Two Aspects of Enumerative Combinatorics: Hyperplane Arrangements and Pattern Avoidance

By

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A thesis submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy

to

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DECLARATION

I declare that the thesis entitled **"Two Aspects of Enumerative Combinatorics: Hyperplane Arrangements and Pattern Avoidance"** submitted by me for the degree of **Doctor of Philosophy in Mathematics** is the record of academic work carried out by me under the guidance of Professor "Priyavrat Deshpande" and this work has not formed the basis for the award of any degree, diploma, associateship, fellowship or other titles in this University or any other University or Institution of Higher Learning.

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CERTIFICATE

I certify that the thesis entitled **"Two Aspects of Enumerative Combinatorics: Hyperplane Arrangements and Pattern Avoidance"** submitted for the degree of **Doctor of Philosophy in Mathematics** by "Krishna Menon P" is the record of research work carried out by him under my guidance and supervision, and that this work has not formed the basis for the award of any degree, diploma, associateship, fellowship or other titles in this University or any other University or Institution of Higher Learning. I further certify that the new results presented in this thesis represent his independent work in a very substantial measure.

Chennai Mathematical Institute
Date: August, 2024.

Priyvrat Deshpande
Thesis Supervisor.

Acknowledgements

It still feels slightly unreal that I am writing the acknowledgments section of my PhD thesis. I have liked mathematics since I was about ten years old, but to actually come this far is something I am grateful to have been able to do. This is definitely not something I have done by myself and there are several people I'd like to thank.

Firstly, I would like to thank my advisor, Priyavrat Deshpande. He introduced me to combinatorics at a time when I was doubtful I'd be able to carry out research in mathematics. His approach to research made it feel down-to-earth and accessible. It is always easy to talk to him and this made my PhD life relatively stress-free. In particular, I am grateful for his advice on my slides for research talks. Making talks accessible, enjoyable, and engaging is a challenging task. But, because of his encouragement, it is a task I now love attempting. I am also thankful to him for listening to my silly worries and invariably calming me down.

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I would especially like to thank my parents who have supported my mathematics career from the start. My father's approach to mathematics (and studies in general) is the reason I got interested in it. My mother's youthfulness taught me to keep enjoying life no matter where I am. The days I spend with them, as well as the calm Sundays in my grandparents' house, allow me take a breather and relax. My grandparents' stories about their lives were always nice to hear, especially my grandmother's stories about her research. I guess I should also thank my brother since he bought me headphones and some anime shirts. He should probably buy me more stuff though. Kidding aside, my family has always been wonderful to me and I'm very grateful for them.

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Krishna Menon P
CMI, August 2024.

Dedicated to my parents

Abstract

This thesis is divided into two parts, where we address problems from two topics in enumerative combinatorics: hyperplane arrangements and pattern avoidance.

1. A hyperplane arrangement in $\mathbb{R}^n$ is a finite collection of affine hyperplanes. Its regions are the connected components of the complement of these hyperplanes. The collection of reflecting hyperplanes of a finite Coxeter group is called a reflection arrangement and it appears in many subareas of combinatorics and representation theory. We focus on the problem of counting regions of reflection arrangements and their deformations. Inspired by the recent work of Bernardi, we show that the notion of 'sketches and moves' can be used to provide a uniform and explicit bijection between regions of (the Catalan deformation of) a reflection arrangement and certain non-nesting partitions.

   We then focus on interpreting the coefficients of the characteristic polynomials of these arrangements. By a theorem of Zaslavsky, the number of regions of a hyperplane arrangement is the sum of the absolute values of the coefficients of its characteristic polynomial. We use the exponential formula to describe a statistic on the non-nesting partitions we defined in such a way that the distribution of this statistic is given by the characteristic polynomial.

   Finally, we study similar questions for a sub-arrangement of type C arrangement called the threshold arrangement and its Catalan and Shi deformations.

2. The study of pattern avoidance in linear permutations has been an active area of research for almost half a century now, starting with the work of Knuth in 1973. More recently, the question of pattern avoidance in circular permutations has gained significant attention. In 2002-03, Callan and Vella independently characterized circular permutations avoiding a single permutation of size 4. Building on their results, Domagalski et al. studied circular pattern avoidance for multiple patterns of size 4. In the second part of this thesis, our main aim is to study circular pattern avoidance of $[4, k]$-pairs, *i.e.*, circular permutations avoiding one pattern of size 4 and another of size k. We do this by using well-studied combinatorial objects to represent circular permutations avoiding

a single pattern of size 4. In particular, we obtain upper bounds for the number of Wilf equivalence classes of $[4, k]$-pairs. Moreover, we prove that the obtained bound is tight when the pattern of size 4 in consideration is $[1342]$. Using ideas from our general results, we also obtain a complete characterization of the avoidance classes for $[4, 5]$-pairs.

List of publications/preprints associated with the thesis

1. Priyavrat Deshpande and Krishna Menon. "A branch statistic for trees: interpreting coefficients of the characteristic polynomial of braid deformations". *Enumer. Comb. Appl.*, 3(1): Paper No. S2R5, 2023.
2. Priyavrat Deshpande and Krishna Menon. "Sketches, moves and partitions: counting regions of deformations of reflection arrangements". arXiv:2308.16653, 2023.
3. Priyavrat Deshpande, Krishna Menon, and W. Sarkar. "Refinements of the braid arrangement and two-parameter Fuss-Catalan numbers". *J. Algebraic Combin.*, 57(3): 687-707, 2023.
4. Priyavrat Deshpande, Krishna Menon, and Anurag Singh. "A combinatorial statistic for labeled threshold graphs". *Enumer. Comb. Appl.*, 1(3): Paper No. S2R22, 2021.
5. Priyavrat Deshpande, Krishna Menon, and Anurag Singh. "Counting regions of the boxed threshold arrangement". *J. Integer Seq.*, 24(5): Art. 21.5.7, 2021.
6. Krishna Menon and Anurag Singh. "Pattern avoidance and dominating compositions". *Enumer. Comb. Appl.*, 2(1): Paper No. S2R4, 2022.
7. Krishna Menon and Anurag Singh. "Pattern avoidance of $[4, k]$-pairs in circular permutations". *Adv. in Appl. Math.*, 138: Paper No. 102346, 2022.
8. Krishna Menon and Anurag Singh. "Dyck paths, binary words, and grassmannian permutations avoiding an increasing pattern". *Ann. Comb.*, 2023.
9. Krishna Menon and Anurag Singh. "Subsequence frequency in binary words". *Discrete Math.*, 347(5): Paper No. 113928, 2024.

Contents

Chapter 0

Introduction

The first line of Chapter 1 of Stanley's Enumerative Combinatorics [55] reads

> "The basic problem of enumerative combinatorics is that of counting the number of elements of a finite set."

As one might guess, this encompasses a broad topic in mathematics that addresses problems in diverse areas. Our goal for any such problem is to obtain a simple, direct, and combinatorial solution. Failing to do so, or when a problem is too general for such a solution to feasibly exist, we sometimes settle for answers involving recursive formulas or expressions for generating functions.

As indicated by the title of this thesis, we will be dealing with two topics in Enumerative Combinatorics: Hyperplane Arrangements and Pattern Avoidance. Both topics have several enumerative questions which have received significant attention over the past few decades.

This chapter contains an overview of the thesis.

Hyperplane Arrangements

A *hyperplane arrangement* $\mathcal{A}$ is a finite collection of affine hyperplanes (i.e., codimension 1 subspaces and their translates) in $\mathbb{R}^n$. For example, an arrangement in $\mathbb{R}^2$ is just a finite collection of lines. One natural question to ask is how many 'pieces' a collection of lines breaks the plane into. In general, a *region* of an arrangement $\mathcal{A}$ is a connected component of $\mathbb{R}^n \setminus \bigcup \mathcal{A}$. Counting regions of arrangements is an active area of research in enumerative combinatorics.

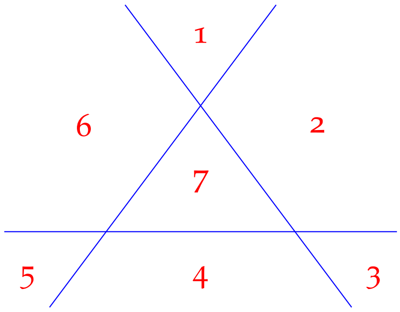


FIGURE 1: An arrangement in $\mathbb{R}^2$ with 7 regions.

One way to count regions is via a bijection. This approach involves finding a combinatorially defined set whose elements are in bijection with the regions of the given arrangement and are easier to count. Such a bijection usually sheds more light on the regions of the arrangement. There are several classes of arrangements where regions correspond to interesting combinatorial objects. For example, permutations, trees, and Dyck paths all appear as regions of certain arrangements. Obtaining bijections for regions of arrangements is an interesting problem in combinatorics (see [6, 8, 20]).

Another way to obtain the number of regions of an arrangement $\mathcal{A}$ is from its *characteristic polynomial* $\chi_{\mathcal{A}}(t)$. The characteristic polynomial is a fundamental combinatorial and topological invariant of the arrangement. Zaslavsky's theorem [60] says that the number of regions of $\mathcal{A}$, denoted $r(A)$, is the sum of the absolute values of the coefficients of $\chi_{\mathcal{A}}(t)$. Hence, if c_i is the coefficient of t^i in $\chi_{\mathcal{A}}(t)$, then we have $r(\mathcal{A}) = \sum |c_i|$. One can now ask if there is a *statistic* on the regions that induces this break-up. That is, is there a nice way to assign numbers to each region such that for all $i \geqslant 0$, there are $|c_i|$ regions with i assigned to them? This question has been addressed for various arrangements (see [18, 19, 33]).

