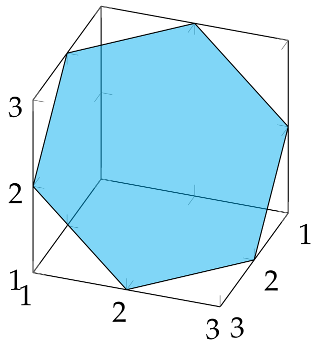


Figure 2.1: Permutohedron $\mathfrak{P}_3$

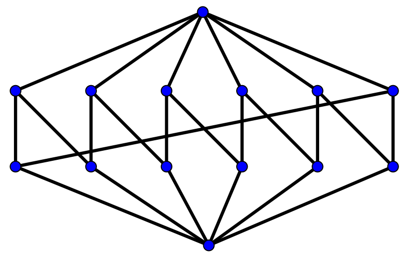


Figure 2.2: Face of $\mathfrak{P}_3$ with $\hat{0}$

Denote the poset $\mathfrak{P}_n$ adjoined with $\hat{0}$ also as $\mathfrak{P}_n$. This forms a lattice. The proper part of this lattice $\bar{\mathfrak{P}}_n = \mathfrak{P}_n - \{\hat{0}, \hat{1}\}$ is the face poset of a regular cell structure on $(n-1)$-sphere. Thus the following lemma follows from Philip hall theorem.

Lemma 2.11. *Given the poset Ω_n of ordered partitions, the Möbius function is*

$$\mu(\hat{0}, \hat{1}) = (-1)^{n-1}.$$

Chapter 3

The Cyclopermutohedron

The cyclopermutohedron of order n, denoted CP_{n+1} was introduced by G. Panina in [8]. It is an $(n-2)$ - dimensional regular CW complex whose cells are labeled by cyclically ordered partitions of $[n+1]$. Recall that for a generic length vector $L := (l_1, l_2, \dots, l_{n+1}) \in \mathbb{R}_+^{n+1}$ the moduli space $\mathscr{M}_L$ of planar polygons is a closed, orientable smooth $(n-2)$-manifold. A remarkable fact is that for every such generic L there is a subcomplex of CP_{n+1} homeomorphic to $\mathscr{M}_L$. Thus, in this sense, the cyclopermutohedron is an "universal object" for moduli spaces of polygonal linkages. Unlike the permutohedron, the cyclopermutohedron cannot be realized as a polytope in any Euclidean space. However, it can be realized as a *virtual polytope*. Intuitively, one can think of a virtual polytope as the boundary of a convex polytope with 'diagonals' inserted between certain faces.

In this chapter, we focus on combinatorics and topology of CP_{n+1}. On the topological side we discuss, in detail, the homology computations using discrete Morse theory as described in [7].

The following notations will be heavily used in this and subsequent chapters:

- A subset of $[n+1]$ containing the element $n+1$ will be called an $n+1$-set. Given a partition of $[n+1]$, the letter N denotes the $n+1$-set.
- The triangle ∇ denotes (a possibly empty) string of singletons arranged in decreasing order.
- Given two subsets I and J, the expression "$I < J$" indicates that $i < j$ for each $i \in I$ and $j \in J$. Similarly, the expression "$k < \nabla$" indicates that k is less than the element in each singleton of ∇.
- The set "$I - \{m\}$" is denoted "$I - m$" and the braces for the singleton will be omitted *i.e.,* the block "$\{s\}$" is denoted by "s".
- The set "$\{a_1, a_2, \dots, a_k\}$" will be denoted "$a_1 a_2 \dots a_k$" when there is no ambiguity.

Definition 3.1. A **cyclically ordered partition** of the set $[n+1]$ is an equivalence class of ordered partitions with the relation that two ordered partitions equivalent if one can be obtained from the other by a cyclic permutation of its blocks. That is, $(I_1, \dots, I_k) \sim$

$(I_2, \ldots, I_k, I_1) \sim \cdots \sim (I_k, I_1, \ldots, I_{k-1})$. When dealing with such partitions, we will always assume that the block containing $n+1$ appears last.

Definition 3.2. For a fixed $n > 2$, the regular cell complex cyclopermutohedron CP_{n+1} is defined as follows.

- For $k = 0, 1, \ldots, n-2$, the k-dimensional cells (k-cells, for short) of the complex CP_{n+1} are labeled by (all possible) cyclically ordered partitions of the set $[n+1]$ into $(n-k+1)$ non-empty parts.
- A (closed) cell F contains a cell F' whenever the label of F' refines the label of F. Here, by refinement we mean orientation preserving refinement.

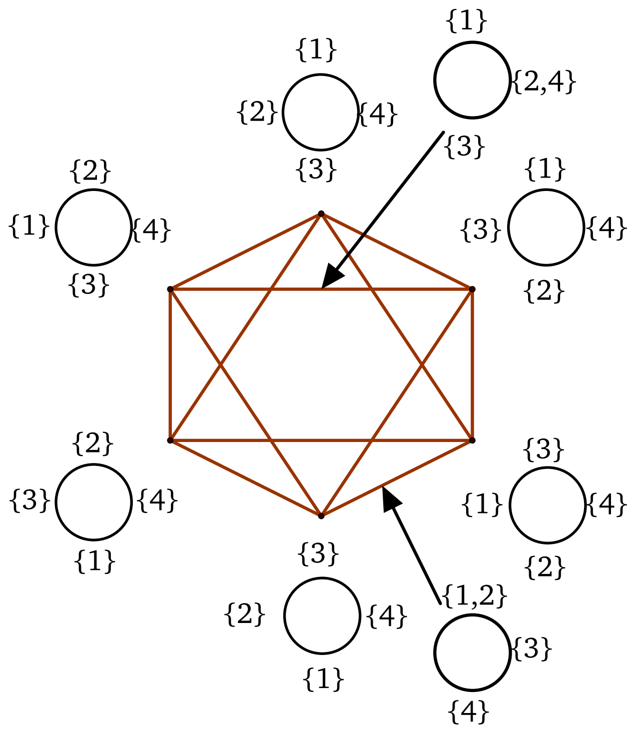


Figure 3.1: The Complex CP_4

In particular, this means that the vertices of the complex CP_{n+1} are labeled by cyclic orderings on the set $[n+1]$. Two vertices are joined by an edge whenever their labels differ on a permutation of two neighboring entries. Example $n = 3$ can be seen in Fig. 3.1.

3.1 Combinatorial description of cellular homology

In this section, we present a purely combinatorial description for the boundary maps of the cellular chain complex of CP_{n+1}in terms of partitions of sets.

3.1.1 Canonical orientation of cells

Note that the vertices of CP_{n+1} are in bijection with the elements of the group S_n. Two vertices of CP_{n+1} are joined by an edge whenever their labels differ on a permutation of two neighbor entries. Given a vertex v in $\alpha = (I_1, I_2, \ldots, I_l)$, there are $\dim(\alpha)$ many vertices of α that are connected to v by an edge. We call such vertices *neighbors* of v and we put an order on these in the following way. We get the first neighbor of v by interchanging the first two entries of v that belong to the same I_i, the second neighbor we get by interchanging the second two entries of v that belong to the same I_i etc. This ordering is called the *orientation* of the cell related to the vertex v.

Definition 3.3. The *principal vertex* $\mathrm{PV}(\alpha)$ of a cell α is the vertex with the label $(\hat{I}_1, \hat{I}_2, \ldots, \hat{I}_l)$, where $\hat{I}_j$ is a partition of the set I_j into singletons coming in increasing order. The orientation of the cell α related to its principal vertex $\mathrm{PV}(\alpha)$ is called the *canonical orientation* of α.

Unless otherwise stated, by an orientation of a cell, we mean the canonical orientation.

Example 3.1. For the cell $\alpha = (\{1\}\{2,4,5\}\{3\}\{6,7,8\})$, the principal vertex $\mathrm{PV}(\alpha)$ is given by $(\{1\}\{2\}\{4\}\{5\}\{3\}\{6\}\{7\}\{8\})$ and the α-neighbors of $\mathrm{PV}(\alpha)$ are ordered as follows:
$v_1 = (1,4,2,5,3,6,7,8)$, $v_2 = (1,2,5,4,3,6,7,8)$, $v_3 = (1,2,4,5,3,7,6,8)$, etc.

3.1.2 Boundary maps

There is a free and transitive action of the group S_{n+1} on the vertices of CP_{n+1}. Note that the action preserves the canonical ordering. For example, let $\sigma = (12345, 6, 7) \in \mathrm{CP}_7$, and $w = (24) \in S_7$.

$$\begin{aligned}
v_0 &= (1,2,3,4,5,6,7) \to (1,4,3,2,5,6,7) \\
v_1 &= (2,1,3,4,5,6,7) \to (4,1,3,2,5,6,7) \\
v_2 &= (1,3,2,4,5,6,7) \to (1,3,4,2,5,6,7) \\
v_3 &= (1,2,4,3,5,6,7) \to (1,2,3,4,5,6,7) \\
v_4 &= (1,2,3,5,4,6,7) \to (1,4,3,5,2,6,7)
\end{aligned}$$

Given a pair of cells $\tau^{p-1} < \sigma^p$ in CP_{n+1}, let v_σ and v_τ denote the principal vertices of σ and τ respectively. Let $(v_1, \ldots, v_p)$ be the ordering on neighbors of v_σ in σ. Since v_σ and v_τ also represent elements of S_{n+1}, there exists some permutation $g_{\sigma,\tau} \in S_n$ such that $g_{\sigma,\tau} v_\sigma = v_\tau$. Moreover, there is exactly one index $i_\tau \in \{1, 2, 3, \ldots, p\}$ such that $g_{\sigma,\tau} v_{i_\tau}$ is not adjacent to v_τ in τ.

Let Δ_m be the free abelian group with basis the m-cells σ^m_α of CP_{n+1}. We define the boundary homomorphism $\partial_m : \Delta_m \to \Delta_{m-1}$ by specifying its values on the basis elements:

$$\begin{aligned}
\partial_m \sigma &= \sum_{\tau^{m-1} < \sigma^m} \mathrm{sign}(g_{\sigma,\tau}) \cdot (-1)^{i_\tau} \cdot \tau \\
\langle \partial_m \sigma, \tau \rangle &:= \mathrm{sign}(g_{\sigma,\tau}) \cdot (-1)^{i_\tau}
\end{aligned}$$

Lemma 3.1. *Let* $\sigma = (I_1, I_2, \ldots, I_k)$ *and* $\tau = (I_1, \ldots, J_1, J_2, \ldots, I_k)$ *with some* $p \in \{1, 2, \ldots, k\}$ *such that* $I_p = J_1 \cup J_2$. *For every* $i \in \{1, 2, \ldots, k\}$, *denote* $|I_i| = r_i$. *Then*

$$\langle \partial\sigma, \tau \rangle = (-1)^{\sum_{i=1}^{p-1} r_i + |J_1| - (p-1)} \cdot \mathrm{sign}(g_{\sigma,\tau})$$

Proof. Without loss of generality assume $\mathrm{PV}(\sigma) = \mathrm{PV}(\tau) = (1, 2, \ldots, n+1) = v_0$. The neighbors of $\mathrm{PV}(\sigma)$ are ordered as follows:

$$\begin{aligned}
v_0 &= (1, 2, \ldots, n+1) \\
v_1 &= (2, 1, 3, \ldots, n+1) \\
v_2 &= (1, 3, 2, \ldots, n+1) \\
&\vdots \\
v_{r_1 - 1} &= (1, 2, \ldots, r_1, r_1 - 1, \ldots, n+1) \\
v_{r_1} &= (1, 2, \ldots, r_1, r_1 + 2, r_1 + 1, \ldots, n+1) \\
&\vdots \\
v_{r_{p-1} + |J_1| - (p-1) - 2} &= (1, 2, \ldots, r_{p-1} + |J_1| - 1, r_{p-1} + |J_1| - 2, \ldots, n+1) \\
v_{r_{p-1} + |J_1| - (p-1) - 1} &= (1, 2, \ldots, r_{p-1} + |J_1|, r_{p-1} + |J_1| - 1, \ldots, n+1) \\
v_{r_{p-1} + |J_1| - (p-1)} &= (1, 2, \ldots, r_{p-1} + |J_1| + 1, r_{p-1} + |J_1|, \ldots, n+1) \\
&\vdots
\end{aligned}$$

From the list, it is clear that the index i_τ such that v_{i_τ} is not a vertex of τ is $r_{p-1} + |J_1| - (p-1)$, since the interchanging is consistent with only σ and not with τ. A similar argument works for the case where $\mathrm{PV}(\tau) \neq (1, 2, \ldots, n+1)$. Observe the fact that the missing index corresponding to the cell which

- has same partition structure (i.e., similar block structure) as τ,
- contained in the boundary of σ and
- has same principal vertex as σ,

is the unique index i_τ such that $g_{\sigma,\tau} v_{i_\tau}$ is not adjacent to v_τ in τ. Briefly, a missing index is taken to another missing index by the permutation g_τ. □

Theorem 3.2. *The composition* $\Delta_m \xrightarrow{\partial_m} \Delta_{m-1} \xrightarrow{\partial_{m-1}} \Delta_{m-2}$ *is zero.*

Proof. Let $\sigma^{k+2} = (I_1, I_2, \ldots, I_{n-1-k})$, $\tau^k = (J_1, \ldots, J_{n+1-k})$, then it is enough to show that $\langle \partial^2\sigma, \tau \rangle = 0$. If $\tau^k < \sigma^{k+2}$ then σ and τ satisfy exactly one of the following relations

1. $\exists i, j, k \in [n+1-k]$ and $\exists p \in [n-1-k]$ such that $J_i \cup J_j \cup J_k = I_p$,
2. $\exists i, j, s, t \in [n+1-k]$ and $\exists p, q \in [n-1-k]$ such that $J_i \cup J_j = I_s$ and $J_k \cup J_l = I_t$.