In Part I of this thesis, we answer these questions for an interesting class of arrangements called *Catalan deformations of reflection arrangements*. Let Φ be a (not necessarily reduced) crystallographic root system and let Φ^+ be a choice of positive roots (relevant definitions can be found in [31]). The reflection (or Coxeter) arrangement $\mathcal{A}(\Phi)$ corresponding to Φ consists of hyperplanes with the defining equations

$$(\alpha, x) = 0 \quad \text{for } \alpha \in \Phi^+.$$

Note that these are the same hyperplanes that are fixed by the Weyl group of Φ. Reflection arrangements appear in many subareas of combinatorics and representation theory.

A deformation of an arrangement $\mathcal{A}$ is an arrangement each of whose hyperplanes is parallel to some hyperplane in $\mathcal{A}$. Our main focus is certain deformations of reflection arrangements called *Catalan deformations*. For brevity, we sometimes write 'type Φ Catalan arrangement' or 'Catalan arrangement of type Φ'. The type Φ Catalan arrangement consists of hyperplanes with defining equations

$$(\alpha, x) = -1, 0, 1 \quad \text{for } \alpha \in \Phi^+.$$

We show that the regions of these arrangements are in bijection with certain labeled non-nesting partitions. We then use the exponential formula to describe a statistic on these partitions such that distribution is given by the coefficients of the characteristic polynomial. We also use similar ideas to tackle another interesting class of arrangements called threshold deformations (see Chapter 4).

The results in Part I are from [16, 17], which are both joint work with Priyavrat Deshpande. An overview of these results is given in Section 0.1.

Pattern Avoidance

Pattern avoidance is a relatively recent topic in combinatorics which has been garnering a lot of attention. For a class of combinatorial objects, we first define what it means for one object to be *contained* in another. When an object A contains an object B, we usually refer to B as a *pattern* and say that A contains the pattern B. If an object A does not contain the pattern B, we say that A *avoids* B. The usual question in pattern avoidance is: Given a set of patterns, describe or count the objects that avoid them.

Permutations are the most popular objects where pattern avoidance is studied. We represent permutations in one-line notation. For $n \geqslant m \geqslant 1$, a permutation $\sigma = \sigma_1 \cdots \sigma_n$ *contains* a permutation (or pattern) $\pi = \pi_1 \cdots \pi_m$ if there exists a subsequence $1 \leqslant h(1) < h(2) < \cdots < h(m) \leqslant n$ such that for any $1 \leqslant i, j \leqslant m$, $\sigma_{h(i)} < \sigma_{h(j)}$ if and only if $\pi_i < \pi_j$. In this case $\sigma_{h(1)} \cdots \sigma_{h(m)}$ is said to be *order isomorphic* to π. We say that the permutation σ *avoids* π if it does not contain π.

We will be dealing with pattern avoidance in *circular permutations*. A circular permutation $[\pi]$ is the set of all rotations of a permutation $\pi = \pi_1 \cdots \pi_n$, *i.e.*,

$$[\pi] = \{\pi_1 \cdots \pi_n, \pi_2 \cdots \pi_n \pi_1, \ldots, \pi_n \pi_1 \cdots \pi_{n-1}\}.$$

We say that a circular permutation $[\sigma]$ *contains* a circular permutation (or pattern) $[\pi]$ if there exists a rotation σ' of σ such that σ' contains π linearly. If there is no rotation of σ containing π, we say that $[\sigma]$ *avoids* $[\pi]$. For instance, $[14523]$ contains $[1234]$ because the permutation 23145 (which is a rotation of 14523) has the subsequence 2345 which is order isomorphic to 1234.

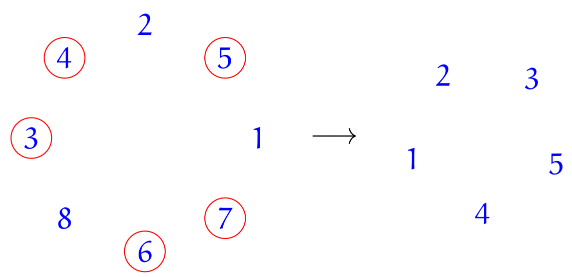


FIGURE 2: The circular permutation $[17683425]$ contains the pattern $[12354]$.

The study of pattern avoidance in linear permutations has been an active area of research for almost half a century now, starting with the work of Knuth in 1973 [36]. More recently, the question of pattern avoidance in circular permutations has gained significant attention. In 2002-03, Callan [12] and Vella [58] independently characterized circular permutations avoiding a single permutation of size 4. Building on their results, Domagalski et al. [21] studied circular pattern avoidance for multiple patterns of size 4. Other notions of pattern avoidance in circular permutations have also been explored [26, 27, 38, 40].

In Part II of this thesis, we study avoidance of a pattern of size 4 along with another pattern of arbitrary length. If the other pattern is of length k, we call this 'avoidance of a $[4, k]$-pair'. These result are from [42], which is joint work with Anurag Singh. An overview of these results is given in Section 0.2.

0.1 Catalan Deformations of Reflection Arrangements

We use a fairly simple but effective method to obtain bijective proofs for the number of regions of Catalan deformations. This method was used by Bernardi in [8, Section 8] to obtain bijections for the regions of several deformations of the braid arrangement. This idea, that we call 'sketches and moves', is to consider an arrangement $\mathcal{B}$ whose regions we wish to count as a sub-arrangement of an arrangement $\mathcal{A}$. This is done in such a way that the regions of $\mathcal{A}$ are well-understood and are usually total orders on certain symbols. These total orders are what we call *sketches*. Since $\mathcal{B} \subseteq \mathcal{A}$, the regions of $\mathcal{B}$ partition the regions of $\mathcal{A}$ and hence define an equivalence on sketches. We define operations called *moves* on sketches to describe the equivalence classes. In regions of $\mathcal{A}$, moves correspond to crossing hyperplanes in $\mathcal{A} \setminus \mathcal{B}$ (see Figure 3).

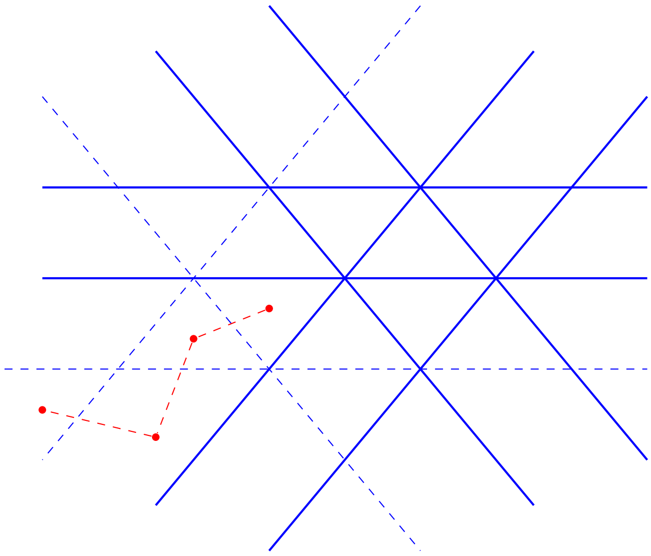


Figure 3: Bold lines form $\mathcal{B}$ and the dotted lines form $\mathcal{A} \setminus \mathcal{B}$. Equivalent $\mathcal{A}$ regions can be connected by changing one $\mathcal{A} \setminus \mathcal{B}$ inequality at a time.

After obtaining a bijection using the idea described above, we give a combinatorial interpretation to the coefficients of the characteristic polynomials of these arrangements. The proofs of these interpretations use a result (Proposition 3.21) that gives a relation between the exponential generating functions for the characteristic polynomials and those for the number of regions. We only present these results for the type C and type D Catalan deformations in this chapter.

Counting regions

The type D Catalan arrangement is a sub-arrangement of that of type C. Hence, we are first going to study the regions of the type C Catalan arrangement and then use the 'sketches and moves' idea mentioned above to study the type D Catalan arrangement.

Type C Catalan

Fix $n \geqslant 1$. The type C Catalan arrangement $\mathcal{C}_n$ in $\mathbb{R}^n$ is the arrangement with hyperplanes

$$\begin{aligned} 2X_i &= -1, 0, 1 \\ X_i + X_j &= -1, 0, 1 \\ X_i - X_j &= -1, 0, 1 \end{aligned}$$

for all $1 \leqslant i < j \leqslant n$. Setting $X_i = x_i + \frac{1}{2}$, we get that the regions of $\mathcal{C}_n$ correspond to *valid total orders* on

$$\{x_i + s \mid i \in [n],\ s \in \{0,1\}\} \cup \{-x_i - s \mid i \in [n],\ s \in \{0,1\}\}$$

where $[n] := \{1, 2, \ldots, n\}$. A total order on these symbols is *valid* if there exists a point in $\mathbb{R}^n$ that satisfies it.

Such orders will be represented by using the symbol $\alpha_i^{(s)}$ for $x_i + s$ and $\alpha_{-i}^{(-s)}$ for $-x_i - s$ for all $i \in [n]$ and $s \in \{0,1\}$. Let $C(n)$ be the set

$$\{\alpha_i^{(s)} \mid i \in [n],\ s \in \{0,1\}\} \cup \{\alpha_i^{(s)} \mid -i \in [n],\ s \in \{-1,0\}\}.$$

Hence, we use orders on the letters of $C(n)$ to represent regions of $\mathcal{C}_n$.

Example 0.1. The total order

$$x_1 < -x_2 - 1 < x_1 + 1 < x_2 < -x_2 < -x_1 - 1 < x_2 + 1 < -x_1$$

is represented as $\alpha_1^{(0)}\ \alpha_{-2}^{(-1)}\ \alpha_1^{(1)}\ \alpha_2^{(0)}\ \alpha_{-2}^{(0)}\ \alpha_{-1}^{(-1)}\ \alpha_2^{(1)}\ \alpha_{-1}^{(0)}$.