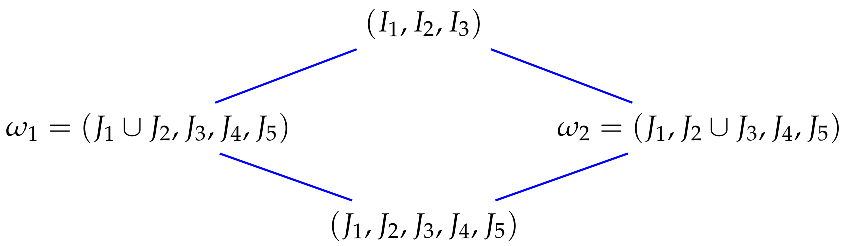


Figure 3.2: The interval $[\tau, \sigma]$ in case 1.

Case 1: Without loss of generality assume $J_1 \cup J_2 \cup J_3 = I_1$, so only I_1 is involved in the computation of $\langle \partial^2 \sigma, \tau \rangle$. We can also assume that σ has the minimum number of blocks, i.e., $s = 3$. Thus, $\sigma = (I_1, I_2, I_3)$ and $\tau = (J_1, J_2, J_3, J_4, J_5)$ and we have

- $\langle \partial\sigma, \omega_1 \rangle = \text{sign}(g_1) \cdot (-1)^{|J_1|+|J_2|}$
- $\langle \partial\sigma, \omega_2 \rangle = \text{sign}(g_2) \cdot (-1)^{|J_1|}$
- $\langle \partial\omega_1, \tau \rangle = \text{sign}(g_3) \cdot (-1)^{|J_1|}$
- $\langle \partial\omega_2, \tau \rangle = \text{sign}(g_4) \cdot (-1)^{|J_1|+|J_2|-1}$

where, g_i's represent the permutation involved in the comparison of principal vertices. Note that there is a unique permutation which takes $\text{PV}(\sigma)$ to $\text{PV}(\tau)$, so $g_3 \circ g_1 = g_4 \circ g_2$. This shows that $\langle \partial^2 \sigma, \tau \rangle = 0$.

Case 2: Without loss of generality assume that $J_1 \cup J_2 = I_1$ and $J_3 \cup J_4 = I_2$, so only I_1 and I_2 are involved in the computation of $\langle \partial^2 \sigma, \tau \rangle$. Further assume that σ has the minimum number of blocks, i.e., $s = 3$. Thus, $\sigma = (I_1, I_2, I_3)$ and $\tau = (J_1, J_2, J_3, J_4, J_5, J_6)$ and we have the following

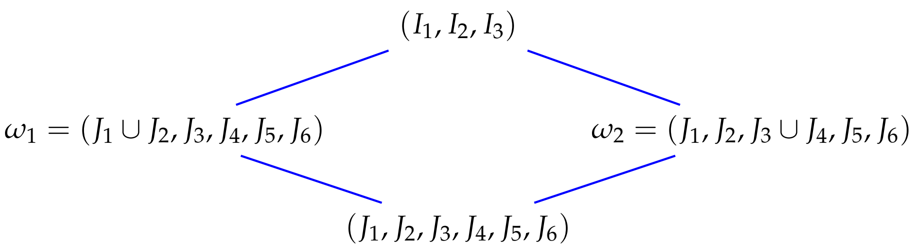


Figure 3.3: The interval $[\tau, \sigma]$ in case 2.

- $\langle \partial\sigma, \omega_1 \rangle = \text{sign}(g_1) \cdot (-1)^{|J_1|+|J_2|+|J_3|-1}$
- $\langle \partial\sigma, \omega_2 \rangle = \text{sign}(g_2) \cdot (-1)^{|J_1|}$
- $\langle \partial\omega_1, \tau \rangle = \text{sign}(g_3) \cdot (-1)^{|J_1|}$
- $\langle \partial\omega_2, \tau \rangle = \text{sign}(g_4) \cdot (-1)^{|J_1|+|J_2|+|J_3|-2}$

where, g_i's represent the permutation involved in the comparison of principal vertices. Note that there is a unique permutation which takes $\mathrm{PV}(\sigma)$ to $\mathrm{PV}(\tau)$, so $g_3 \circ g_1 = g_4 \circ g_2$. This shows that $\langle \partial^2\sigma, \tau \rangle = 0$. □

Theorem 3.3. *Let $\tau^{p-1} < \sigma^p$, then*

$$[\sigma : \tau] = \langle \partial\sigma, \tau \rangle \tag{3.1}$$

i.e., the coefficient of τ in the image of σ under the boundary homomorphism is precisely the incidence number $[\sigma : \tau]$.

Proof. We will prove this inequality using induction on the dimension of cells. If dimension of σ is 2, then the boundary complex is exactly one of the following.

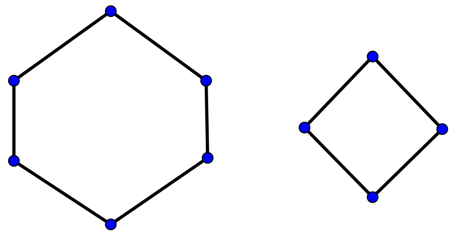


Figure 3.4: Possible boundaries of a 2-cell

By computing the incidence numbers explicitly, Eq. (3.1) can be proven easily.

Now assume the induction hypothesis that the Eq. (3.1) is true for all cells of dimension less than or equal to k. Let $\sigma = (I_1, I_2 \ldots I_{n-k}), \tau = (I_1, I_2 \ldots J_1, J_2 \ldots I_{n-k})$ with $J_1 \cup J_2 = I_p$ and $|I_i| = r_i$. Without loss of generality we can assume that the $\mathrm{PV}(\sigma) = (1, 2, 3, \ldots, n+1)$.

Step 1: If τ has the same principal vertex as σ, then $J_1 = (r_{p-1}+1, r_{p-1}+2, \ldots, r_{p-1}+t)$ and $J_2 = (r_{p-1}+t+1, r_{p-1}+t+2, \ldots, r_p)$. Let $\tilde{\tau} = (I_1, I_2, \ldots, \tilde{J}_1, \tilde{J}_2, \ldots, I_{n-k})$ where $\tilde{J}_1 = (r_{p-1}+1, r_{p-1}+2, \ldots, r_{p-1}+t-1)$ and $\tilde{J}_2 = (r_{p-1}+t, r_{p-1}+t+1, \ldots, r_p)$.

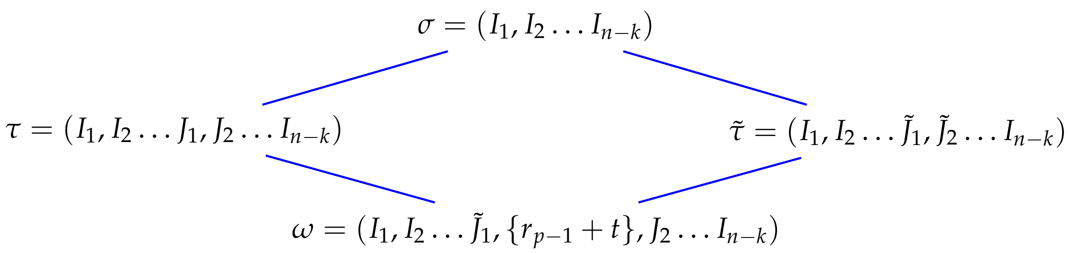


Figure 3.5

By induction hypothesis we know $[\tau : \omega] \cdot [\tilde{\tau} : \omega] = 1$. Since the square of the boundary map vanishes in the cellular chain complex, we have $[\sigma : \tau] \cdot [\sigma : \tilde{\tau}] = -1$. Let $\gamma =$

$(1, I_1 - \{1\}, I_2, \ldots, I_{n-k})$. Then fixing the value of $[\sigma : \gamma]$ fixes the value for every τ whose principal vertex is same as σ. we fix this value to be -1.

Step 2: If $\mathrm{PV}(\sigma) \neq \mathrm{PV}(\tau)$ then it is enough to consider the cells where permutation required to take one to the other is just a transposition. Then $\tau = (I_1, I_2, \ldots, J_1, J_2, \ldots, I_{n-k})$ where $J_1 = (r_{p-1}+1, r_{p-1}+2, \ldots, r_{p-1}+t+1)$, $J_2 = (r_{p-1}+t, r_{p-1}+t+2, \ldots, r_p)$. Let $\tilde{\tau} = (I_1, I_2, \ldots, \tilde{J}_1, \tilde{J}_2, \ldots, I_{n-k})$ where $\tilde{J}_1 = (r_{p-1}+1, r_{p-1}+2, \ldots, r_{p-1}+t-1)$ and $\tilde{J}_2 = (r_{p-1}+t, r_{p-1}+t+1, \ldots, r_p)$.

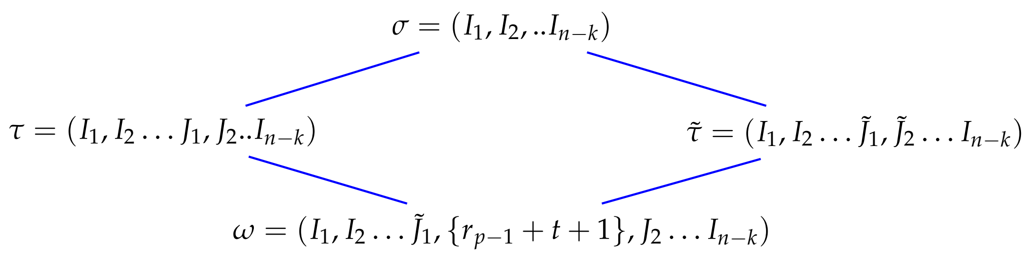


Figure 3.6

By induction hypothesis and step 1, we know the values $[\sigma : \tilde{\tau}], [\tau : \omega], [\tilde{\tau} : \omega]$ and hence we can compute the $[\sigma : \tau]$ by using the fact the square of the boundary map vanishes in cellular chain complex.

This should be equal to the value defined above in definition because we have already showed $\partial^2 = 0$. □

Here are some examples that illustrate the proof above:

Example 3.2. Let $\alpha = (1, 23, 45, 6)$, $\beta = (1, 2, 3, 45, 6)$. The neighbors of $\mathrm{PV}(\alpha)$ are ordered as $v_1 = (1, 3, 2, 4, 5, 6)$ and $v_2 = (1, 2, 3, 5, 4, 6)$. Since $\mathrm{PV}(\alpha) = \mathrm{PV}(\beta)$, $g_{\alpha,\beta} = Id$. The vertex $\mathrm{PV}(\beta)$ has only one neighbor which is $(1, 2, 3, 5, 4, 6)$ and hence the missing vertex is v_1. This shows that $\langle \partial\alpha, \beta \rangle = -1$

Example 3.3. Let $\alpha = (1, 23, 45, 6)$, $\beta_1 = (1, 2, 3, 45, 6)$, $\beta_2 = (1, 23, 4, 5, 6)$ and $\gamma = (1, 2, 3, 4, 5, 6)$. Observe $\gamma < \beta_1, \beta_2$ and $\beta_1, \beta_2 < \alpha$. Clearly $\langle \partial\alpha, \beta_1 \rangle = -1, \langle \partial\alpha, \beta_2 \rangle = 1$ and $\langle \partial\beta_1, \gamma \rangle = -1, \langle \partial\beta_2, \gamma \rangle = -1$. Thus we have $\langle \partial\alpha, \beta_1 \rangle \cdot \langle \partial\beta_1, \gamma \rangle = 1$ and $\langle \partial\alpha, \beta_2 \rangle \cdot \langle \partial\beta_2, \gamma \rangle = -1$ showing that $\langle \partial^2\alpha, \gamma \rangle = 0$.

3.2 A discrete Morse function for the cyclopermutahedron

This section follows closely the construction of a discrete Morse function on CP_{n+1} described by I. Nekrasov, G. Panina and A. Zhukova in [7].

Step 1. We pair together two cells

$$\alpha = (\ldots 1, I \ldots) \text{ and } \beta = (\ldots 1 \cup I \ldots)$$

if $n + 1 \notin I$.

We proceed for all $2 \leq k < n$, assuming that the k-th step is:

Step k. We pair together two cells

$$\alpha = (\ldots k, I \ldots) \text{ and } \beta = (\ldots k \cup I \ldots)$$

if the following holds:

1. α and β were not paired at any of the previous steps.
2. $n+1 \notin I$.
3. $k < I$.

Example 3.4. The cell $(2, 43, 1, 56)$ is paired with the cell $(243, 1, 56)$ on the second step. The cell $(4, 5, 3, 1, 26)$ is paired with the cell $(45, 3, 1, 26)$ on the fourth step. The cell $(4, 3, 2, 1, 56)$ is not paired.

Lemma 3.4. *The above pairing is a discrete Morse function.*

Proof. By construction, no cell can be a part of more than one such pairing. The possible paths are of the form

$$a_0, b_0, a_1, b_1, \ldots, a_p, b_p, a_{p+1},$$

where each pair (a_i, b_i) is of the form described above.
Claim: $a_{p+1} \neq a_0$
If a_{p+1} is to equal a_0, then $\exists s \in [n]$ that must move into the $(n+1)$-set and re-emerge on the left by a splitting of the $(n+1)$-set, but that is forbidden by the rules. □

Lemma 3.5. *The critical cells of the above defined Morse function are exactly all the cells of the following two types:*
Type 1. *Cells labeled by* $(\nabla, \{n+1, \ldots\})$, *where* ∇ *is a string of singletons coming in decreasing order.*
Type 2. *Cells labeled by* $(i, I, \{n+1, \ldots\})$, *where* $i < I$.

Proof. let $\sigma = (I_1, I_2, \ldots, I_k)$ be a critical cell and $n+1 \in I_k$.

1. If $k > 3$,
   - then the block I_1 has to be a singleton, otherwise choose the minimum element, say $m \in I_1$. Then σ is paired to the cell $\tau = (m, I_1 - m, \ldots, I_k)$. The block I_1 and I_2 satisfy the relation $I_1 > I_2$, otherwise
     - σ is paired to the cell $\tau = (I_1 \cup I_2, \ldots, I_k)$ if $I_1 < I_2$ and
     - If $\exists j \in I_2$ such that $j < I_1$, then choose the minimum element of I_2, say m. Then σ is paired to the cell $\tau = (I_1, m, I_2 - m, \ldots, I_k)$.
   - The block I_2 has to be a singleton, otherwise choose the minimum element, say $m' \in I_1$. Then σ is paired to the cell $\tau = (I_1, m, I_2 - m, \ldots, I_k)$. The block I_2 and I_3 satisfy the relation $I_2 > I_3$, otherwise
     - σ is paired to the cell $\tau = (I_1, I_2 \cup I_3, \ldots, I_k)$ if $I_1 < I_2$ and

– If $\exists j \in I_3$ such that $j < I_2$, then choose the minimum element of I_3, say m. Then σ is paired to the cell $\tau = (I_1, I_2, m, I_3 - m, \dots, I_k)$.

Similarly, It can be proved that blocks I_j for $j \neq k$ are singletons arranged in descending order.

2. If $k = 3$, then the block I_1 has to be a singleton, otherwise choose the minimum element, say $m \in I_1$. Then σ is paired to the cell $\tau = (m, I_1 - m, \dots, I_k)$.

   - If $I_1 < I_2$, then this is a critical cell of type 2.
   - If $I_1 > I_2$, then the block I_2 has to be a singleton, otherwise choose the minimum element, say $m^{'} \in I_1$. Then σ is paired to the cell $\tau = (I_1, m, I_2 - m, I_3)$.
   - If $\exists j \in I_2$ such that $j < I_1$, then choose the minimum element of I_2, say m. Then σ is paired to the cell $\tau = (I_1, m, I_2 - m, I_3)$.

□

Lemma 3.6. *There are no critical gradient paths that end at critical cells of type 2.*

Proof. Critical cells of type 2 have the maximal possible dimension. □

Lemma 3.7. *The following three cases describe all the gradient paths between critical cells:*

1. *Let $\beta = (\nabla^{'}, \{n+1, \dots\})$ and $\alpha = (\nabla, \{n+1, \dots\})$ be two cells of type 1. Then there are two gradient paths from β to α iff $\nabla^{'} = \nabla \cup k$ for some k.*

2. *Let $\beta = (\{i\}, \{j, k\}, \{n+1, \dots\})$ and $\alpha = (\{k\}, \{j\}, \{i\}, \{n+1, \dots\})$ be cells of type 2 and 1 respectively. Then there are two gradient paths from β to α.*

3. *Let $\beta = (\{i\}, \{j\}, \{n+1, \dots\})$ and $\alpha = (\nabla, \{n+1, \dots\})$ be cell of type 2 and type 1 respectively. Then there are two gradient paths from β to α iff ∇ consists of three singletons, two of which are $\{i\}$ and $\{j\}$.*

Proof. See [7, Lemma 9] for the proof. □

Lemma 3.8 (The good path lemma)**.** *Let (τ_1, τ_2, σ) be a triple such that $\tau_1 = (X, \{k\}, I, Y)$ and $\tau_2 = (X, I, \{k\}, Y)$ are two $(p-1)$-cells in the boundary of the p-cell $\sigma = (X, \{k\} \cup I, Y)$ in* CP_{n+1}*, where X and Y are sequences of sets. Then*

$$[\tau_1 : \sigma][\tau_2 : \sigma] = -1. \tag{3.2}$$

Proof. Let $I = \{i_1, \dots, i_s\}$ with $i_1 < \dots < i_r < k < i_{r+1} < \dots < i_s$. Then,

$$v_0 = PR(\sigma) = (\Delta_X, i_1, \dots, i_r, k, i_{r+1}, \dots, i_s, \Delta_Y).$$
$$v_{\tau_1} = PR(\tau_1) = (\Delta_X, k, i_1, \dots, i_s, \Delta_Y).$$
$$v_{\tau_2} = PR(\tau_2) = (\Delta_X, i_1, \dots, i_s, k, \Delta_Y).$$

In the above expressions, given any sequence Z of sets, Δ_Z denotes the partition of Z into singletons (such that a sequence of singletons arising from a single set is contiguous and in ascending order). Thus $g_{\sigma,\tau_1} = (k, i_1, \ldots, i_r)$ and $g_{\sigma,\tau_2} = (k, i_s, \ldots, i_{r+1})$. Now, if $X = A_1 A_2 \ldots A_a$, then denote by $\|X\|$ the quantity $\sum_{i=1}^{a}(\|A_i\| - 1)$. Then $i_{\tau_1} = \|X\| + 1$ and $i_{\tau_2} = \|X\| + s$. Now we obtain

$$\begin{aligned}
[\tau_1 : \sigma] &= \operatorname{sign}(g_{\sigma,\tau_1}) \cdot (-1)^{p+i_{\tau_1}} \\
&= (-1)^r \cdot (-1)^{p+\|X\|+1} \\
&= (-1)^{p+\|X\|+r+1}, \\
[\tau_2 : \sigma] &= \operatorname{sign}(g_{\sigma,\tau_2}) \cdot (-1)^{p+i_{\tau_2}} \\
&= (-1)^{s-r} \cdot (-1)^{p+\|X\|+s} \\
&= (-1)^{p+\|X\|-r}
\end{aligned}$$

and the result follows. □

Definition 3.4. Suppose we have a path $b_0, a_1, b_1, \ldots, a_t, b_t, a_{t+1}$, where each triple (a_i, a_{i+1}, b_i) is as above for $1 \le i \le t$. We call such a path a *good path*.

The following lemma follows directly from the proof of Lemma 3.7.

Lemma 3.9. *The gradient paths between critical cells in Lemma 3.7 are good paths.*

The above results lead to a rather simple proof for the vanishing of boundary maps in the Morse complex, which clearly mean that the homology groups of CP_{n+1} are torsion free and the Betti numbers are exactly equal to number of critical cells.

Theorem 3.10. *The boundary operators of the Morse complex vanish.*

Proof. From Section 1.3.3, recall that

$$\langle \tilde{\partial}\sigma, \tau \rangle = \sum_{\tilde{\sigma}_p < \sigma} \langle \partial\sigma, \tilde{\sigma} \rangle \sum_{c \in \Gamma(\tilde{\sigma},\tau)} w(c)$$

Where, $\Gamma(\tilde{\sigma}, \tau)$ denote the set of all gradient paths from $\tilde{\sigma}$ to τ.

By Lemma 3.9 the paths between critical cells are good paths. Hence $\forall \tilde{\sigma} < \sigma$, $\forall c \in \Gamma(\tilde{\sigma}, \tau)$, we have $w(c) = 1$. Let us denote these two paths as $C := \sigma, a_1, b_1, \ldots, a_t, b_t, \tau$ and $D = \sigma, \alpha_1, \beta_1, \ldots, \alpha_t, \beta_t, \tau$. Then $\langle \tilde{\partial}\sigma, \tau \rangle = \langle \partial\sigma, a_1 \rangle + \langle \partial\sigma, \alpha_1 \rangle$. Since the triple (a_1, σ, α_1) also forms a good pair, $[\sigma : a_1] \cdot [\sigma : \alpha_1] = -1$ implying $\tilde{\partial} = 0$. □

Corollary 3.11. *The homology of CP_{n+1} is torsion free and the Betti numbers are given as follows*

$$b_i = \begin{cases} 2^n + \frac{2^n - 3n - 2}{2}, & i = n-2; \\ \binom{n}{i}, & 0 \le i < n-2. \end{cases}$$

3.3 Combinatorics of cyclopermutohedron

In the remaining chapter we study combinatorial properties like Whitney numbers of the first and the second kind, Möbius function etc.

3.3.1 Whitney numbers of the Second kind

Denote by $T(n,k)$ the number of cyclically ordered partitions of n into k blocks. In cyclopermutoherdron CP_{n+1}, the Whitney numbers of the second kind W_k is equal to $T(n,k)$. Clearly $T(n,k)$ is $(k-1)!S(n,k)$, where $S(n,k)$ is a Stirling number of the second kind.

Claim 3.12. $T(n,k) = (k-1)T(n-1,k-1) + kT(n-1,k), T(1,1) = 1$

Proof. Use the recursive relation mentioned in Definition 1.5.

$$\begin{aligned}
S(n+1,k) &= S(n-1,k-1) + kS(n-1,k) \\
(k-1)!S(n+1,k) &= (k-1)!S(n-1,k-1) + k(k-1)!S(n-1,k) \\
T(n,k) &= (k-1)T(n-1,k-1) + kT(n-1,k)
\end{aligned}$$

□

Definition 3.5 (Special Falling factorial)**.** Define $\langle x\rangle_k = \frac{x(x-1)(x-2)\ldots(x-k+1)}{(k-1)!}$.

The set $\mathbb{F}[x]$ of all polynomials in the indeterminate x with coefficients in the field $\mathbb{F}$ forms a vector space over $\mathbb{F}$. The sets $B_1 = \{1, x, x^2, \ldots\}$ and $B_2 = \{1, \langle x\rangle_1, \langle x\rangle_2, \ldots\}$ are both bases for $\mathbb{F}[x]$. Then the following proposition asserts that the (infinite) matrix $\mathcal{T} := [T(n,k)]_{k,n\in\mathbb{N}}$ is the transition matrix between the basis B_2 and the basis B_1.

Proposition 3.13. *With the notation as above we have* $x^n = \sum_{k=0}^{n} T(n,k)\langle x\rangle_k$.

Proof. Let $M(n,k)$ be the transition matrix between the basis B_2 and the basis B_1. We will prove that the entries of this matrix satisfy the same recursive relation as $T(n,k)$'s. We do this by inducting on n. When $n = 1, k = 1$, $\langle x\rangle_1 = x$ and so $M(1,1) = 1 = T(1,1)$

$$
\begin{aligned}
x^n &= x \cdot x^{n-1} \\
&= x \cdot \sum_{k=0}^{n-1} M(n-1,k)\langle x \rangle_k \\
&= \sum_{k=0}^{n-1} M(n,k)x \cdot \langle x \rangle_k \\
&= \sum_{k=0}^{n-1} M(n,k)(x-k+k) \cdot \langle x \rangle_k \\
&= \sum_{k=0}^{n-1} M(n,k)(x-k) \cdot \langle x \rangle_k + x^n \sum_{k=0}^{n} M(n,k)k \cdot (x)_k \\
&= \sum_{k=0}^{n-1} M(n,k)k \cdot \langle x \rangle_{k+1} + x^n \sum_{k=0}^{n} M(n,k)k \cdot \langle x \rangle_k \\
&= \sum_{k=1}^{n} M(n,k)(x-k) \cdot \langle x \rangle_k + x^n \sum_{k=0}^{n} M(n,k)k \cdot \langle x \rangle_k \\
\sum_{k=0}^{n} M(n,k)\rangle x \rangle_k &= \sum_{k=1}^{n} M(n,k)(x-k) \cdot \langle x \rangle_k + x^n \sum_{k=0}^{n} M(n,k)k \cdot \langle x \rangle_k
\end{aligned}
$$

by comparing the coefficients we get the required recurrence relation same as $T(n,k)$. □

3.3.2 Whitney numbers of the first kind

Let C be a $(k+1)$ cell represented by a cyclically ordered partition $I = (I_1, I_2, \ldots, I_{n-k})$ and $|I_j| = a_j$. All the cells of the k-skeleton that should be incident to C form a subcomplex of the k-skeleton which is combinatorially isomorphic to the boundary complex of the Cartesian product of permutohedra $\partial(\Pi_{a_1} \times \cdots \times \Pi_{a_k})$. Obviously, it is a k-dimensional sphere, and there is a unique way to attach the boundary C to the sphere. Thus, we get $\mu(0,C) = (-1)^k$.

Proposition 3.14. *The following expression holds:*

$$w_k = T(n+1, n+1-k)(-1)^{k-1}. \tag{3.3}$$

Proof. Let C be a k-cell represented by a cyclically ordered partition $I = (I_1, I_2, \ldots, I_{n+1-k})$, then by above $\mu(\hat{0}, C) = (-1)^{n+1-k}$. Since the möbius function depends only on the rank, it is sufficient to compute the number of k-cells. This is given by $T(n+1, n+1-k)$. □

Recall, $\chi(\mathrm{CP}_{n+1}) = (-1)^n(2^n - 2)$, so $\widetilde{\chi}(\mathrm{CP}_{n+1}) = (-1)^n(2^n-2) - 1$

Corollary 3.15. *The following expression holds:*

$$\sum_{k=1}^{n+1} T(n+1,k)(-1)^k = 0$$

Proof.

$$\mu(\hat{0},\hat{1}) = -\sum_{k=3}^{n+1} T(n+1,k)(-1)^{n-k} - 1$$
$$= (-1)^{n+1}\sum_{k=1}^{n+1} T(n+1,k)(-1)^k + (-1)^{n+1}T(n+1,1) + (-1)^n T(n+1,2) - 1$$
$$= (-1)^{n+1}\sum_{k=1}^{n+1} T(n+1,k)(-1)^k + (-1)^{n+1} + (-1)^n(2^n-1) - 1$$
$$(-1)^n(2^n-2) - 1 = (-1)^{n+1}\sum_{k=1}^{n+1} T(n+1,k)(-1)^k - 1 + (-1)^n(2^n-2)$$
$$\text{thus it follows } (-1)^{n+1}\sum_{k=1}^{n+1} T(n+1,k)(-1)^k = 0$$

□

3.3.3 Intervals in the poset

Given a regular cell complex $\mathcal{K}$ and a p-cell $\sigma \in \mathcal{K}$, the proper part of $[\hat{0},\sigma]$ is homeomorphic to a $(p-1)$-sphere. More generally by a theorem of Björner in [3] all intervals in the face poset of a regular cell complex are shellable. The following theorem shows that for cyclopermutohedra the maximal intervals are in fact L-shellable.