Considering $-x_i$ as x_{-i}, the letter $\alpha_i^{(s)}$ represents $x_i + s$ for any $\alpha_i^{(s)} \in C(n)$. For any $\alpha_i^{(s)} \in C(n)$, we use $\overline{\alpha_i^{(s)}}$ to represent the letter $\alpha_{-i}^{(-s)}$, which we call the *conjugate* of $\alpha_i^{(s)}$. We now describe which orders on $C(n)$ correspond to regions of $\mathcal{C}_n$.

Definition 0.2. A *symmetric sketch* is an order on the letters in $C(n)$ such that the following hold for any $\alpha_i^{(s)}, \alpha_j^{(t)} \in C(n)$:

1. If $\alpha_i^{(s)}$ appears before $\alpha_j^{(t)}$, then $\overline{\alpha_j^{(t)}}$ appears before $\overline{\alpha_i^{(s)}}$.
2. If $\alpha_i^{(s-1)}$ appears before $\alpha_j^{(t-1)}$, then $\alpha_i^{(s)}$ appears before $\alpha_j^{(t)}$.
3. $\alpha_i^{(s-1)}$ appears before $\alpha_i^{(s)}$.

Proposition 0.3 (Proposition 2.8). *An order on the letters of $C(n)$ corresponds to a region of $\mathcal{C}_n$ if and only if it is a symmetric sketch.*

It is not too difficult to count symmetric sketches. In fact, one can show that a symmetric sketch corresponds to a pair consisting of a signed permutation and a certain lattice path (see Proposition 2.13). This allows us to count the regions of the type C Catalan arrangement.

Theorem 0.4. *The number of symmetric sketches and hence regions of $\mathcal{C}_n$ is*

$$2^n n! \binom{2n}{n}.$$

We use a certain objects called *labeled symmetric non-nesting partition* to represent symmetric sketches. We obtain a labeled symmetric non-nesting partition from a symmetric sketch by joining the letters $\alpha_i^{(0)}$ and $\alpha_i^{(1)}$ and similarly $\alpha_{-i}^{(-1)}$ and $\alpha_{-i}^{(0)}$ with arcs and replacing each letter in the sketch with its subscript. As can be seen in Figure 4, such diagrams are symmetric and have arcs that are non-nesting. The labels are also symmetric (with sign changes) about the center. Any such diagram is a labeled symmetric non-nesting partition and these objects are in bijection with symmetric sketches.

Example 0.5. To the symmetric sketch

$$\alpha_3^{(0)}\alpha_2^{(0)}\alpha_{-1}^{(-1)}\alpha_3^{(1)}\alpha_1^{(0)}\alpha_2^{(1)}|\alpha_{-2}^{(-1)}\alpha_{-1}^{(0)}\alpha_{-3}^{(-1)}\alpha_1^{(1)}\alpha_{-2}^{(0)}\alpha_{-3}^{(0)}$$

we associate the labeled symmetric non-nesting partition in Figure 4.

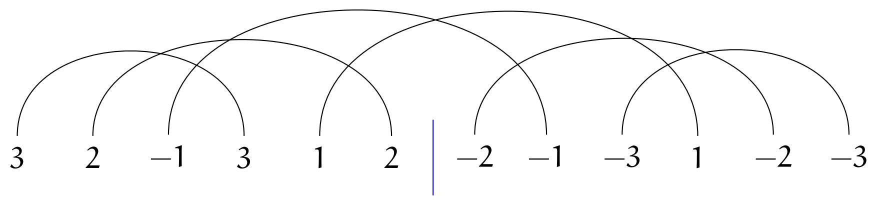


FIGURE 4: Labeled symmetric non-nesting partition associated to the symmetric sketch in Example 0.5.

Type D Catalan

Fix $n \geqslant 2$. The type D Catalan arrangement $\mathcal{D}_n$ in $\mathbb{R}^n$ has hyperplanes

$$X_i + X_j = -1, 0, 1$$
$$X_i - X_j = -1, 0, 1$$

for $1 \leqslant i < j \leqslant n$. Figure 5 shows $\mathcal{D}_2$ as a sub-arrangement of $\mathcal{C}_2$. It also shows how the regions of $\mathcal{D}_2$ partition the regions of $\mathcal{C}_2$.

We use the idea of moves to count the regions of $\mathcal{D}_n$ by considering it as a sub-arrangement of $\mathcal{C}_n$. The hyperplanes from $\mathcal{C}_n$ that are missing in $\mathcal{D}_n$ are

$$2X_i = -1, 0, 1$$

for all $i \in [n]$. We now describe the type D Catalan moves on symmetric sketches (regions of $\mathcal{C}_n$), which we call $\mathcal{D}$ moves. The $\mathcal{D}$ moves correspond to crossing exactly one hyperplane in $\mathcal{C}_n$ that is not in $\mathcal{D}_n$. Studying the correspondence between sketches and regions of $\mathcal{C}_n$, the $\mathcal{D}$ moves are:

1. Swapping the $2n^{\text{th}}$ and $(2n+1)^{\text{th}}$ letter.
2. Swapping $\alpha_i^{(0)}$ and $\alpha_{-i}^{(-1)}$ if they are adjacent (along with $\alpha_i^{(1)}$ and $\alpha_{-i}^{(0)}$) for some $i \in [n]$.

To count the regions of $\mathcal{D}_n$, we have to count the number of $\mathcal{D}$ equivalence classes of symmetric sketches where two sketches are $\mathcal{D}$ equivalent if one can be obtained from the other via a series of $\mathcal{D}$ moves. In Figure 5, the two labeled regions of $\mathcal{C}_2$ lie in the same region of $\mathcal{D}_2$ and hence are $\mathcal{D}$ equivalent. They are related by swapping of the fourth and fifth letters of their sketches, which is a $\mathcal{D}$ move.

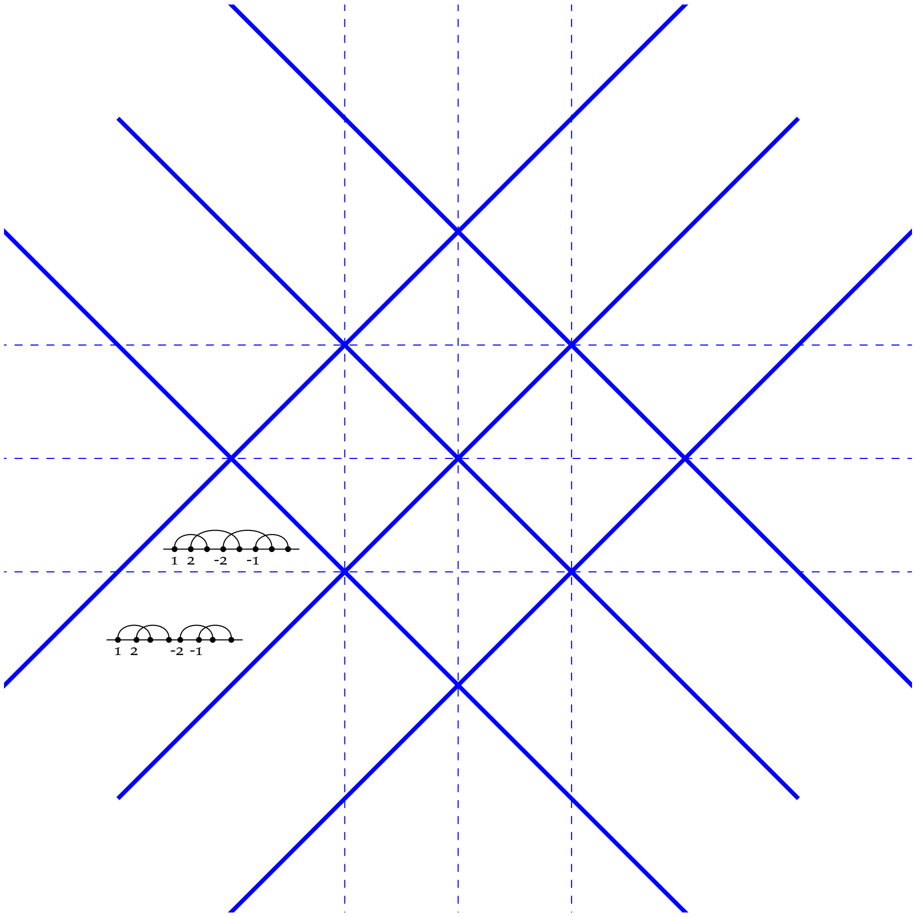


FIGURE 5: The arrangement $\mathcal{C}_2$ with the hyperplanes in $\mathcal{D}_2$ in bold. Two regions of $\mathcal{C}_2$ are labeled with their symmetric labeled non-nesting partitions.

The fact about these moves that helps with the count is that a series of $\mathcal{D}$ moves does not change the sketch too much. Hence we can list the sketches that are $\mathcal{D}$ equivalent to a given sketch and this allows us to count the number of $\mathcal{D}$ equivalence classes.

Theorem 0.6 (Theorem 2.30)**.** *The number of $\mathcal{D}$ equivalence classes on symmetric sketches and hence the number of regions of $\mathcal{D}_n$ is*

$$2^{n-1} \cdot \frac{(2n-2)!}{(n-1)!} \cdot (3n-2).$$

Although the number of regions of $\mathcal{D}_n$ (and other Catalan deformations) is well-known, the bijective proofs presented in Section 2.3 seem to be new.

Statistic on regions

Fix $n \geqslant 1$. We now describe a statistic on labeled symmetric non-nesting partitions whose distribution is given by the coefficients of $\chi_{\mathcal{C}_n}(t)$.

Given a labeled symmetric non-nesting partition, we first break it up into a 'bounded' part and two 'unbounded' parts. This is done as indicated in Figure 6. The bounded part is the interlinked piece of the non-nesting partition that crosses the center of symmetry. The remaining part of the diagram consists of two non-nesting partitions on each side of the bounded part. These arc diagrams are identical apart from their labels which are negatives of one another. These form the unbounded parts.