Theorem 3.16. *Maximal intervals in* CP_{n+1} *are L-shellable.*

Proof. It is sufficient to consider intervals of the form $[(1,2,3,\dots,n+1),(I_1,I_2,I_3)]$ because any other interval can be obtained from this interval by applying suitable permutation. Moreover, for calculation purposes we'll reverse the order and show that the dual intervals $[(I_1,I_2,I_3),(1,2,3,\dots,n+1)]$ are shellable.

Now, define an edge-labelling λ inductively in the following manner.

Step 0:

Let $\alpha = (I_1,I_2,I_3)$, $\beta = (J_1,J_2,J_3,J_4)$ and $\alpha \prec \beta$. Also, denote $|I_i| = n_i$, $\forall i = 1,2,3$. If $J_1 = (1,2,\dots,m)$, $J_2 = (m+1,\dots,n_1)$ and $J_1 \cup J_2 = I_1$, then define $\lambda(\alpha,\beta) = m$. If $J_2 = (n_1+1,n_1+2,\dots,n_1+l)$, $J_3 = (n_1+l+1,\dots,n_2)$ and $J_2 \cup J_3 = I_2$, then define $\lambda(\alpha,\beta) = l+(n_1-1)$.

Similarly, if $J_t = (n_{t-1}+1,n_{t-1}+2,\dots,n_{t-1}+l)$, $J_{t+1} = (n_{t-1}+l+1,\dots,n_t)$ and $J_t \cup J_{t+1} = I_t$, then define $\lambda(\alpha,\beta) = l+(n_1-1)+(n_2-1)+\dots+(n_{t-1}-1)$.

Step 1:

Let $\alpha = (I_1,I_2,I_3,I_4)$, $\beta = (J_1,J_2,J_3,J_4,J_5)$ and $\alpha \prec \beta$. Also, let $|I_i| = n_i, \forall i = 1,2,3,4$. If $J_1 = (1,2,\dots,m)$, $J_2 = (m+1,\dots,n_1)$ and $J_1 \cup J_2 = I_1$, then define $\lambda(\alpha,\beta) = m$. If $J_2 = (n_1+1,n_1+2,\dots,n_1+l)$, $J_3 = (n_1+l+1,\dots,n_2)$ and $J_2 \cup J_3 = I_2$, then define $\lambda(\alpha,\beta) = l+(n_1-1)$.

Similarly, if $J_t = (n_{t-1}+1,n_{t-1}+2,\dots,n_{t-1}+l)$, $J_{t+1} = (n_{t-1}+l+1,\dots,n_t)$ and $J_t \cup J_{t+1} = I_t$, then define $\lambda(\alpha,\beta) = l+(n_1-1)+(n_2-1)+\dots+(n_{t-1}-1)$.

Step k**:** $(k \geq 2)$
Let $\alpha = (I_1, I_2, I_3, \ldots, I_{k+3})$, $\beta = (J_1, J_2, J_3, J_4, \ldots, J_{k+4})$ and $\alpha \prec \beta$. Also, denote $|I_i| = n_i, \forall i = 1, 2, \ldots, k+3$. If $J_1 = (1, 2, \ldots, m), J_2 = (m+1, \ldots, n_1)$ and $J_1 \cup J_2 = I_1$, then define $\lambda(\alpha, \beta) = m$. If $J_2 = (n_1+1, n_1+2, \ldots, n_1+l)$, $J_3 = (n_1+l+1, \ldots, n_2)$ and $J_2 \cup J_3 = I_2$, then define $\lambda(\alpha, \beta) = l + (n_1 - 1)$.
Similarly, if $J_t = (n_{t-1}+1, n_{t-1}+2, \ldots n_{t-1}+l)$, $J_{t+1} = (n_{t-1}+l+1, \ldots, n_t)$ and $J_t \cup J_{t+1} = I_t$, then define $\lambda(\alpha, \beta) = l + (n_1 - 1) + (n_2 - 1) + \cdots + (n_{t-1} - 1)$.

Denote by M the chain $(I_1, I_1, I_3) \prec (1, I_1 - \{1\}, I_2, I_3) \prec (1, 2, I_1 - \{1, 2\}, I_2, I_3) \cdots \prec (1, 2, \ldots, n_1, I_2, I_3) \prec (1, 2, \ldots, n_1, n_1+1, I_2 - \{n_1+1\}, I_3) \prec \cdots \prec (1, 2, \ldots, n+1)$. Clearly, $\lambda(a, b) = 1 \ \forall a, b \in M$.

Let $\alpha = (I_1, I_2, I_3)$, $\beta = (J_1, J_2, J_3, J_4)$ and $\alpha \prec \beta$. Also, denote $|I_i| = n_i, \forall i = 1, 2, 3$. If $J_1 = (1, 2, \ldots, m)$, $J_2 = (m+1, \ldots, n_1)$ and $J_1 \cup J_2 = I_1$, then

$$\lambda(\alpha, \beta) = 1 \iff \beta = (1, I_1 - \{1\}, I_2, I_3). \tag{3.4}$$

Thus M is the unique unrefinable rising chain. □

Example 3.5. Consider a maximal chain, $\alpha_0 = (123, 456, 78) \prec (1, 23, 456, 78) \prec (1, 2, 3, 456, 78) \prec (1, 2, \ldots, 4, 56, 78) \prec \cdots \prec (1, 2, \ldots, 7, 8) = \alpha_6$. Then, $\lambda(\alpha_0, \alpha_1) = \lambda(\alpha_1, \alpha_2) = \cdots = \lambda(\alpha_5, \alpha_6) = 1$.

Example 3.6. Consider another maximal chain, $\alpha_0 = (123, 456, 78) \prec (12, 3, 456, 78) \prec (1, 2, 3, 456, 78) \prec (1, .., 45, 6, 78) \prec \cdots \prec (1, 2, \ldots, 7, 8) = \alpha_6$. Then, $\lambda(\alpha_0, \alpha_1) = 2$, $\lambda(\alpha_1, \alpha_2) = 1$, $\lambda(\alpha_2, \alpha_3) = 2$, $\lambda(\alpha_3, \alpha_4) = \lambda(\alpha_4, \alpha_5) = \lambda(\alpha_5, \alpha_6) = 1$.

Theorem 3.17. *Let* $T = \{x \in \mathrm{CP}_{n+1} : x > (1, 2, 3, \ldots, n+1)\}$. *Then we have the following homotopy equivalence*

$$\Delta(T) \simeq \bigvee_{\frac{n(n-1)}{2}} S^{n-3}. \tag{3.5}$$

Proof. If $\alpha = (I_1, I_2, \ldots, I_k) \in T$, then $I_1 < I_2 < \cdots < I_k$. From Section 3.2, recall the pairing defined on CP_{n+1}.

Claim: If α and β are paired in CP_{n+1}, then $\alpha \in T \iff \beta \in T$
Let $\alpha = (\ldots, I, \{i\}, J, \ldots)$ be paired with $\beta = (\ldots, I, \{i\} \cup J, \ldots)$, $K = \{i\} \cup J$. If $\alpha \in T$, then $I < i < J \implies I < K$. Thus $\beta \in T$. Similarly, the converse holds.

So, the critical cells of f restricted to T are of type 2 with one exception $\alpha = (12, 3, 4, \ldots, n+1)$ which was earlier paired to $(1, 2, 3, \ldots, n+1)$ in CP_{n+1}. Thus by Corollary 1.10, the $\Delta(T)$ is homotopy equivalent to a CW-complex with one 0-cell and some $(n-3)$-cells.

The critical cells of type-2 contained in T will be of the form:

1. $(1, \{2, 3, \ldots, i\}, \{n+1, n, \ldots, i+1\})$,

2. $(2, \{3, 4, \ldots, i\}, \{1, n+1, \ldots, i+1\})$
   ⋮

k. $(k, \{k+1, k+2, \ldots, i\}, \{1, 2, \ldots, k-1, n+1, \ldots, i+1\})$

Thus, there are $(n-1)+(n-2)+\cdots+1=\frac{n(n-1)}{2}$ copies of $(n-3)$-cells proving the homotopy type of $\Delta(T)$ is $\bigvee_{\frac{n(n-1)}{2}} S^{n-3}$. □

Chapter 4

The Bicyclopermutohedron

In this Chapter we first construct a certain quotient of CP_{n+1} then define a discrete Morse function on it and use it compute the mod-2 homology.

Recall that for a generic length vector $L := (l_1, l_2, \ldots, l_{n+1}) \in \mathbb{R}_+^{n+1}$ the moduli space $\mathscr{M}_L$ of planar polygons admits a natural free $\mathbb{Z}_2$ action; wherein each polygon is mapped to its reflection about the X-axis. The quotient space $\mathscr{M}_L/\mathbb{Z}_2$, denoted $\bar{\mathscr{M}}_L$, is the space of polygons viewed up to the action of all isometries.

Consider a $\mathbb{Z}_2$ action on CP_{n+1} given by the involution.

$$\begin{aligned} r : \mathrm{CP}_{n+1} &\longrightarrow \mathrm{CP}_{n+1} \\ (I_1, I_2, \ldots, I_{k-1}, I_k) &\mapsto (I_{k-1}, \ldots, I_2, I_1, I_k). \end{aligned} \tag{4.1}$$

Essentially the action identifies cyclically ordered partitions that are obtained by cyclically permuting blocks in either direction. Clearly this action is fixed point free and we have the quotient $\mathrm{CP}_{n+1}/\mathbb{Z}_2$ which we name the *bi-cyclopermutohedron* and denote it by QP_{n+1}. See Figure 4.1 for an example when $n = 3$.

Note that the involution defined in Equation (4.1) mimics the above reflection. Moreover the complex QP_{n+1} is the "universal object" for the moduli space $\bar{\mathscr{M}}_L$ in the same sense as CP_{n+1} is for $\mathscr{M}_L$.

Definition 4.1. The regular CW complex bi-cyclopermutohedron QP_{n+1} is defined as:

- For $k = 0, 1, \ldots, n-2$, the k-cells of QP_{n+1} are labeled by (all possible) bi-cyclically ordered partitions of the set $[n+1]$ into $(n-k+1)$ non-empty parts.
- A closed cell $\bar{F}$ contains a cell F' whenever the label of F' refines that of $\bar{F}$.

We begin by introducing some notions that are useful when dealing with equivalence classes of bi-cyclically ordered partitions. The aim is to show how to choose a nice representative for these equivalence classes. These ideas were originally introduced by Adhikari in his Masters' thesis [1].

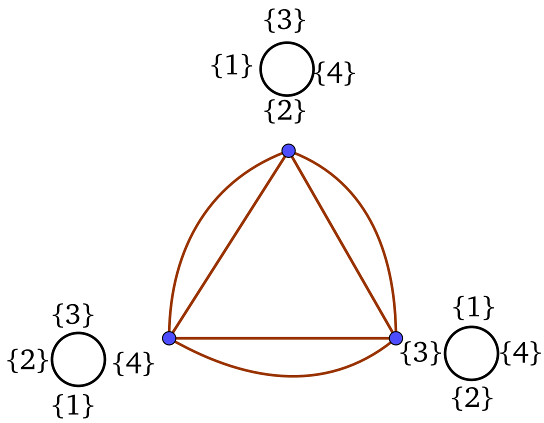


Figure 4.1: The complex QP_4

Definition 4.2. Let $\lambda = (I_1, I_2, \ldots, I_k)$ be a cyclically ordered partition of $[n+1]$. Let j be the greatest element outside the $n+1$ set and $j \in I_l$ for $1 \leq l < k$. Further, let i be the greatest element outside $n+1$ set and I_l and $i \in I_m$ for $1 \leq m < k$ and $m \neq l$. Then λ is said to be of class (i, j) if $m \leq l$ and of class (j, i) otherwise. The class of λ is denoted $\mathrm{cl}(\lambda)$.

Definition 4.3. Let λ be a cell of $\mathrm{cl}(i, j)$. λ is called an *ascending cell* if $i < j$ and *descending* otherwise.

Lemma 4.1. *The cells of* CP_{n+1} *can be partitioned into two classes: one with ascending cells and the other with descending cells. Involution defined in Equation* (4.1) *establishes bijection between these two classes.*

$$\{\textit{Ascending Cells}\} \longleftrightarrow \{\textit{Descending Cells}\}$$

Therefore, each cell in the quotient complex QP_{n+1} is an equivalence class containing an ascending cell and a descending cell(each the reflection of the other). So it does not make sense to talk about ascending and descending cell in QP_{n+1}. Henceforth, unless otherwise mentioned, all the cells will be considered ascending.