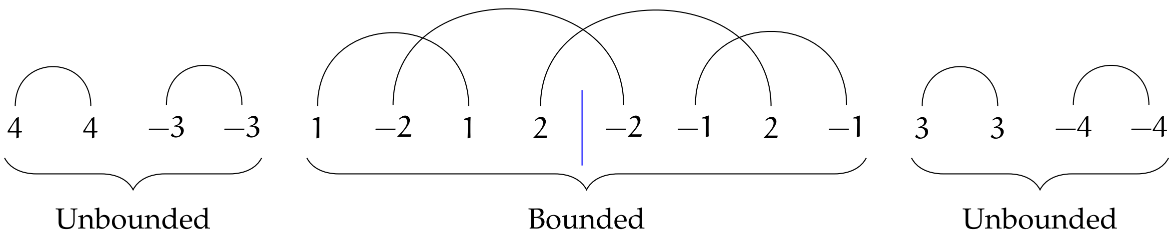


FIGURE 6: Bounded and unbounded parts of a symmetric arc diagram.

We now focus on just the unbounded part that is on the *right* of the bounded part. We break this unbounded part into *compartments* as follows: Ignore the signs of the labels. Find the interlinked piece with the smallest label. That interlinked piece along with all interlinked pieces to its left form the first compartment. Now delete the first compartment, and repeat the same procedure to obtain the second compartment. This process is repeated until the entire diagram is broken into compartments.

A *positive compartment* is one whose last element has a positive label. For example, the labeled symmetric non-nesting partition in Figure 6 has two compartments only one of which is positive.

Example 0.7. Suppose the arc diagram in Figure 7 is the unbounded part on the right side of some symmetric non-nesting partition. This diagram has two compartments. The first consists of the first interlinked piece and the second consists of the second and third interlinked pieces. But only the first compartment is positive since its last element has label 6 which is positive.

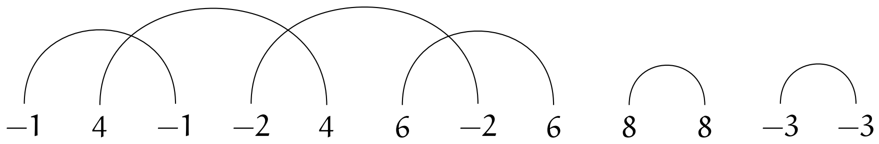


FIGURE 7: The unbounded part of a symmetric non-nesting partition that has 1 positive compartment.

Using Proposition 3.21, properties of labeled symmetric non-nesting partitions, and combinatorial operations on exponential generating functions, we get the following result.

Theorem 0.8 (Theorem 3.28). *The absolute value of the coefficient of* t^j *in* $\chi_{\mathcal{C}_n}(t)$ *is the number of labeled symmetric non-nesting partitions with* j *positive compartments.*

A similar statistic works for the type D Catalan arrangement. Recall that each region of $\mathcal{D}_n$ corresponds to a $\mathcal{D}$ equivalence class (which is a collection of regions of $\mathcal{C}_n$). One can choose a region of $\mathcal{C}_n$ (equivalently, a labeled symmetric non-nesting partition) from each $\mathcal{D}$ equivalence class to be its representative. We call these representatives 'type D labeled symmetric non-nesting partitions'. These representatives can be chosen a such a way that the following result holds.

Theorem 0.9 (Theorem 3.30). *The absolute value of the coefficient of* t^j *in* $\chi_{\mathcal{D}_n}(t)$ *is the number of type* D *labeled symmetric non-nesting partitions with* j *positive compartments.*

0.2 Pattern Avoidance in Circular Permutations

For any $n \geqslant 1$, we denote the set of all circular permutations of $[n] := \{1, 2, \ldots, n\}$ by $[\mathfrak{S}_n]$. For example, $[\mathfrak{S}_3] = \{[123], [132]\}$. For a given set $\{[\pi_1], \ldots, [\pi_k]\}$ of circular permutations, we say that $[\sigma]$ avoids $\{[\pi_1], \ldots, [\pi_k]\}$ if $[\sigma]$ avoids $[\pi_i]$ for each $i \in [k]$. For simplicity, we use $[\pi_1, \ldots, \pi_k]$ to denote this set of patterns. The set of permutations in $[\mathfrak{S}_n]$ that avoid $[\pi_1, \ldots, \pi_k]$ is denoted by $\mathrm{Av}_n[\pi_1, \ldots, \pi_k]$, *i.e.*,

$$\mathrm{Av}_n[\pi_1, \ldots, \pi_k] = \{[\sigma] \in [\mathfrak{S}_n] : [\sigma] \text{ avoids } [\pi_i] \text{ for each } 1 \leqslant i \leqslant k\}.$$

Also, $\mathrm{Av}[\pi_1, \ldots, \pi_k]$ will denote the set of *all* circular permutations avoiding $[\pi_1, \ldots, \pi_k]$. If $[\pi_i]$ contains $[\pi_j]$ for some distinct $i, j \in [k]$, then omitting $[\pi_i]$ from the sets of patterns does not affect the avoidance class. Hence, we can assume that the permutations in any set of patterns avoid each other.

An important notion in the study of pattern avoidance is the Wilf equivalence on sets of patterns. Two sets $[\pi_1, \ldots, \pi_k]$ and $[\tau_1, \ldots, \tau_\ell]$ of circular permutations are called *(circular) Wilf equivalent*, denoted by $[\pi_1, \ldots, \pi_k] \equiv [\tau_1, \ldots, \tau_\ell]$, if $\#\mathrm{Av}_n[\pi_1, \ldots, \pi_k] = \#\mathrm{Av}_n[\tau_1, \ldots, \tau_\ell]$ for each $n \geqslant 1$. Here, # stands for the cardinality of a set. For $[\pi] = [\pi_1 \cdots \pi_n]$, the *trivial Wilf equivalences* are those of the form

$$[\pi] \equiv [\pi^r] \equiv [\pi^c] \equiv [\pi^{rc}]$$

where $[\pi^r] = [\pi_n \cdots \pi_1]$ is the *reversal* of $[\pi]$, $[\pi^c] = [(n+1-\pi_1)\cdots(n+1-\pi_n)]$ is the *complement* of $[\pi]$ and $[\pi^{rc}] = [(n+1-\pi_n)\cdots(n+1-\pi_1)]$ is the *reverse complement* of $[\pi]$. Similarly, we have trivial Wilf equivalences on sets of patterns. For example, $[1342, 12345] \equiv [1342^r, 12345^r] = [1243, 15432]$ is a trivial Wilf equivalence.

We study circular permutations avoiding two patterns $\{[\sigma], [\tau]\}$, where $[\sigma]$ is of size 4 and $[\tau]$ is of size k. For simplicity, we say that such pairs of patterns are $[4, k]$-pairs. Observe that, using trivial Wilf equivalences among circular permutations of size 4, it is enough to study those pairs where the pattern of size 4 is $[1342]$, $[1324]$, or $[1432]$.

Avoiding $[1342, k]$-pairs

Our first collection of results involves the study of circular permutations avoiding pairs $[1342, \tau]$ where $[\tau] \in \mathrm{Av}[1342]$. We show that the circular permutations avoiding $[1342]$ are in bijection with binary words, subject to equivalences given by

$$0^{a+1}1^b \sim 1^{b+1}0^a \text{ for all } a, b \geqslant 0.$$

We also show that pattern containment in the permutations corresponds to subsequence containment in the corresponding binary words. These results are the cyclic analogue of the results in [46]. In this case, we obtain the exact number of Wilf equivalence classes.

Theorem 0.10 (Theorem 8.10)**.** *For any $k \geqslant 4$, there are exactly $\lceil \frac{k}{2} \rceil$ Wilf equivalence classes of $[1342, k]$-pairs.*

We also obtain closed form formulas for the sequence $(\#\mathrm{Av}_n[1342,\sigma])_{n\geqslant 1}$ for various $[\sigma] \in \mathrm{Av}[1342]$. For example, using ι_n and δ_n to denote the increasing permutation $12\cdots n$ and decreasing permutation $n\cdots 21$ respectively, we have the following result.

Theorem 0.11 (Proposition 8.11)**.** *For any* $k \geqslant 1$*, we have* $[1342,\iota_{k+1}] \equiv [1342,\delta_{k+1}]$ *and for any* $n \geqslant k$*,*

$$\#\mathrm{Av}_{n+1}[1342,\iota_{k+1}] = \binom{n-1}{k-2} - (k-1) + \sum_{i=0}^{k-2}\binom{n}{i}.$$

Avoiding $[1324, k]$-pairs

Next, we focus on pairs of the form $[1324,\sigma]$ where $[\sigma] \in \mathrm{Av}[1324]$. To study avoidance of such pairs, we first establish a bijection between the elements of $\mathrm{Av}_n[1324]$ and *circled compositions* of n (see Theorem 7.17).

Definition 0.12. A *circled composition* of n is a pair (a, C) where $a = (a_1,\ldots,a_k)$ is a composition of n with k parts, and C is a subset of $[k]$ such that

1. both 1 and k are contained in C, and
2. for any $i \in C$, we have $a_i = 1$.

A circled composition (a, C) is represented by writing the composition a and circling all the terms of a whose indices are in C.

We also define a notion of pattern avoidance in circled compositions that coincides with pattern avoidance in the corresponding circular permutations (see Definition 8.18 and Theorem 8.20). We then prove various Wilf equivalences among circled compositions and therefore among $[1324, k]$-pairs. These equivalences can be summarized as follows.

Theorem 0.13 (Theorem 8.32)**.** *Any circled composition of* n *is Wilf equivalent to a circled composition of* n *of one of the following forms.*

1. *The circled composition* $\textcircled{1}^{n}$.