Definition 4.4. A cell $\bar{\lambda}$ is said to be of *class* $\{i, j\}$ if one of the preimages under the quotient map is of class (i, j). We denote the class by $\mathrm{cl}(\bar{\lambda})$.

Clearly the class $\{i, j\}$ is equal to class $\{j, i\}$. Now, we define a sense of hierarchy on the cells of QP_{n+1}.

Definition 4.5. Let $\bar{\lambda}$ and $\bar{\lambda}'$ be two cells of class $\{i, j\}$ and $\{i', j'\}$ respectively. If $\min\{i, j\} \leq \min\{i', j'\}$ and $\max\{i, j\} \leq \max\{i', j'\}$, then $\bar{\lambda}'$ is said to be *higher* than $\bar{\lambda}$.

Lemma 4.2. *If* $\bar{\alpha}, \bar{\beta} \in \mathrm{QP}_{n+1}$ *and* $\bar{\alpha}$ *is contained in the boundary of* $\bar{\beta}$*, then* $\bar{\alpha}$ *is higher than* $\bar{\beta}$.

Proof. Let $\text{cl}(\bar{\beta}) = \{i, j\}$ and $i < j$. Since $\bar{\alpha}$ is in the boundary of $\bar{\beta}$, each block of $\bar{\alpha}$ is a subset of a block of $\bar{\beta}$. Therefore, the largest element in $\bar{\alpha}$ outside the $n+1$ set (say j') has to be greater than or equal to j, *i.e.*, $j' \geq j$. Similarly, the second largest element outside the $n+1$ set and the set containing j' has to be greater than or equal to i. □

4.1 A discrete Morse function and mod-2 homology

Here, we first define a discrete Morse function on QP_{n+1} inductively.
Step 1: Pair $\alpha = (\ldots, 1, I, \ldots)$ and $\beta = (\ldots, 1 \cup I, \ldots)$ in CP_{n+1} if the following conditions hold:

1. $n+1 \notin I$.
2. α is ascending.
3. $\text{cl}(\alpha) = \text{cl}(\beta)$.

Note that, the conditions 2 and 3 together imply that β also is an ascending cell.
Step k: Pair $\alpha = (\ldots, k, I, \ldots)$ and $\beta = (\ldots, k \cup I, \ldots)$ if the following conditions hold.

1. $n+1 \notin I$.
2. α and β have not yet been paired.
3. α is ascending.
4. $\text{cl}(\alpha) = \text{cl}(\beta)$

After the $(n-2)^{\text{nd}}$ step, we have-
The final step: If α and β have been paired in CP_{n+1}, then match $\bar{\alpha}$ with $\bar{\beta}$ in QP_{n+1} (here $\bar{\alpha}$ with $\bar{\beta}$ represents the image of α and β under the map $\pi : CP_{n+1} \to \text{QP}_{n+1}$).
Following is a straightforward observation about the above matching.

Lemma 4.3. *If there is a gradient path*

$$\bar{\beta}_0, \bar{\alpha}_1, \bar{\beta}_1, \ldots, \bar{\alpha}_p,$$

then $\bar{\alpha_p}$ is higher than $\bar{\beta}_0$.

Proof. Since we only match the cells in the same class, we have $\text{cl}(\bar{\alpha}_i) = \text{cl}(\bar{\beta}_i)$ for each $i \in \{1, 2, \ldots, p-1\}$. Moreover, using Lemma 4.2, we get that $\text{cl}(\bar{\alpha}_j) \geq \text{cl}(\bar{\beta}_{j-1})$ for each $j \in \{1, \ldots, p\}$. Thus, the result follows. □

Theorem 4.4. *The pairing on QP_{n+1}, as described above is a discrete Morse function.*

Proof. On the contrary, assume that the matching defined is not acyclic, *i.e.* there is a path

$$\bar{\alpha}_0, \bar{\beta}_0, \bar{\alpha}_1, \bar{\beta}_1, \ldots, \bar{\alpha}_p$$

with $p > 1$ and $\bar{\alpha}_0 = \bar{\alpha}_p$. Since $\bar{\alpha}_0$ and $\bar{\beta}_0$ are matched, they are in the same class, Therefore, using Lemma 4.3, we get that $\mathrm{cl}(\bar{\alpha}_0) = \mathrm{cl}(\bar{\beta}_0) = \mathrm{cl}(\bar{\alpha}_1) = \ldots = \mathrm{cl}(\bar{\alpha}_p)$.

We now "lift" this cycle to CP_{n+1}. Let a_0 be the ascending cell such that $\pi(a_0) = \bar{\alpha_0}$. Let b_0 the cell with which a_0 is paired (in particular, $\pi(b_0) = \bar{\beta_0}$). Next, suppose a_1 is ascending with $\pi(a_1) = \bar{\alpha_1}$. Note that $cl(a_0) = cl(b_0) = cl(a_1) = (i, j)$ for some $i < j$. If a_1 is not in the boundary of b_0, then it must be in the boundary of $r(b_0)$ (for otherwise a_1 would not be in the boundary of b_0). But since $cl(r(b_0)) = (j, i)$, we have a cell of class (i, j) in the boundary of a cell of class (j, i), which is impossible. Hence a_1 is in the boundary of b_0. Continuing thus, we obtain a path $a_0, b_0, a_1, b_1, \ldots, a_p$ with a_i and b_i ascending for each i (and, in particular, $a_0 = a_p$). Thus the cycle in QP_{n+1} lifts to the cycle in CP_{n+1}. The matching on the ascending cells is, however, a subset of the matching of CP_{n+1} described in Section 3.2, and hence the cycle cannot exist. □

Notation: Let λ denote the unique ascending representative of $\bar{\lambda} \in \mathrm{QP}_{n+1}$.

Theorem 4.5. *The critical cells of the discrete Morse function on* QP_{n+1} *are the images under* π *of the cells of the type (i,I,∇,N) with* $\nabla < i < I$.

Proof. Assume $\bar{\lambda} = (I_1, I_2, \ldots, I_k)$ is critical and $\mathrm{cl}(\bar{\lambda})=\{i, j\}$.

Claim 1: $i \in I_1$.

Proof. Assume, without loss of generality that $i \in I_2$.

1. If $|I_1| = 1$, then by construction $I_1 < i \leq I_2$. Hence, the cell $\lambda = (I_1, I_2, \ldots, I_k)$ can be matched with $\lambda' = (I_1 \cup I_2, \ldots, I_k)$ as they have the same class type.

2. Let $|I_1| > 1$ and denote the minimum of I_1 by m. Then the cells $\lambda = (I_1, I_2, \ldots, I_k)$ and $\lambda' = (m, I_1 - \{m\}, I_2, \ldots, I_k)$ can be matched as they have the same class type.

Claim 2: $I_1 = \{i\}$

Proof. Assume on the contrary that $|I_1| > 1$. Denote the minimum of I_1 be m. Then the cells λ and $\lambda' = (m, I_1 - \{m\}, \ldots, I_k)$ can be matched.

Claim 3: $j \in I_2$.

Proof. Assume without loss of generality that $j \in I_3$.

1. $|I_2| = 1$.

   (a) If $I_2 < I_3$, then the cell $\lambda = (\{i\}, I_2, \ldots, I_k)$ and $\lambda' = (\{i\}, I_2 \cup I_3, \ldots, I_k)$ can be matched as they have the same class type.

   (b) If $I_2 \not< I_3$, then $\exists m \in I_3$ such that $m < I_2$ The cell $\lambda = (\{i\}, I_2, \ldots, I_k)$ and $\lambda' = (\{i\}, I_2, m, I_3 - \{m\}, \ldots, I_k)$ can be matched as they have the same class type.

2. Let $|I_2| > 1$ and denote the minimum of I_2 by m. Then the cells $\lambda = (\{i\}, I_2, \ldots, I_k)$ and $\lambda' = (\{i\}, m, I_2 - \{m\}, \ldots, I_k)$ can be matched as they have the same class type.

Claim 4: $i < I_2$.

Proof. If $\exists m \in I_2$ such that $m < i$ then Then the cells $\lambda = (\{i\}, I_2, \ldots, I_k)$ and $\lambda' = (\{i\}, m, I_2 - \{m\}, \ldots, I_k)$ can be matched as they have the same class type.

A similar argument shows that all other subsets $I_3, \dots, I_{k-1}$ are singletons arranged in decreasing order. $\square$

Proposition 4.6. *Let $\bar{\alpha} = (i, I, \nabla, N)$ and $\bar{\beta} = (j, J, \nabla', N')$. If there is a path from $\bar{\alpha}$ to $\bar{\beta}$ then either $N \cap J = \varnothing$ or $I \cap J = \varnothing$.*

Proof. let $x \in N \cap J$ and $t \in I \cap J$. Clearly $x > j \geq i$ and $t > j \geq i$. Denote the maximum element of I by m and $m > j$. Let the path from $\bar{\alpha}$ to $\bar{\beta}$ be

$$\bar{\alpha} = \bar{\alpha}_0, \bar{\beta}_0, \bar{\alpha}_1, \bar{\beta}_1, \dots, \bar{\alpha}_p, \bar{\beta}_p = \bar{\beta}$$

During the course of the path, x leaves the set N, say at $\bar{\beta}_k$. Then min $\mathrm{cl}(\bar{\beta}_k) \geq \min\{x, m\} \geq \min\{x, t\} > j = \min \mathrm{cl}(\bar{\beta})$. This contradicts the fact that the class increases along a gradient path. $\square$

The following theorem about the paths between critical cells is crucial in computing the homology of QP_{n+1}.

Theorem 4.7. *Let $\bar{\alpha} = (i, I, \nabla, N)$ and $\bar{\beta} = (j, J, \nabla', N')$ be two critical cells. If there is a path from $\bar{\alpha}$ to $\bar{\beta}$, then $\bar{\beta}$ takes exactly one of the following form*

1. (a) *If $J = I$ and $N' = N - t$. Then*

$$\bar{\beta} = \begin{cases} (t, I, i, \nabla, N'), & \text{if } i < t < I, \\ (i, I, \nabla \cup t, N'), & \text{if } t < i. \end{cases}$$

   (b) *If $N' = N - t$ and $|I| = 1$. Then*

$$\bar{\beta} = \begin{cases} (I, t, i, \nabla, N'), & \text{if } t > I, \\ (t, I, i, \nabla, N'), & \text{if } i < t < I, \\ (i, I, t \cup \nabla, N'), & \text{if } t < i. \end{cases}$$

2. *If $N = N'$ and $J = I - j$. Then*

$$\bar{\beta} = (j, J, i, \nabla, N).$$

Proof. We will prove this explicitly *i.e.,* by following the paths from $\bar{\alpha}$ to $\bar{\beta}$. Let ∇ be $\{a_1\}, \{a_2\}, \dots, \{a_l\}$ with $a_1 > a_2 > \cdots > a_l$

1. (a) i. For $\bar{\alpha} = (i, I, \nabla, N)$, $\beta = (t, I, i, \nabla, N')$ and $i < t < I$, we've the paths

$$\begin{gathered}
(i, I, a_1, a_2, \ldots, a_l, N) \\
(i, I, a_1, a_2, \ldots, a_l, t, N-t) \\
(t, a_l, \ldots, a_1, I, i, N-t) \\
(t, a_l, \ldots, a_1 \cup I, i, N-t) \\
(t, a_l, \ldots, I, a_1, i, N-t) \\
\vdots \\
(t, I, a_l, \ldots, a_1, i, N-t) \\
(t, I, a_l, \ldots, \{a_1, i\}, N-t) \\
(t, I, a_l, \ldots, i, a_1, N-t) \\
\vdots \\
(t, I, i, a_l, \ldots, a_1, N-t) \\
(t, I, i, a_l, \ldots, \{a_2, a_1\}, N-t) \\
(t, I, i, a_l, \ldots, a_0, a_1, N-t) \\
\vdots \\
(t, I, i, a_1, a_2, \ldots, a_l, N-t)
\end{gathered} \tag{4.2}$$

and

$$\begin{gathered}
(i, I, a_1, a_2, \ldots, a_l, N) \\
(i, I, a_1, a_2, \ldots, a_l, N-t, t) \\
(t, i \cup I, a_1, a_2, \ldots, a_l, N-t) \\
(t, I, i, a_1, a_2, \ldots, a_l, N-t).
\end{gathered} \tag{4.3}$$

ii. For $\bar{\alpha} = (i, I, \nabla, N)$, $\beta = (i, I, \nabla \cup t, N')$ and $a_p > t > a_{p+1}$, we've the paths

$$\begin{gathered}
(i, I, a_1, a_2, \ldots, a_l, N) \\
(i, I, a_1, a_2, \ldots, a_l, t, N-t) \\
(i, I, a_1, a_2, \ldots, \{a_l, t\}, N-t) \\
(i, I, a_1, a_2, \ldots, t, a_l, N-t) \\
(i, I, a_1, a_2, \ldots, \{a_{l-1}, t\}, a_l, N-t) \\
\vdots \\
(i, I, a_1, a_2, \ldots, \{a_{p+1}, t\}, \ldots, a_l, N-t) \\
(i, I, a_1, a_2, \ldots, t, a_{p+1}, \ldots, a_l, N-t)
\end{gathered} \tag{4.4}$$

and

$$\begin{gathered}(i, I, a_1, a_2, \dots, a_l, N) \\ (i, I, a_1, a_2, \dots, a_l, N-t, t) \\ (\{t, i\}, I, a_1, a_2, \dots, a_l, N-t) \\ (i, t, I, a_1, a_2, \dots, a_l, N-t) \\ (i, t \cup I, a_1, a_2, \dots, a_l, N-t) \\ (i, I, t, a_1, a_2, \dots, a_l, N-t) \\ (i, I, \{t, a_1\}, a_2, \dots, a_l, N-t) \\ (i, I, a_1, t, a_2, \dots, a_l, N-t) \\ \vdots \\ (i, I, a_1, a_2, \dots, \{t, a_p\}, \dots, a_l, N-t) \\ (i, I, a_1, a_2, \dots, a_p, t, \dots, a_l, N-t).\end{gathered} \tag{4.5}$$