2. *A circled composition*

$$\textcircled{1}^{k_0} \quad a_1 \quad a_2 \quad \cdots \quad a_k \quad \textcircled{1}^{k_1} \quad a_{k+1} \quad \textcircled{1}^{k_2}$$

where $k_0 \geqslant k_2$, $a_1 \geqslant a_2 \geqslant \cdots \geqslant a_k \geqslant a_{k+1}$, *and* $a_1, \ldots, a_{k+1}, k_0, k_2 \neq 2$.

3. *A circled composition*

$$\textcircled{1}^{k_0} \quad a_1 \quad a_2 \quad \cdots \quad a_k \quad \textcircled{1}^{k_2}$$

where $k_0 \geqslant k_2$, $k_0, k_2 \neq 2$, $a_1 \geqslant a_2 \geqslant \cdots \geqslant a_k$, *and if* $k \geqslant 2$, *then* $a_1, \ldots, a_k \neq 2$.

As in the case of [1342], we have closed form formulas for the sequence $(\#\mathrm{Av}_n[1324, \sigma])_{n \geqslant 1}$ for various $[\sigma] \in \mathrm{Av}[1324]$. For example, we have the following result.

Theorem 0.14 (Proposition 8.33)**.** *For* $n \geqslant 2$ *and* $k \geqslant 1$,

$$\#\mathrm{Av}_n[1324, \delta_{k+2}] = \sum_{i=0}^{k-1} \binom{n-2+i}{2i}.$$

Avoiding [1432, k]-pairs

Our final set of results is about pairs of patterns of the form $[1432, \sigma]$ for $[\sigma] \in \mathrm{Av}[1432]$. We use the characterization of Av[1432] given in [58].

Definition 0.15. A *Grassmannian permutation* is a permutation which has at most one descent.

Combining [58, Corollary 2.10] and [58, Proposition 3.6], we get the following result.

Theorem 0.16 ([58])**.** *Let* $[\sigma] \in [\mathfrak{S}_n]$ *be a permutation written so that* σ *ends with* n. *Then* $[\sigma]$ *avoids* [1432] *if and only if* σ *is either a Grassmannian permutation or the inverse of a Grassmannian permutation.*

We represent both Grassmannian permutations as well as their inverses as binary words starting with 0 and describe the analogue for pattern avoidance in such words. Note that such words can be represented as compositions and we define $B(n_1, n_2, \ldots)$

to be the binary word $0^{n_1}1^{n_2}\cdots$. The number of *runs* of a binary word is the number of parts of the corresponding composition. For a given binary word w, we use $[G(w)]$ to represent the corresponding Grassmannian permutation and $[IG(w)]$ for the corresponding inverse Grassmannian permutation.

We have the following result on Wilf equivalence among $[1432, k]$-pairs.

Theorem 0.17 (Theorem 8.56)**.** *Any pair $[1432, \sigma]$ is Wilf equivalent to a pair $[1432, \tau]$ where $[\tau]$ has one of the following forms:*

1. *$[G(w)]$ where $w = 0^a1^b0^c$ where $a \geqslant c$.*
2. *$[G(w)]$ where w is an alternating binary word starting with 0 having at least 4 runs.*
3. *$[G(w)]$ where $w = B(n_1, n_2, \ldots, n_k, 1^r)$ has at least 4 runs, $r \geqslant 0$, $n_1, n_k \neq 1$, and $(n_k, \ldots, n_2, n_1) \leqslant_{lex} (n_1, n_2, \ldots, n_k)$. Here $\leqslant_{lex}$ denotes the lexicographic order.*
4. *$[IG(w)]$ where $w = 0^{n_1}1^{n_2}\cdots 1^{n_k}0^m$ is not an alternating binary word, has at least 5 runs, $m \geqslant 1$, and $(n_k, \ldots, n_2, n_1) \leqslant_{lex} (n_1, n_2, \ldots, n_k)$.*

We also obtain formulas and generating functions for avoidance class sizes of various $[1432, k]$-pairs. For example, we prove the following result.

Theorem 0.18 (Proposition 8.57)**.** *For any binary word w of length k, starting with 0, having at least 5 runs, and ending with 1, we have for $n \geqslant 5$,*

$$\#\mathrm{Av}_n[1432, IG(w)] = 2^{n-1} - (n-1) + \sum_{i=4}^{k-2} \binom{n-1}{i}.$$

In the specific case $k = 5$, using the results from the three types of avoidance mentioned above, we are able to enumerative all avoidance classes for $[4, k]$-pairs (see Table 9.1). This also gives us that there are exactly 14 Wilf equivalence classes of $[4, 5]$-pairs.

0.3 Organization of the thesis

In this section we give an overview of the chapters that follow.

Part I

1. In Chapter 1, we cover the basic definitions and important results that we will need in the topic of hyperplane arrangements. We also bijectively count the regions as well as give combinatorial interpretations to the coefficients of the characteristic polynomials for some not-too-complicated arrangements.

2. In Chapter 2, we first recall the 'sketches and moves' idea to bijectively count regions of arrangements. We then use this idea to obtain non-nesting partitions that correspond to the regions of Catalan deformations of reflection arrangements. We also directly count these objects and hence obtain bijective proofs for the number of regions of these arrangements.

3. In Chapter 3, we interpret the coefficients of the characteristic polynomials of various arrangements by defining appropriate statistics on their regions. We start with doing so for a large class of deformations of the braid arrangement, which were studied in great detail by Bernardi [8]. We then do so for the Catalan deformations studied in Chapter 2. We use combinatorial interpretations of the exponential generating function of these characteristic polynomials to define our statistics.

4. In Chapter 4, we use ideas similar to the ones in the previous chapters to study another class of arrangements called deformations of the threshold arrangement.

5. We end this part of the thesis with some directions for future research in Chapter 5.

Part II

6. In Chapter 6, we define the notion of patterns in permutations and mention a few basic results on permutation patterns. We then cover the basic definitions and background we will need to study pattern avoidance in circular permutations.

7. In Chapter 7, we study avoidance of a single circular pattern of size 4. We split our results into three sections based on the pattern being avoided. For each case, we show that the circular permutations in the avoidance classes can be represented using well-known combinatorial objects such as binary words and compositions. We also reprove some results from [21] using these representations.

8. In Chapter 8, we use the representations developed in Chapter 7 to study avoidance of $[4, k]$-pairs in circular permutations. Just as before, we split our results into three sections based on the pattern of size 4. In each section we use our general results to study the particular case of $k = 5$, i.e., avoidance of $[4, 5]$-pairs.

9. Finally, we end with Chapter 9 where some directions for future research are mentioned.

Part I

Hyperplane Arrangements

Chapter 1

Preliminaries

In this chapter, we will cover the basic definitions and results related to hyperplane arrangements. The interested reader is referred to [54] for more information. We also assume reader's familiarity with the notion of posets, formal power series, and related terminologies. The main reference for which is [55].

For any two integers a, b, we use $[a, b]$ to denote the set $\{c \in \mathbb{Z} \mid a \leqslant c \leqslant b\}$. For a positive integer n, for brevity, we use $[n]$ to denote $[1, n]$.

1.1 Basic definitions

Definition 1.1 (Hyperplane arrangement)**.** A *hyperplane arrangement* is a finite set of affine hyperplanes in $\mathbb{F}^n$, where $\mathbb{F}$ is a field. An affine hyperplane is a translate of a codimension 1 subspace of $\mathbb{F}^n$.

We sometimes write just 'arrangement' instead of 'hyperplane arrangement'. We will be mainly focused on when $\mathbb{F} = \mathbb{R}$.

Definition 1.2 (Region)**.** A *region* of an arrangement $\mathcal{A}$ in $\mathbb{R}^n$ is a connected component of $\mathbb{R}^n \setminus \bigcup_{H \in \mathcal{A}} H$. The number of regions of $\mathcal{A}$ is denoted by $r(\mathcal{A})$.

Definition 1.3 (Rank)**.** The *rank* of an arrangement $\mathcal{A}$, denoted by $\text{rank}(\mathcal{A})$, is the dimension of the space spanned by the normals to its hyperplanes.

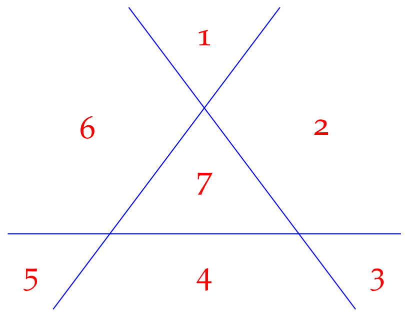


FIGURE 1.1: An arrangement in $\mathbb{R}^2$ with 7 regions.

Definition 1.4 (Bounded region)**.** A region of an arrangement $\mathcal{A}$ is said to be *bounded* if its intersection with the span of the normals of the hyperplanes in $\mathcal{A}$ is bounded. The number of bounded regions of $\mathcal{A}$ is denoted by $b(\mathcal{A})$.

Example 1.5. Figure 1.1 shows an arrangement in $\mathbb{R}^2$ with 7 regions. It has 1 bounded region, which is labeled 7.

Definition 1.6 (Intersection poset)**.** The poset of non-empty intersections of hyperplanes in an arrangement $\mathcal{A}$, ordered by reverse inclusion, is called its *intersection poset*. It is denoted by $L_{\mathcal{A}}$.

The ambient space of the arrangement (i.e. $\mathbb{R}^n$) is an element of the intersection poset. It is considered as the intersection of none of the hyperplanes.

Example 1.7. Note that the lines l_1 and l_3 in the second example of Figure 1.2 do not intersect. Such empty intersections are not included in $L_{\mathcal{A}}$.

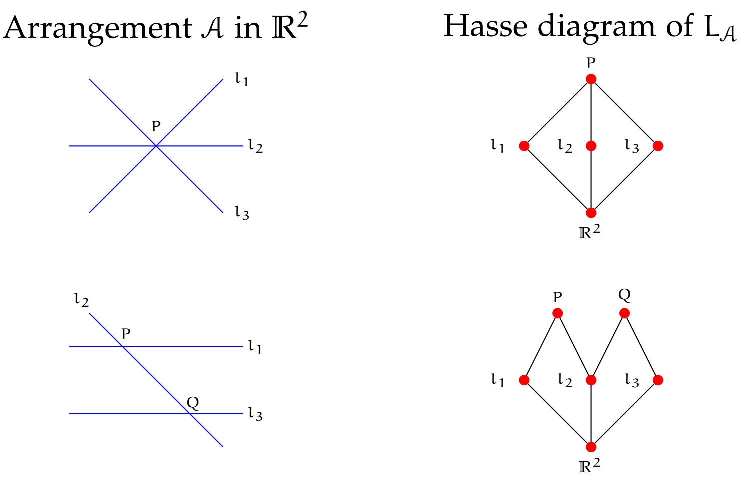


FIGURE 1.2: Examples of intersection posets.

Definition 1.8 (Möbius function)**.** For an arrangement $\mathcal{A}$ in $\mathbb{R}^n$, its *Möbius function* $\mu : L_{\mathcal{A}} \to \mathbb{Z}$ is given by

$$\mu(x) = \begin{cases} 1, & \text{if } x = \mathbb{R}^n \\ -\sum_{y<x} \mu(y), & \text{otherwise.} \end{cases}$$

Definition 1.9 (Characteristic polynomial)**.** The *characteristic polynomial* of an arrangement $\mathcal{A}$ is the generating function of the Möbius values of $L_{\mathcal{A}}$ weighted by dimension, *i.e.*

$$\chi_{\mathcal{A}}(t) := \sum_{x \in L_{\mathcal{A}}} \mu(x) t^{\dim(x)}.$$

Example 1.10. The numbers next to elements of $L_{\mathcal{A}}$ in Figure 1.3 are their Möbius values.

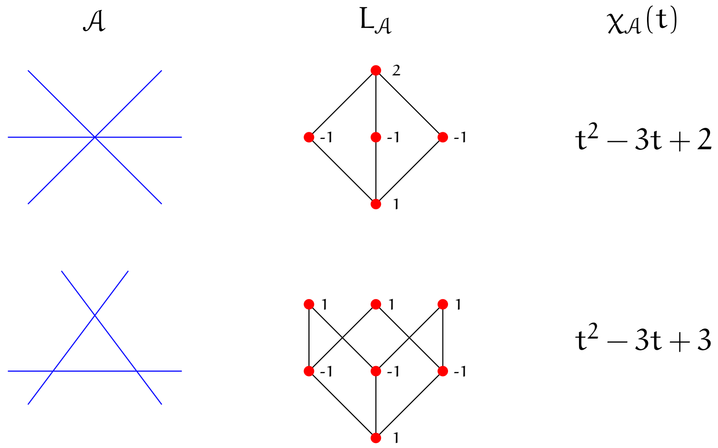


Figure 1.3: Examples of characteristic polynomials.

Definition 1.11 (Restriction)**.** The *restriction* of an arrangement $\mathcal{A}$ to some $x \in L_{\mathcal{A}}$ is the arrangement $\mathcal{A}^x$ in x with hyperplanes $\{H \cap x \mid H \in \mathcal{A},\ x \not\subseteq H\}$.

Definition 1.12 (Face)**.** A *face* of an arrangement $\mathcal{A}$ is a region of $\mathcal{A}^x$ for some $x \in L_{\mathcal{A}}$. The *dimension* of a face is the dimension of its affine span.

The regions of an arrangement are themselves faces (regions of $\mathcal{A}^{\mathbb{R}^n} = \mathcal{A}$). In fact, they are the maximum-dimensional faces.

Example 1.13. The numbers inside the circles in Figure 1.4 are the dimensions of the faces.

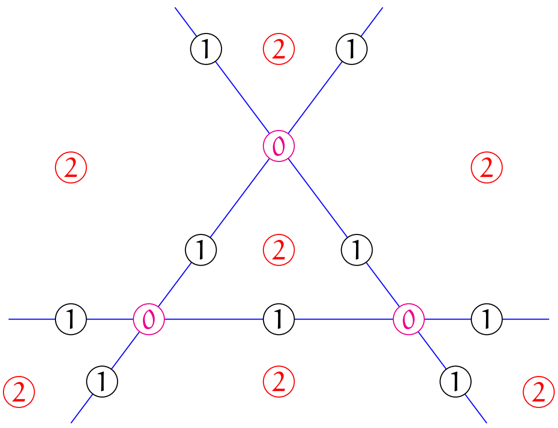


FIGURE 1.4: Faces of an arrangement in $\mathbb{R}^2$.

Before going further we note some results on arrangements that are consequences of basic Euclidean geometry.

- Any hyperplane H in $\mathbb{R}^n$ is a set of the form

$$\{\boldsymbol{x} \in \mathbb{R}^n \mid P_H(\boldsymbol{x}) = 0\}$$

where $P_H(\boldsymbol{x}) = a_1x_1 + a_2x_2 + \cdots + a_nx_n + c$ for some constants $a_1, \ldots, a_n, c \in \mathbb{R}$. We say that P_H is a *defining polynomial* of H. It is unique up to multiplication by a nonzero scalar.

- The regions of an arrangement $\mathcal{A}$ are precisely the non-empty intersections of sets of the form

$$\{\boldsymbol{x} \in \mathbb{R}^n \mid P_H(\boldsymbol{x}) > 0\} \text{ or } \{\boldsymbol{x} \in \mathbb{R}^n \mid P_H(\boldsymbol{x}) < 0\}$$

where we have one such set for each $H \in \mathcal{A}$.

- The faces of an arrangement $\mathcal{A}$ are precisely the non-empty intersections of sets of the form

$$\{\boldsymbol{x} \in \mathbb{R}^n \mid P_H(\boldsymbol{x}) = 0\} \text{ or } \{\boldsymbol{x} \in \mathbb{R}^n \mid P_H(\boldsymbol{x}) > 0\} \text{ or } \{\boldsymbol{x} \in \mathbb{R}^n \mid P_H(\boldsymbol{x}) < 0\}$$

where we have one such set for each $H \in \mathcal{A}$.

We will focus on regions of arrangements and will not be dealing with faces. Results on faces of arrangements can be found in [37] and the references therein.

1.2 Important results

The first major theorem in the theory of arrangements was due to Zaslavsky in 1975 [60].

Theorem 1.14. *Let $\mathcal{A}$ be an arrangement in $\mathbb{R}^n$. The number of regions of $\mathcal{A}$ is given by*

$$r(\mathcal{A}) = (-1)^n \chi_{\mathcal{A}}(-1)$$

and similarly, the number of bounded regions is given by

$$b(\mathcal{A}) = (-1)^{\text{rank}(\mathcal{A})} \chi_{\mathcal{A}}(1).$$

Remark 1.15. The above theorem can be proved by induction on the number of hyperplanes in the arrangement using Deletion-Restriction arguments.

To apply combinatorial methods, we will be focused on certain "nice" arrangements.

Definition 1.16 (Rational arrangement)**.** An arrangement in $\mathbb{R}^n$ with every hyperplane H having a defining polynomial P_H in $\mathbb{Z}[x_1, \dots, x_n]$ is called a *rational arrangement*.

Even if $P_H \in \mathbb{Q}[x_1, \dots, x_n]$, we can multiply it by an integer to obtain an integer-coefficient defining polynomial for H, which explains the term 'rational arrangement'. Also, for such arrangements we can obtain related arrangements in vector spaces over finite fields.

Definition 1.17 (Reduction mod q)**.** Let $\mathcal{A}$ be a rational arrangement in $\mathbb{R}^n$. For any prime q, we obtain an arrangement $\mathcal{A}_q$ in $\mathbb{Z}_q^n$ by reducing mod q the coefficients of the defining polynomials of the hyperplanes in $\mathcal{A}$.

We now have the vocabulary required to state a very convenient method to compute the characteristic polynomials of rational arrangements. This method was developed by Athanasiadis in 1996 [3].

Theorem 1.18 (The finite field method)**.** *Let $\mathcal{A}$ be a rational hyperplane arrangement in $\mathbb{R}^n$. For large primes q,*

$$\chi_{\mathcal{A}}(q) = \#(\mathbb{Z}_q^n \setminus \bigcup_{H \in \mathcal{A}_q} H).$$

Note that a polynomial of degree n is determined by its value at $n+1$ points. Hence, the finite field method converts the problem of calculating the characteristic polynomial of rational arrangements to a counting problem. Combined with Zaslavsky's theorem, we get a nice method of getting the number of regions of rational arrangements.

1.3 Main question

In this section, we discuss the main problem we will be addressing in the upcoming chapters: 'Interpreting coefficients of the characteristic polynomial'. To understand what this means, we first state a nice property of the characteristic polynomial of an arrangement.

Proposition 1.19. *[54, Corollary 3.4] For any arrangment $\mathcal{A}$ in $\mathbb{R}^n$, we have*

$$\chi_{\mathcal{A}}(t) = \sum_{i=0}^{n} (-1)^{n-i} c_i t^i$$

where c_i is a non-negative integer for all $0 \leqslant i \leqslant n$.

Remark 1.20. The above result follows using the fact that every interval of the intersection poset of an arrangement is a geometric lattice.

Combining this result with Zaslavsky's theorem, we see that the coefficients of the characteristic polynomial of an arrangement give a breakup of the number of its regions. Precisely, we have

$$r(\mathcal{A}) = (-1)^n \chi_{\mathcal{A}}(-1) = \sum_{i=0}^{n} c_i.$$

Hence, one could ask if there is a *statistic* on the regions whose distribution is given by the coefficients of the characteristic polynomial. That is, one could ask for a nice way to assign a number from $[0, n]$ to each region of $\mathcal{A}$ such that there are precisely c_i regions that get assigned the number i for each $i \in [0, n]$.