(b) For $\bar{\alpha} = (i, I, \nabla, N)$, $\beta = (I, t, i, \nabla, N')$ and $t > I$, we've the paths

$$\begin{gathered}(i, I, a_1, a_2, \dots, a_l, N) \\ (i, I, a_1, a_2, \dots, a_l, N-t, t) \\ (t, i, I, a_1, a_2, \dots, a_l, N-t) \\ (a_l, a_{l-1}, \dots, a_1, I, i, t, N-t) \\ (a_l, a_{l-1}, \dots, a_1, I, i \cup t, N-t) \\ (a_l, a_{l-1}, \dots, a_1, I, t, i, N-t) \\ (a_l, a_{l-1}, \dots, a_1 \cup I, t, i, N-t) \\ (a_l, a_{l-1}, \dots, I, a_1, t, i, N-t) \\ \vdots \\ (I, a_l, a_{l-1}, \dots, a_1, t, i, N-t) \\ (I, a_l, a_{l-1}, \dots, \{a_2, a_1\}, t, i, N-t) \\ (I, a_l, a_{l-1}, \dots, a_1, a_2 t, i, N-t) \\ \vdots \\ (t, I, i, a_1, a_2, \dots, a_l, N-t)\end{gathered} \tag{4.6}$$

and

$$
\begin{array}{c}
(i, I, a_1, a_2, \dots, a_l, N) \\
(i, I, a_1, a_2, \dots, a_l, t, N - \{t\}) \\
(i \cup I, a_1, a_2, \dots, a_l, t, N - \{t\}) \\
(I, i, a_1, a_2, \dots, a_l, t, N - \{t\}) \\
\vdots \\
(I, t, i, a_1, a_2, \dots, a_l, N - \{t\}).
\end{array} \tag{4.7}
$$

The proofs for the other two cases are similar to (a).

2. For $\bar{\alpha} = (i, I, \nabla, N)$, $\beta = (j, I - \{j\}, i, \nabla, N)$ and $i < j$, we've the paths

$$
\begin{array}{c}
(i, I, a_1, a_2, \dots, a_l, N) \\
(i, j, I - j, a_1, a_2, \dots, a_l, N) \\
(\{i, j\}, I - j, a_1, a_2, \dots, a_l, N) \\
(j, i, I - j, a_1, a_2, \dots, a_l, N) \\
(j, i \cup I - j, a_1, a_2, \dots, a_l, N) \\
(j, I - j, i, a_1, a_2, \dots, a_l, N)
\end{array} \tag{4.8}
$$

and

$$
\begin{array}{c}
(i, I, a_1, a_2, \dots, a_l, N) \\
(i, I - j, j, a_1, a_2, \dots, a_l, N) \\
(a_l, a_{l-1}, \dots, a_1, j, I - j, i, N) \\
(a_l, a_{l-1}, \dots, \{a_1, j\}, I - j, i, N) \\
(a_l, a_{l-1}, \dots, j, a_1, I - j, i, N) \\
\vdots \\
(j, a_l, a_{l-1}, \dots, a_1, I - j, i, N) \\
(j, a_l, a_{l-1}, \dots, a_1 \cup I - j, i, N) \\
(j, a_l, a_{l-1}, \dots, I - j, a_1, i, N) \\
\vdots \\
(j, I - j, i, a_1, a_2, \dots, a_l, N).
\end{array} \tag{4.9}
$$

□

The $\mathbb{Z}_2$-homology of QP_{n+1} can be computed directly from Theorem 4.7.

Theorem 4.8. *The $\mathbb{Z}_2$-homology of QP_{n+1} is given as follows*

$$
H_i(\mathrm{QP}_{n+1}, \mathbb{Z}_2) = \begin{cases} \bigoplus_{\xi(n,i)} \mathbb{Z}_2, & 0 \le i \le n-2; \\ 0, & \text{otherwise.} \end{cases}
$$

Where, $\xi(n,i)$ denotes the sum $\sum_{k=0}^{i}\binom{n}{k}$.

Proof. One can infer from Theorem 4.7 that between any two critical cells in consecutive dimensions either there is no path between them or there are exactly two paths. This implies that the boundary maps in the Morse complex of QP_{n+1} with $\mathbb{Z}_2$- coefficients are zero. So, the mod-2 Betti numbers are given by the number of critical cells. Once the dimension is fixed, say i, the $(n+1)$-set completely determines the critical cell and it contains at most $i+1$ elements. □

4.2 The integral homology of the quotient

To compute the $\mathbb{Z}$-homology we need a well-defined notion of orientation on the cells of QP_{n+1}. So, we induce an orientation on each cell of QP_{n+1} from its ascending representative in CP_{n+1}. But, this is not sufficient to compute the $\mathbb{Z}$-homology because the paths between some critical cells involve identification of ascending and descending cells. If $\{\sigma,\tau\} = \bar{\sigma} \in QP_{n+1}$ and σ ascending, a compatible way of inducing an orientation on the cell σ from canonical orientation of τ is required and is defined as follows.

Let $\mathcal{L}$ denote the ordered neighbors of the vertex $PV(\tau)$ as defined in Section 3.1.1. The ordered vertices $\mathcal{L}'$ obtained from $\mathcal{L}$ by the action of r on individual elements induce a orientation on σ.

Now, we need to compute the difference in the orientation induced by each representative on $\bar{\sigma}$. The following examples demonstrate the existence of a closed-expression for the difference in the induced orientations.

Example 4.1. Let $n=6$, $\sigma = \{1,2,3\}\{4,5\}\{6,7\}$ and $\tau = \{4,5\}\{1,2,3\}\{6,7\}$ are two cells in CP_7 such that $\bar{\sigma} = \bar{\tau}$ in QP_7.

Neighbors of $PV(\sigma)$	Neighbors of $PV(\tau)$
$v_0 = (\{1\}\{2\}\{3\}\{4\}\{5\}\{6\}\{7\})$	$w_0 = (\{4\}\{5\}\{1\}\{2\}\{3\}\{6\}\{7\})$
$v_1 = (\{2\}\{1\}\{3\}\{4\}\{5\}\{6\}\{7\})$	$w_1 = (\{5\}\{4\}\{1\}\{2\}\{3\}\{6\}\{7\})$
$v_2 = (\{1\}\{3\}\{2\}\{4\}\{5\}\{6\}\{7\})$	$w_2 = (\{4\}\{5\}\{2\}\{1\}\{3\}\{6\}\{7\})$
$v_3 = (\{1\}\{2\}\{3\}\{5\}\{4\}\{6\}\{7\})$	$w_3 = (\{4\}\{5\}\{1\}\{3\}\{2\}\{6\}\{7\})$
$v_4 = (\{1\}\{2\}\{3\}\{4\}\{5\}\{7\}\{6\})$	$w_4 = (\{4\}\{5\}\{1\}\{2\}\{3\}\{7\}\{6\})$

Let ξ be the permutation which takes the vertex $PV(\sigma)$ to the vertex $r(PV(\tau))$ *i.e.,* $\xi = (13)(45)(76)$. This permutation has been chosen to preserve the entries of each block.

$\xi(v_0) = (\{6\}\{3\}\{2\}\{1\}\{5\}\{4\}\{7\})$	$r(w_0) = (\{6\}\{3\}\{2\}\{1\}\{5\}\{4\}\{7\})$
$\xi(v_1) = (\{6\}\{3\}\{2\}\{1\}\{4\}\{5\}\{7\})$	$r(w_1) = (\{2\}\{3\}\{1\}\{5\}\{4\}\{7\}\{6\})$
$\xi(v_2) = (\{6\}\{3\}\{1\}\{2\}\{5\}\{4\}\{7\})$	$r(w_2) = (\{3\}\{1\}\{2\}\{5\}\{4\}\{7\}\{6\})$
$\xi(v_3) = (\{6\}\{2\}\{3\}\{1\}\{5\}\{4\}\{7\})$	$r(w_3) = (\{3\}\{2\}\{1\}\{4\}\{5\}\{7\}\{6\})$
$\xi(v_4) = (\{3\}\{2\}\{1\}\{5\}\{4\}\{6\}\{7\})$	$r(w_4) = (\{3\}\{2\}\{1\}\{5\}\{4\}\{6\}\{7\})$

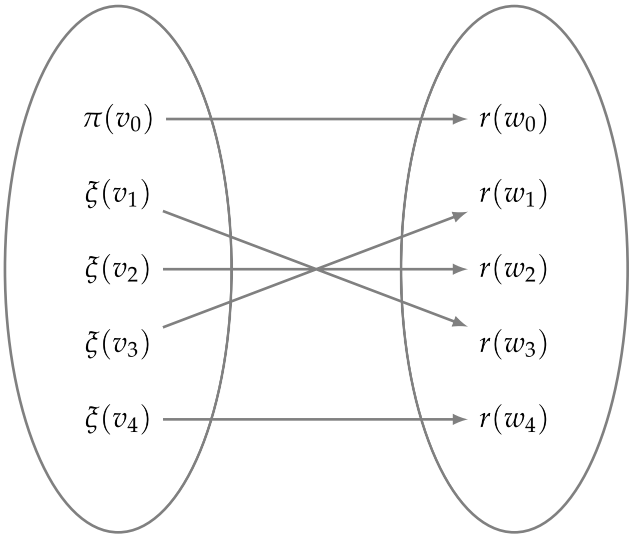


Figure 4.2

Comparing the orientations induced on the cell $\bar{\sigma} = \bar{\tau}$ by the cells σ and τ involves exactly two permutations. The permutation ξ and the permutation from the comparing the orientation induced on $\sigma \in \mathrm{CP}_7$ by the vertices v_0 and $\xi(w_0)$, refer Fig. 4.2.

Example 4.2. Let $n = 8$, $\sigma = \{1,2,3\}\{4,5\}\{6,7,8,9\}$ and $\tau = \{4,5\}\{1,2,3\}\{6,7,8,9\}$ are two cells in CP_9 such that $\bar{\sigma} = \bar{\tau}$ in QP_9.

Neighbors of $\mathrm{PV}(\sigma)$	Neighbors of $\mathrm{PV}(\tau)$
$v_0 = (\{1\}\{2\}\{3\}\{4\}\{5\}\{6\}\{7\}\{8\}\{9\})$	$w_0 = (\{4\}\{5\}\{1\}\{2\}\{3\}\{6\}\{7\}\{8\}\{9\})$
$v_1 = (\{2\}\{1\}\{3\}\{4\}\{5\}\{6\}\{7\}\{8\}\{9\})$	$w_1 = (\{5\}\{4\}\{1\}\{2\}\{3\}\{6\}\{7\}\{8\}\{9\})$
$v_2 = (\{1\}\{3\}\{2\}\{4\}\{5\}\{6\}\{7\}\{8\}\{9\})$	$w_2 = (\{4\}\{5\}\{2\}\{1\}\{3\}\{6\}\{7\}\{8\}\{9\})$
$v_3 = (\{1\}\{2\}\{3\}\{5\}\{4\}\{6\}\{7\}\{8\}\{9\})$	$w_3 = (\{4\}\{5\}\{1\}\{3\}\{2\}\{6\}\{7\}\{8\}\{9\})$
$v_4 = (\{1\}\{2\}\{3\}\{4\}\{5\}\{7\}\{6\}\{8\}\{9\})$	$w_4 = (\{4\}\{5\}\{1\}\{2\}\{3\}\{7\}\{6\}\{8\}\{9\})$
$v_5 = (\{1\}\{2\}\{3\}\{4\}\{5\}\{6\}\{8\}\{7\}\{9\})$	$w_5 = (\{4\}\{5\}\{1\}\{2\}\{3\}\{6\}\{8\}\{7\}\{9\})$
$v_6 = (\{1\}\{2\}\{3\}\{4\}\{5\}\{6\}\{7\}\{9\}\{8\})$	$w_6 = (\{4\}\{5\}\{1\}\{2\}\{3\}\{6\}\{7\}\{9\}\{8\})$

Let ξ be the permutation which takes the vertex $\mathrm{PV}(\sigma)$ to the vertex $r(\mathrm{PV}(\tau))$ *i.e.*, $\xi = (13)(45)(69)(78)$.