For the arrangements we study in this thesis, we first define certain combinatorial objects that correspond to the regions of the arrangement and then define an appropriate statistic on these objects. For most of this thesis, we focus on an interesting class of arrangements called *Catalan deformations of reflection arrangements*.

The Catalan arrangement of type A in $\mathbb{R}^n$ is given by

$$\mathcal{A}_n = \{x_i - x_j = -1, 0, 1 \mid 1 \leqslant i < j \leqslant n\}.$$

This arrangement and its sub-arrangements have been studied in great detail (for example, see [8]). It is well-known that the number of regions of $\mathcal{A}_n$ where $x_1 < x_2 < \cdots < x_n$ (also known as the dominant regions) is given by the Catalan number $\frac{1}{n+1}\binom{2n}{n}$.

Using this, it is easy to see that

$$r(\mathcal{A}_n) = \frac{n!}{n+1}\binom{2n}{n}.$$

Let Φ be a (not necessarily reduced) crystallographic root system and let Φ^+ be a choice of positive roots (relevant definitions can be found in [31]). The reflection (or Coxeter) arrangement $\mathcal{A}(\Phi)$ corresponding to Φ consists of hyperplanes with the defining equations

$$(\alpha, x) = 0 \quad \text{for } \alpha \in \Phi^+.$$

Note that these are the same hyperplanes that are fixed by the Weyl group of Φ.

A *deformation* of an arrangement $\mathcal{A}$ is an arrangement each of whose hyperplanes is parallel to some hyperplane in $\mathcal{A}$. Our main focus is certain deformations of reflection arrangements called *Catalan deformations*. For brevity, we sometimes write Catalan arrangement of type Φ. We have already defined the Catalan arrangement of type A above. The defining equations of Catalan arrangements of other types are as follows:

- The Catalan arrangement of type B in $\mathbb{R}^n$ is given by

$$\{x_i = -1, 0, 1 \mid i \in [n]\} \cup \{x_i + x_j = -1, 0, 1 \mid 1 \leqslant i < j \leqslant n\} \cup \mathcal{A}_n.$$

- The Catalan arrangement of type C in $\mathbb{R}^n$ is given by

$$\{2x_i = -1, 0, 1 \mid i \in [n]\} \cup \{x_i + x_j = -1, 0, 1 \mid 1 \leqslant i < j \leqslant n\} \cup \mathcal{A}_n.$$

- The Catalan arrangement of type D in $\mathbb{R}^n$ is given by

$$\{x_i + x_j = -1, 0, 1 \mid 1 \leqslant i < j \leqslant n\} \cup \mathcal{A}_n.$$

- The Catalan arrangement of type BC in $\mathbb{R}^n$ (defined in [5]) is the union of the type B and type C Catalan arrangements in $\mathbb{R}^n$.

In addition to these, we also consider the *extended* Catalan arrangements of type C; consisting of hyperplanes of the form $(\alpha, x) = k$ for $k \in [-m, m]$ for a fixed integer $m \geqslant 1$.

1.4 Examples

Before moving on to Catalan deformations, we first interpret the coefficients of the characteristic polynomials of some fairly simple arrangements. These arrangements have nice combinatorial objects that correspond to their regions. We define an appropriate statistic on these objects whose distribution is given by the characteristic polynomial of the corresponding arrangement.

Boolean arrangement

The Boolean arrangement is one of the first examples one encounters when studying hyperplane arrangements. The Boolean arrangement in $\mathbb{R}^n$ consists of the coordinate hyperplanes, *i.e.*,

$$\mathcal{B}_n = \{x_i = 0 \mid i \in [n]\}.$$

Any region of $\mathcal{B}_n$ is of the form

$$R_A := \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \mid x_i > 0 \text{ for } i \in A \text{ and } x_i < 0 \text{ for } i \in [n] \setminus A\}$$

for some $A \subseteq [n]$. Note that R_A is non-empty for any subset A of $[n]$. Hence, the regions of $\mathcal{B}_n$ correspond to subsets of $[n]$ and we have $r(\mathcal{B}_n) = 2^n$.

To compute the characteristic polynomial, we use the finite field method (Theorem 1.18). This tells us the for large primes q, we have

$$\begin{aligned}\chi_{\mathcal{B}_n}(q) &= \#\{(x_1, x_2, \dots, x_n) \in \mathbb{Z}_q^n \mid x_i \neq 0 \text{ for all } i \in [n]\} \\ &= (q-1)^n.\end{aligned}$$

This shows us that the characteristic polynomial of $\mathcal{B}_n$ is

$$\begin{aligned}\chi_{\mathcal{B}_n}(t) &= (t-1)^n \\ &= \sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} t^i.\end{aligned}$$

The statistic 'number of elements' on the set of all subsets of $[n]$ has distribution given by the coefficients of $\chi_{\mathcal{B}_n}(t)$. This is because the number of subsets of $[n]$ with i elements is $\binom{n}{i}$, which is the absolute value of the coefficient of t^i in $\chi_{\mathcal{B}_n}(t)$.

Grid arrangement

We now consider a deformation of the Boolean arrangement. Given a positive integer m, we define the *grid arrangement* in $\mathbb{R}^n$ as

$$\mathcal{B}_{m,n} = \{x_i = 0, 1, \dots, m-1 \mid i \in [n]\}.$$

This is an arrangement with mn hyperplanes, each of which is parallel to some coordinate hyperplane.

The regions of this arrangement are in bijection with tuples of the form $(c_1, c_2, \dots, c_n)$ where $c_i \in [0, m]$ for each $i \in [n]$. Here, for each $i \in [n]$, c_i specifies the inequalities for x_i:

1. If $c_i = 0$, we set $x_i < 0$.
2. If $c_i \in [m-1]$, we set $x_i < c_i$ and $x_i > c_i - 1$.
3. If $c_i = m$, we set $x_i > m-1$.

Hence, we have $r(\mathcal{B}_{m,n}) = (m+1)^n$.

By Theorem 1.18, we have that for large primes q,

$$\begin{aligned}\chi(\mathcal{B}_{m,n})(q) &= \#\{(x_1, x_2, \dots, x_n) \in \mathbb{Z}_q^n \mid x_i \neq 0, 1, \dots, m-1 \text{ for all } i \in [n]\} \\ &= (q-m)^n.\end{aligned}$$

This shows that the characteristic polynomial of the grid arrangement is

$$\begin{aligned}\chi(\mathcal{B}_{m,n})(t) &= (t-m)^n \\ &= \sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} m^{n-i} t^i.\end{aligned}$$

The statistic 'number of 0s' on the tuples we used to represent the regions has distribution given by the coefficients of $\chi(\mathcal{B}_{m,n})(t)$. This is because there are precisely $\binom{n}{i} m^{n-i}$ tuples where the number of 0s is exactly i.

Braid arrangement

The braid arrangement in $\mathbb{R}^n$ is given by

$$\mathcal{B}_n = \{x_i - x_j = 0 \mid 1 \leqslant i < j \leqslant n\}.$$

Note that this is the reflection arrangement corresponding to the root system A_{n-1}.

Any region of $\mathcal{B}_n$ is of the form

$$\{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n \mid x_{\sigma(1)} < x_{\sigma(2)} < \dots < x_{\sigma(n)}\}$$

for some permutation σ of $[n]$. This is because validly choosing $x_i < x_j$ or $x_i > x_j$ for all $1 \leqslant i < j \leqslant n$ gives a total order on the coordinates. So we see that there is a bijection between the regions of $\mathcal{B}_n$ and the permutations of $[n]$ and hence $r(\mathcal{B}_n) = n!$.

By Theorem 1.18, for large primes q, we have

$$\begin{aligned}\chi_{\mathcal{B}_n}(q) &= \#\{(x_1, x_2, \dots, x_n) \in \mathbb{Z}_q^n \mid x_i \neq x_j \text{ for all distinct } i, j \in [n]\} \\ &= q \times (q-1) \times (q-2) \times (q-(n-1)).\end{aligned}$$

This shows that the characteristic polynomial of the braid arrangement is $\chi_{\mathcal{B}_n}(t) = \prod_{i=0}^{n-1}(t-i)$.

It is not too difficult to prove that the coefficient of t^i in $\prod_{i=0}^{n-1}(t+i)$ is the number of permutations on $[n]$ that have i cycles (for example, see [55, Proposition 1.3.7]). Hence, the distribution of the statistic 'number of cycles' on the set of all permutations of $[n]$ is given by the coefficients of $\chi_{\mathcal{B}_n}(t)$.

Chapter 2

Sketches, moves, and partitions

In this chapter, we describe combinatorial objects (certain labeled non-nesting partitions) that correspond to regions of the Catalan deformations of various types. The idea we use the obtain these bijections, which we call 'sketches and moves', was used by Bernardi [8, Section 8] to study deformations of the braid arrangement.

The results in this chapter are from [17, Sections 2–4], which is joint work with Priyavrat Deshpande.

2.1 Sketches and moves

In his paper [8], Bernardi describes a method to count the regions of any deformation of the braid arrangement using certain objects called *boxed trees*. He also obtains explicit bijections with certain trees for several deformations. The general strategy to establish the bijection is to consider an arrangement $\mathcal{B}$ whose regions we wish to count as a sub-arrangement of an arrangement $\mathcal{A}$ whose regions are well-understood. The regions of $\mathcal{B}$ then define an equivalence on the regions of $\mathcal{A}$. This is done by declaring two regions of $\mathcal{A}$ to be equivalent if they lie inside the same region of $\mathcal{B}$. Now counting the number of regions of $\mathcal{B}$ is the same as counting the number of equivalence classes of this equivalence on the regions of $\mathcal{A}$. This is usually done by choosing a canonical representative for each equivalence class, which also gives a bijection between the regions of $\mathcal{B}$ and certain regions of $\mathcal{A}$.