$\xi(v_0) = (\{8\}\{7\}\{6\}\{3\}\{2\}\{1\}\{5\}\{4\}\{9\})$	$r(w_0) = (\{8\}\{7\}\{6\}\{3\}\{2\}\{1\}\{5\}\{4\}\{9\})$
$\xi(v_1) = (\{8\}\{7\}\{6\}\{2\}\{3\}\{1\}\{5\}\{4\}\{9\})$	$r(w_1) = (\{8\}\{7\}\{6\}\{3\}\{2\}\{1\}\{4\}\{5\}\{9\})$
$\xi(v_2) = (\{8\}\{7\}\{6\}\{3\}\{1\}\{2\}\{5\}\{4\}\{9\})$	$r(w_2) = (\{8\}\{7\}\{6\}\{3\}\{1\}\{2\}\{5\}\{4\}\{9\})$
$\xi(v_3) = (\{8\}\{7\}\{6\}\{3\}\{2\}\{1\}\{4\}\{5\}\{9\})$	$r(w_3) = (\{8\}\{7\}\{6\}\{2\}\{3\}\{1\}\{5\}\{4\}\{9\})$
$\xi(v_4) = (\{7\}\{6\}\{3\}\{2\}\{1\}\{5\}\{4\}\{8\}\{9\})$	$r(w_4) = (\{8\}\{6\}\{7\}\{3\}\{2\}\{1\}\{5\}\{4\}\{9\})$
$\xi(v_5) = (\{7\}\{8\}\{6\}\{3\}\{2\}\{1\}\{5\}\{4\}\{9\})$	$r(w_5) = (\{7\}\{8\}\{6\}\{3\}\{2\}\{1\}\{5\}\{4\}\{9\})$
$\xi(v_6) = (\{8\}\{6\}\{7\}\{3\}\{2\}\{1\}\{5\}\{4\}\{9\})$	$r(w_6) = (\{7\}\{6\}\{3\}\{2\}\{1\}\{5\}\{4\}\{8\}\{9\})$

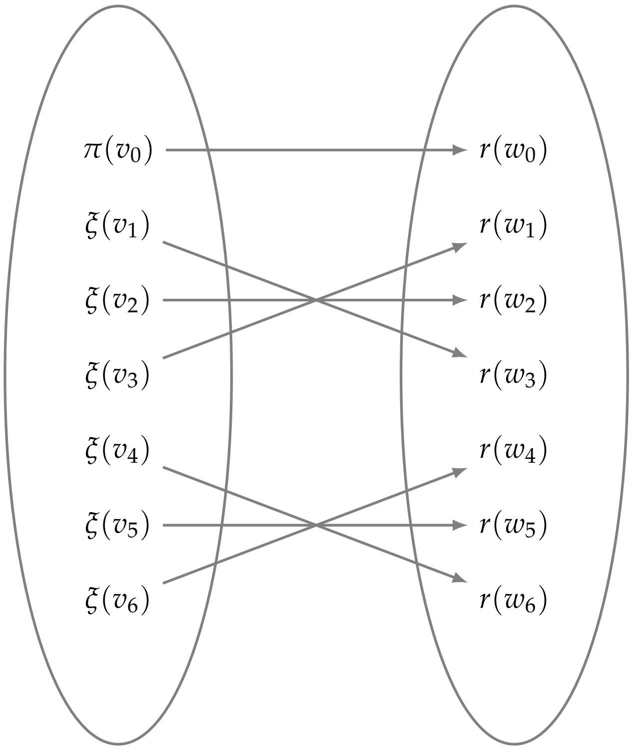


Figure 4.3

Comparing the orientations induced on the cell $\bar{\sigma} = \bar{\tau}$ by the cells σ and τ involves exactly two permutations. The permutation ξ and the permutation from the comparing the orientation induced on $\sigma \in \mathrm{CP}_9$ by the vertices v_0 and $\xi(w_0)$, refer Fig. 4.3.

From the examples, it is clear that the permutation involved in comparing the orientations induced are of the type

$$(1, 2, 3, \ldots, k) \to (k, k-1, \ldots, 2, 1).$$

and the following function is useful in computing the sign of such permutations.

Definition 4.6. Define a function, sgn : $[n] \cup \{0\} \to \{1, -1\}$ as follows, given $s \in [n]$

$$\mathrm{sgn}(s) = \begin{cases} (-1)^{\frac{s-1}{2}}, & \text{if s is odd,} \\ (-1)^{\frac{s}{2}}, & \text{if s is even.} \end{cases}$$

Let $\sigma = (I_1, I_2, \ldots, I_k, I_\omega)$ and $\tau = r(\sigma) = (I_k, I_{k-1}, \ldots, I_1, I_\omega)$ be two cells in CP_{n+1}. Observe that

- The number of neighbours of a particular vertex $\bar{v}_0$ in $\bar{\sigma}$ is same as number of neighbours of v_0 in σ.
- The neighbours of vertex v_0 in σ are naturally in 1-1 correspondence with the neighbours $r(v_0)$ in $r(\sigma)$.

Theorem 4.9. *Let* $\sigma = (I_1, I_2, \ldots, I_k, I_\omega)$ *and* $\tau = r(\sigma) = (I_k, I_{k-1}, \ldots, I_1, I_\omega)$. *Then the difference in the orientations induced on the cell* $\bar{\sigma} = \bar{\tau}$ *in* QP_{n+1} *by* σ *and* τ *in* CP_{n+1} *is given by the expression*

$$\operatorname{sgn}(|A| - |I_\omega| + 1) \cdot \operatorname{sgn}(|I_\omega| - 1) \cdot \prod_{i=1}^{k} \operatorname{sgn}(|I_i|) \cdot \operatorname{sgn}(|I_\omega|), \tag{4.10}$$

where $|A|$ *is the number of neighbors of* $\mathrm{PV}(\sigma)$.

Proof. Without loss of generality assume $PV(\sigma) = (1, 2, \ldots, n+1)$ i.e., the blocks $I_1 = \{1, 2, \ldots, a_1\}, \ldots, I_j = \{a_{j-1}+1, a_{j-1}+2, \ldots, a_j\}$ for every j such that $1 \le j \le k$ or $j = \omega$.

The neighbors of $\mathrm{PV}(\sigma)$ are ordered as follows.

$$\begin{aligned}
v_0 &= (\{1\}\{2\}\{3\}\ldots\{n+1\}) \\
v_1 &= (\{2\}\{1\}\{3\}\ldots\{n+1\}) \\
v_2 &= (\{1\}\{3\}\{2\}\ldots\{n+1\}) \\
&\vdots \\
v_{a_1-1} &= (\{1\}\ldots\{a_1\}\{a_1-1\}\ldots\{n+1\}) \\
&\vdots \\
v_{a_k-k-1} &= (\{1\}\ldots\{a_k-1\}\{a_k\}\ldots\{n+1\}) \\
v_{a_k-k} &= (\{1\}\ldots\{a_k+2\}\{a_k+1\}\ldots\{n+1\}) \\
&\vdots \\
v_{|A|} &= (\{1\}\{2\}\ldots\{n+1\}\{n\})
\end{aligned} \tag{4.11}$$

The neighbors of $\mathrm{PV}(\tau)$ are ordered as follows.

$$\begin{aligned}
w_0 &= (\{a_{k-1}+1\}\{a_{k-1}+2\}\ldots\{a_k\}\{a_{k-2}\}\ldots\{n+1\}) \\
w_1 &= (\{a_{k-1}+2\}\{a_{k-1}+1\}\ldots\{a_k\}\{a_{k-2}\}\ldots\{n+1\}) \\
&\vdots \\
w_{a_k-a_{k-1}} &= (\{a_{k-1}+1\}\{a_{k-1}+2\}\ldots\{a_{k-2}+1\}\{a_{k-2}\}\ldots\{n+1\}) \\
&\vdots \\
w_{|A|-a_k-a_{k-1}-1} &= (\{a_{k-1}+1\}\{a_{k-1}+2\}\ldots\{2\}\{1\}\ldots\{n+1\}) \\
&\vdots \\
w_{|A|-a_k-1} &= (\{a_{k-1}+1\}\{a_{k-1}+2\}\ldots\{a_1\}\{a_1-1\}\ldots\{n+1\}) \\
w_{|A|-a_k} &= (\{a_{k-1}+1\}\{a_{k-1}+2\}\ldots\{a_1\}\{a_k+2\}\{a_k+1\}\ldots\{n+1\}) \\
&\vdots \\
w_{|A|} &= (\{a_{k-1}+1\}\{a_{k-1}+2\}\ldots\{n+1\}\{n\})
\end{aligned} \tag{4.12}$$

Now apply r on the neighbours of $\mathrm{PV}(\tau)$ to obtain an ordered collection of vertices in σ. This would enable us to compare the orientations induced by 4.11 and 4.12 on $\bar{\sigma}$.

$$\begin{aligned}
r(w_0) &= (\{n\}\{n-1\}\ldots\{a_k+1\}\{a_1\}\ldots\{a_{k-1}+1\}\{n+1\}) \\
r(w_1) &= (\{n\}\{n-1\}\ldots\{a_k+1\}\{a_1\}\ldots\{a_{k-1}+1\}\{a_{k-1}+2\}) \\
&\vdots \\
r(w_{a_k-a_{k-1}}) &= (\{n\}\{n-1\}\ldots\{a_{k-2}\}\{a_{k-2}+1\}\ldots\{a_{k-1}+1\}\{n+1\}) \\
&\vdots \\
r(w_{|A|-a_k-a_{k-1}-1}) &= (\{n\}\{n-1\}\ldots\{1\}\{2\}\ldots\{a_{k-1}+1\}\{n+1\}) \\
&\vdots \\
r(w_{|A|-a_k-1}) &= (\{n\}\{n-1\}\ldots\{a_1-1\}\{a_1\}\ldots\{a_{k-1}+1\}\{n+1\}) \\
r(w_{|A|-a_k}) &= (\{n\}\{n-1\}\ldots\{a_1+1\}\{a_k+2\}\{a_k\}\ldots\{a_{k-1}+1\}\{n+1\}) \\
&\vdots \\
r(w_{|A|}) &= (\{n-1\}\{n-2\}\ldots\{a_{k-1}+1\}\{n\}\{n+1\})
\end{aligned} \tag{4.13}$$

Let ξ be the permutation which takes the vertex $\mathrm{PV}(\sigma)$ to the vertex $r(\mathrm{PV}(\tau))$.

$$\begin{aligned}
\xi(v_0) &= (\{a_1\}\{a_1-1\}\ldots\{2\}\{1\}\{a_2\}\ldots\{n+1\}\ldots\{a_k+1\}) \\
\xi(v_1) &= (\{a_1\}\{a_1-1\}\ldots\{1\}\{2\}\ldots\{n+1\}\ldots\{a_k+1\}) \\
\xi(v_2) &= (\{a_1\}\{a_1-1\}\ldots\{2\}\{3\}\{1\}\ldots\{n+1\}\ldots\{a_k+1\}) \\
&\vdots \\
\xi(v_{a_1-1}) &= (\{a_1-1\}\{a_1\}\ldots\{2\}\{1\}\ldots\{n+1\}\ldots\{a_k+1\}) \\
&\vdots \\
\xi(v_{a_k-k-1}) &= (\{a_1\}\{a_1-1\}\ldots\{a_k\}\{a_k-1\}\ldots\{n+1\}\{a_k+1\}) \\
\xi(v_{a_k-k}) &= (\{a_1\}\{a_1-1\}\ldots\{a_k+1\}\{a_k+2\}\ldots\{n+1\}\{a_k+1\}) \\
&\vdots \\
\xi(v_{|A|}) &= (\{a_1\}\{a_1-1\}\ldots\{n\}\{n+1\}\ldots\{a_k+1\})
\end{aligned} \tag{4.14}$$

It is clear from above that if $\sigma = (I_1, I_2, \ldots, I_k, I_\omega)$ the sign of the permutation ξ is $\prod_{i=1}^{k} \mathrm{sgn}(|I_i|) \cdot \mathrm{sgn}(|I_\omega|)$. The sign of permutation coming from comparing the induced orientations on $\sigma \in \mathrm{CP}_{n+1}$ by the vertices v_0 and $\xi(w_0)$ is $\mathrm{sgn}(|A| - |I_\omega| + 1) \cdot \mathrm{sgn}(|I_\omega| - 1)$. Thus the total sign to be taken into account is $\mathrm{sgn}(|A| - |I_\omega| + 1) \cdot \mathrm{sgn}(|I_\omega| - 1) \cdot \prod_{i=1}^{k} \mathrm{sgn}(|I_i|) \cdot \mathrm{sgn}(|I_\omega|)$. □

Table 4.1: Integer homology, $H_i(\mathrm{QP}_{n+1})$

		i $\rightarrow$							
		0	1	2	3	4	5	6	7
$n+1$	4	$\mathbb{Z}$	$\mathbb{Z}^4$						
$\downarrow$	5	$\mathbb{Z}$	$\mathbb{Z}_2^5$	$\mathbb{Z}^6$					
	6	$\mathbb{Z}$	$\mathbb{Z}_2^6$	$\mathbb{Z}^{10}$	$\mathbb{Z}^{26}$				
	7	$\mathbb{Z}$	$\mathbb{Z}_2^7$	$\mathbb{Z}^{15}$	$\mathbb{Z}_2^{42}$	$\mathbb{Z}^{15}$			
	8	$\mathbb{Z}$	$\mathbb{Z}_2^8$	$\mathbb{Z}^{21}$	$\mathbb{Z}_2^{64}$	$\mathbb{Z}^{35}$	$\mathbb{Z}^{120}$		
	9	$\mathbb{Z}$	$\mathbb{Z}_2^9$	$\mathbb{Z}^{28}$	$\mathbb{Z}_2^{93}$	$\mathbb{Z}^{70}$	$\mathbb{Z}_2^{219}$	$\mathbb{Z}^{28}$	
	10	$\mathbb{Z}$	$\mathbb{Z}_2^{10}$	$\mathbb{Z}^{36}$	$\mathbb{Z}_2^{130}$	$\mathbb{Z}^{126}$	$\mathbb{Z}_2^{382}$	$\mathbb{Z}^{84}$	$\mathbb{Z}^{502}$

The following observations are helpful in computing the $\mathbb{Z}$-homology of QP_{n+1}.

1. There exists no path or exactly two paths between critical cells whose dimension differ by one. Thus the matrices corresponding to the boundary maps contain only 2's and 0's depending on whether the orientation induced by the paths match or not.

2. These are good paths, except some paths involves a identification of a cell σ with $r(\sigma)$, where the orientation change involved is given by Eq. (4.10).

Definition 4.7. Two rectangular matrices $A, B \in M_{n\times m}(\mathbb{Z})$ are called equivalent if they can be transformed into one another by a combination of elementary row and column operations.

Definition 4.8 (2-full rank)**.** Let $f : \mathbb{Z}^m \to \mathbb{Z}^n$ be a group homomorphism. The map f is *2-full rank*, denoted $2\mathcal{F}$, if it is equivalent to a scalar matrix with scalar multiple 2.

Proposition 4.10. *The boundary maps in the Morse complex of* QP_{n+1} *are either 2-full rank or null maps. i.e., if the Morse complex* $\mathcal{M}_.$ *on* QP_{n+1} *is*

$$0 \longrightarrow \mathcal{M}_{n-2} \xrightarrow{\tilde{\partial}_{n-2}} \mathcal{M}_{n-3} \xrightarrow{\tilde{\partial}_{n-3}} \dots \xrightarrow{\tilde{\partial}_1} \mathcal{M}_0 \to 0$$

then the boundary maps

$$\tilde{\partial}_\mu \equiv \begin{cases} \mathbf{0}, & \text{if } \mu \text{ is odd;} \\ 2\mathcal{F}, & \text{if } \mu \text{ is even.} \end{cases}$$

Proof. If the sign correction for the identification involved in the path is positive (resp. negative), then by Lemma 3.8 and Lemma 3.9 the coefficient $\langle \partial\alpha, \beta\rangle = 0$ (resp. $\langle \partial\alpha, \beta\rangle = 2$).

Claim 1: $\tilde{\partial}_1 = 0$.

Proof. Let $\alpha = (i, I, \nabla, N)$ and $\beta = (j, J, \nabla', N')$ be two critical cells contained in $\mathcal{M}_1$ and $\mathcal{M}_0$ respectively.

1. If $|N| = 1$, then $|I| = 2$ and $N' = N = \{n+1\}$. Otherwise there will be no path between the cells giving $\langle \tilde{\partial}\alpha, \beta\rangle = 0$. There is an identification of the cell $(i, I - j, j, \nabla, N)$ with its image under the map r during the path. All the blocks of this cell are singletons, thus the sign correction given by the Eq. (4.10) is 1.

2. If $|N| = 2$, then $|I| = 1$, $N' \subset N$ and $|N'| = 1$. Otherwise an argument similar to above shows that $\langle \tilde{\partial}\alpha, \beta \rangle = 0$. There is an identification of the cell $(i, I, \nabla, N - t, t)$ or $(i, I, \nabla, t, N - t)$ with its image under the map r during the path. All the blocks of these cells are singletons, thus the sign correction given by the Eq. (4.10) is 1.

Since there is no effect on the orientation induced along the paths by the action, an argument similar to Theorem 3.10 shows that $\tilde{\partial}_1 = 0$.

Claim 2: $\partial_\mu = 0$ when μ is odd.

Proof. Let $\alpha = (i, I, \nabla, N)$ and $\beta = (j, J, \nabla', N')$ be two critical cells contained in $\mathcal{M}_\mu$ and $\mathcal{M}_{\mu-1}$ respectively. Also, let d denote the number of blocks in α which is equal to $(n + 1 - \mu)$.

1. Let $N = N'$ and $J = I - j$.

   - Let $|N| = k + 1$ for some k odd. There is an identification of the cell $(i, I - j, j, \nabla, N)$ with its image under r during the path. From Eq. (4.10), the sign correction $\mathcal{S}$ is given by $\mathrm{sgn}(|J|) \cdot \mathrm{sgn}(|N|) \cdot \mathrm{sgn}(|J| - 1) \cdot \mathrm{sgn}(|N| - 1)$.
     Observe that the $|J| = n - k - d - 3 = (n + 1 - d) - (k + 4)$. Since $(n + 1 - d) = \mu$ is odd and $(k + 4)$ is odd, $|J|$ is even and

     $$\mathrm{sgn}(|J|) = (-1)^{\frac{n-k-d-3}{2}}.$$

     Similarly,

     $$\mathrm{sgn}(|N|) = (-1)^{\frac{k+1}{2}},$$
     $$\mathrm{sgn}(|J| - 1) = (-1)^{\frac{n-k-d-5}{2}},$$
     $$\mathrm{sgn}(|N| - 1) = (-1)^{\frac{k-1}{2}}.$$

     This shows that the sign correction $\mathcal{S}$ is equal to 1.

   - Let $|N| = k + 1$ for some k even. There is an identification of the cell $(i, I - j, j, \nabla, N)$ with its image under r during the path. From Eq. (4.10), the sign correction $\mathcal{S}$ is given by $\mathrm{sgn}(|J|) \cdot \mathrm{sgn}(|N|) \cdot \mathrm{sgn}(|J| - 1) \cdot \mathrm{sgn}(|N| - 1)$.
     Observe that the $|J| = n - k - d - 3 = (n + 1 - d) - (k + 4)$. Since $(n + 1 - d) = \mu$ is odd and $(k + 4)$ is even, $|J|$ is odd and

     $$\mathrm{sgn}(|J|) = (-1)^{\frac{n-k-d-4}{2}}.$$

     Similarly,

     $$\mathrm{sgn}(|N|) = (-1)^{\frac{k}{2}},$$
     $$\mathrm{sgn}(|J| - 1) = (-1)^{\frac{n-k-d-4}{2}},$$
     $$\mathrm{sgn}(|N| - 1) = (-1)^{\frac{k}{2}}.$$

     This shows that the sign correction $\mathcal{S}$ is equal to 1.

2. Let $I = J$ and $N' = N - t$ for some $t \in N$.

- Let $|N| = k+1$ for some k odd. There is an identification of the cell $(i, I, \nabla, N-t, t)$ or $(i, I, \nabla, t, N-t)$ with its image under the map r during the path. From Eq. (4.10), the sign correction $\mathcal{S}$ is given by $\text{sgn}(|I|) \cdot \text{sgn}(|N'|) \cdot \text{sgn}(|I|-1) \cdot \text{sgn}(|N'|-1)$. Observe that the $|I| = n-k-d-2 = (n+1-d)-(k+3)$. Since $(n+1-d)$ is odd and $(k+3)$ is even, $|I|$ is odd and
$$\text{sgn}(|I|) = (-1)^{\frac{n-k-d-3}{2}}.$$
Similarly,
$$\text{sgn}(|N|) = (-1)^{\frac{k-1}{2}},$$
$$\text{sgn}(|J|-1) = (-1)^{\frac{n-k-d-3}{2}},$$
$$\text{sgn}(|N|-1) = (-1)^{\frac{k-1}{2}}.$$
Clearly, the sign correction $\mathcal{S}$ is equal to 1.

- Let $|N| = k+1$ for some k even. There is an identification of the cell $(i, I, \nabla, N-t, t)$ or $(i, I, \nabla, t, N-t)$ with its image under the map r during the path. From Eq. (4.10), the sign correction $\mathcal{S}$ is given by $\text{sgn}(|I|) \cdot \text{sgn}(|N'|) \cdot \text{sgn}(|I|-1) \cdot \text{sgn}(|N'|-1)$. Observe that the $|I| = n-k-d-2 = (n+1-d)-(k+3)$. Since $(n+1-d)$ is odd and $(k+3)$ is odd, $|I|$ is even and
$$\text{sgn}(|I|) = (-1)^{\frac{n-k-d-2}{2}}.$$
Similarly,
$$\text{sgn}(|N|) = (-1)^{\frac{k}{2}},$$
$$\text{sgn}(|J|-1) = (-1)^{\frac{n-k-d-4}{2}},$$
$$\text{sgn}(|N|-1) = (-1)^{\frac{k-2}{2}}.$$
Clearly, the sign correction $\mathcal{S}$ is equal to 1.

From Theorem 4.5, it is clear that for a fixed $(n+1)$-set, there is a unique cell of dimension μ. For the rest of this section, we arrange the basis elements of $\mathcal{M}_\mu$ as follows: cells are arranged in increasing order of the cardinality of their $(n+1)$-set and if the cardinality of $(n+1)$-set of two different cells is same then they are arranged in lexicographic ordering on their $(n+1)$-set.

Using the above defined ordering on the basis of $\mathcal{M}_\mu$ and $\mathcal{M}_{\mu-1}$ and from Theorem 4.7, we observe that the matrix corresponding to the boundary map $\tilde{\partial}_\mu$ is upper triangular with 2 being the only non-zero entry. We now prove that when μ is even, all the diagonal entries of $\tilde{\partial}_\mu$ are 2.

Claim 3: $\partial_\mu = 2\mathcal{F}$ when μ is even.

Proof. Let $\alpha = (i, I, \nabla, N)$ and $\beta = (j, J, \nabla', N')$ be two critical cells contained in $\mathcal{M}_i$ and $\mathcal{M}_{i-1}$ respectively. Also, let d denote the number of blocks in α which is equal to $(n+1-\mu)$. It is enough to show that $\langle \partial\alpha, \beta \rangle = 2$ whenever $N = N'$.

1. Let $|N| = k+1$ for some k odd. There is an identification of the cell $(i, I-j, j, \nabla, N)$ with its image under r during the path. From Eq. (4.10), the sign correction $\mathcal{S}$ is given by $\text{sgn}(|J|) \cdot \text{sgn}(|N|) \cdot \text{sgn}(|J|-1) \cdot \text{sgn}(|N|-1)$.
   Observe that the $|J| = n-k-d-3 = (n+1-d)-(k+4)$. Since $(n+1-d)$ is even and $(k+4)$ is odd, $|J|$ is odd and
   $$\text{sgn}(|J|) = (-1)^{\frac{n-k-d-4}{2}}.$$
   Similarly,
   $$\begin{aligned}\text{sgn}(|N|) &= (-1)^{\frac{k+1}{2}},\\ \text{sgn}(|J|-1) &= (-1)^{\frac{n-k-d-4}{2}},\\ \text{sgn}(|N|-1) &= (-1)^{\frac{k-1}{2}}.\end{aligned}$$
   Clearly, the sign correction $\mathcal{S}$ is equal to -1.

2. Let $|N| = k+1$ for some k even. There is an identification of the cell $(i, I-j, j, \nabla, N)$ with its image under r during the path. From Eq. (4.10), the sign correction $\mathcal{S}$ is given by $\text{sgn}(|J|) \cdot \text{sgn}(|N|) \cdot \text{sgn}(|J|-1) \cdot \text{sgn}(|N|-1)$. Observe that $|J| = n-k-d-3 = (n+1-d)-(k+4)$. Since $(n+1-d)$ is even and $(k+4)$ is even, $|J|$ is even and
   $$\text{sgn}(|J|) = (-1)^{\frac{n-k-d-3}{2}}.$$
   Similarly,
   $$\begin{aligned}\text{sgn}(|N|) &= (-1)^{\frac{k}{2}},\\ \text{sgn}(|J|-1) &= (-1)^{\frac{n-k-d-5}{2}},\\ \text{sgn}(|N|-1) &= (-1)^{\frac{k}{2}}.\end{aligned}$$

Clearly, the sign correction $\mathcal{S}$ is -1. □

Theorem 4.11. *The $\mathbb{Z}$-homology of QP_{n+1} is given as follows.*
If n is even, then

$$H_i(\text{QP}_{n+1}, \mathbb{Z}) = \begin{cases} \bigoplus_{\xi(n,i)} \mathbb{Z}_2, & \text{if } i \text{ is odd and } 0 \le i \le n-2; \\ \bigoplus_{\binom{n}{i}} \mathbb{Z}, & \text{if } i \text{ is even and } 0 \le i \le n-2; \\ 0, & \text{otherwise.} \end{cases}$$

If n is odd, then

$$H_i(\mathrm{QP}_{n+1}, \mathbb{Z}) = \begin{cases} \bigoplus_{\xi(n,i)} \mathbb{Z}, & \text{if } i = n-2; \\ \bigoplus_{\xi(n,i)} \mathbb{Z}_2, & \text{if } i \text{ is odd and } 0 \leq i < n-2; \\ \bigoplus_{\binom{n}{i}} \mathbb{Z}, & \text{if } i \text{ is even and } 0 \leq i \leq n-2; \\ 0, & \text{otherwise.} \end{cases}$$

Where, $\xi(n,i)$ denotes the sum $\sum_{k=0}^{i} \binom{n}{k}$.

Proof. We will present the proof for the case of n being odd, the proof for the even case is similar in nature.

1. If $i = n-2$, then the Morse complex looks like

$$0 \xrightarrow{0} \mathcal{M}_{n-2} \xrightarrow{0} \mathcal{M}_{n-3}.$$

   Then the homology at $\mathcal{M}_{n-2}$ is $\mathbb{Z}^{\mathrm{rnk}(\mathcal{M}_{n-2})}$.

2. If i is odd and $i \neq n-2$, then the Morse complex looks like

$$\mathcal{M}_{i+1} \xrightarrow{2F} \mathcal{M}_i \xrightarrow{0} \mathcal{M}_{i-1}.$$

   Then the homology at $\mathcal{M}_i$ is $\mathbb{Z}_2^{\mathrm{rnk}(\mathcal{M}_i)}$.

3. If i is even, then the Morse complex looks like

$$\mathcal{M}_{i+1} \xrightarrow{0} \mathcal{M}_i \xrightarrow{2F} \mathcal{M}_{i-1}.$$

   Then the homology at $\mathcal{M}_i$ is $\mathbb{Z}^{\mathrm{rnk}(\mathcal{M}_i)-\mathrm{rnk}(\mathcal{M}_{i-1})}$.

□

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