In particular, a (transitive) deformation of the braid arrangement is a sub-arrangement of the (extended or) m-Catalan arrangement (for some large m) in $\mathbb{R}^n$, whose hyperplanes are

$$\{x_i - x_j = k \mid 1 \leqslant i < j \leqslant n,\ k \in [-m, m]\}.$$

The regions of these arrangements are known to correspond labeled $(m+1)$-ary trees with n nodes (see [8, Section 8.1]). Using the idea mentioned above, one can show that the regions a deformation correspond to certain trees. We should mention that while he obtains direct combinatorial arguments to describe this bijection for some transitive deformations (see [8, Section 8.2]), the proof for the general bijection uses much stronger results (see [8, Section 8.3]).

Coming back to the general strategy, which we aim to generalize in order to apply it to deformations of other types. It is clear that any two equivalent regions of $\mathcal{A}$ have to be on the same side of each hyperplane of $\mathcal{B}$. However, it turns out that this equivalence is the transitive closure of a simpler relation. This follows from the fact that one can reach a region in an arrangement from another by crossing exactly one hyperplane at a time with respect to which the regions lie on opposite sides. We now prove this result, for which we require the following definition.

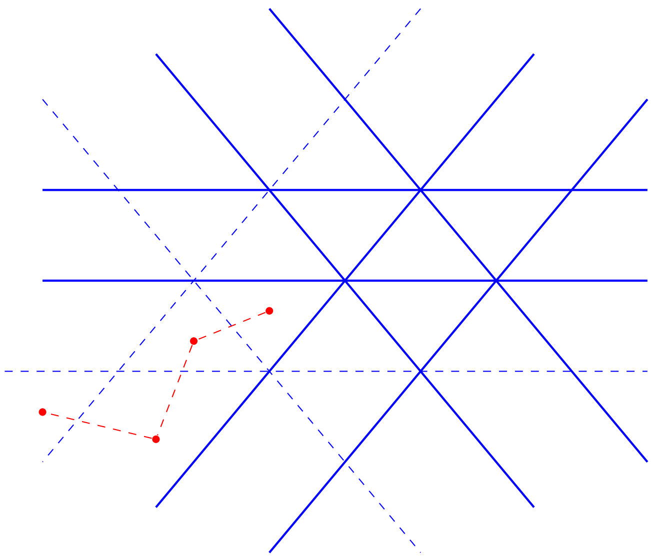


FIGURE 2.1: Bold lines form $\mathcal{B}$ and the dotted lines form $\mathcal{A} \setminus \mathcal{B}$. Equivalent $\mathcal{A}$ regions can be connected by changing one $\mathcal{A} \setminus \mathcal{B}$ inequality at a time.

Definition 2.1. Let R be a region of an arrangement $\mathcal{A}$. A *determining set* of R is a sub-arrangement $\mathcal{D} \subseteq \mathcal{A}$ such that the region of the arrangement $\mathcal{D}$ containing R, denoted $R_{\mathcal{D}}$, is equal to R.

Note that a region of $\mathcal{A}$ always has the entire arrangement $\mathcal{A}$ as a determining set. Also, if a region R' is on the same side as a region R for each hyperplane in a determining set of R, then we must have $R = R'$.

Before going forward, we recall the explicit description of regions of an arrangement mentioned in Chapter 1. First note that any hyperplane H in $\mathbb{R}^n$ is a set of the form

$$\{\boldsymbol{x} \in \mathbb{R}^n \mid P_H(\boldsymbol{x}) = 0\}$$

where $P_H(\boldsymbol{x}) = a_1x_1 + a_2x_2 + \cdots + a_nx_n + c$ for some constants $a_1, \ldots, a_n, c \in \mathbb{R}$. Also, the regions of an arrangement $\mathcal{A}$ are precisely the non-empty intersections of sets of the form

$$\{\boldsymbol{x} \in \mathbb{R}^n \mid P_H(\boldsymbol{x}) > 0\} \text{ or } \{\boldsymbol{x} \in \mathbb{R}^n \mid P_H(\boldsymbol{x}) < 0\}$$

where we have one set for each $H \in \mathcal{A}$. Hence, crossing exactly one hyperplane H in an arrangement corresponds to changing the inequality chosen for H in this description of the region.

Theorem 2.2. *If $\mathcal{D}$ is a minimal determining set of a region R of an arrangement $\mathcal{A}$, then changing the inequality in the definition of R of exactly one $H \in \mathcal{D}$, and keeping all other inequalities of hyperplanes in $\mathcal{A}$ the same, describes a non-empty region of $\mathcal{A}$.*

Before proving this, we will see how it proves the fact mentioned above. Start with two distinct regions R and R' of an arrangement $\mathcal{A}$. We want to get from R to R' by crossing exactly one hyperplane at a time with respect to which the regions lie on opposite sides.

1. Let $\mathcal{D}$ be a minimal determining set of R.
2. Since $R \neq R'$ there is some $H \in \mathcal{D}$ for which R' is on the opposite side as R.
3. Change the inequality corresponding to H in R, call this new region R''.
4. The number of hyperplanes in $\mathcal{A}$ for which R'' and R' lie on opposite sides is less than that for R and R'.
5. Repeat this process to get to R' by changing one inequality at a time.

Proof of Theorem 2.2. Let $H \in \mathcal{D}$. Since $\mathcal{D}$ is a minimal determining set, $\mathcal{E} = \mathcal{D} \setminus \{H\}$ is not a determining set. So R is strictly contained in $R_{\mathcal{E}}$. This means that the hyperplane H intersects $R_{\mathcal{E}}$ and splits it into two open convex sets, one of which is R.

So we can choose a point $p \in H$ that lies inside R_ϵ and an n-ball centered at p that does not touch any other hyperplanes of $\mathcal{A}$ (since $\mathcal{A}$ is finite). One half of the ball lies in R and the other half lies in a region R' of $\mathcal{A}$. Since R' can be reached from R by just crossing the hyperplane H, we get the required result. □

To sum up, we start with an arrangement $\mathcal{B} \subseteq \mathcal{A}$. We know the regions of $\mathcal{A}$ and usually represent them by combinatorial objects we call 'sketches'. We then define 'moves' on these sketches that correspond to changing exactly one inequality of a hyperplane in $\mathcal{A} \setminus \mathcal{B}$. We define sketches to be equivalent if one can be obtained from another through a series of moves. We then count the number of equivalence classes to obtain the number of regions of $\mathcal{B}$. Before using this method to study the Catalan arrangements of various types, we first look at some simpler arrangements.

2.1.1 A simple example

Before using the 'sketches and moves' idea to study the Catalan arrangements of various types, we first apply it to a simpler example: the type D arrangement which is contained in the type C arrangement.

The type C arrangement. This arrangement in $\mathbb{R}^n$ is the set of reflecting hyperplanes of the root system C_n. The defining equations of hyperplanes are

$$\begin{aligned} 2x_i &= 0 \\ x_i + x_j &= 0 \\ x_i - x_j &= 0 \end{aligned}$$

for $1 \leqslant i < j \leqslant n$. Though we could write $x_i = 0$ for the first type of hyperplanes, we think of them as $x_i + x_i = 0$ to define sketches.

We can write the hyperplanes of the type C arrangement as follows:

$$\begin{aligned} x_i &= x_j, \quad && 1 \leqslant i < j \leqslant n \\ x_i &= -x_j, \quad && i, j \in [n]. \end{aligned}$$

Hence, any region of the arrangement is given by a *valid* total order on

$$x_1, \ldots, x_n, -x_1, \ldots, -x_n.$$

A total order is said to be valid if there is some point in $\mathbb{R}^n$ that satisfies it. We will represent x_i by $\overset{+}{i}$ and $-x_i$ by $\overset{-}{i}$ for all $i \in [n]$.

Example 2.3. The region $-x_2 < x_3 < x_1 < -x_1 < -x_3 < x_2$ is represented as $\overset{-}{2}\,\overset{+}{3}\,\overset{+}{1}\,\overset{-}{1}\,\overset{-}{3}\,\overset{+}{2}$.

It can be shown that words of the form

$$\overset{s_1}{i_1} \quad \overset{s_2}{i_2} \quad \cdots \quad \overset{s_n}{i_n} \quad \overset{-s_n}{i_n} \quad \cdots \quad \overset{-s_2}{i_2} \quad \overset{-s_1}{i_1}$$

where $\{i_1, \ldots, i_n\} = [n]$ are the ones that correspond to regions. Such orders are the only ones that can correspond to regions since negatives reverse order. Also, choosing n distinct negative numbers, it is easy to construct a point satisfying the inequalities specified by such a word.

Hence the number of regions of the type C arrangement is $2^n n!$. We will call such words *sketches* (which are basically signed permutations). We will draw a line after the first n symbols to denote the reflection and call the part of the sketch before the line its first half and similarly define the second half.

Example 2.4. $\overset{+}{3}\,\overset{-}{1}\,\overset{-}{2}\,\overset{+}{4} \mid \overset{-}{4}\,\overset{+}{2}\,\overset{+}{1}\,\overset{-}{3}$ is a sketch.

The type D arrangement. This arrangement in $\mathbb{R}^n$ has the hyperplanes

$$x_i + x_j = 0$$
$$x_i - x_j = 0$$

for $1 \leqslant i < j \leqslant n$. We use the 'sketches and moves' idea to count the regions of this arrangement by considering it as a sub-arrangement of the type C arrangement. We will define the moves that we can apply to the sketches (which represent changing exactly one inequality corresponding to a hyperplane not in the type D arrangement) and then choose a canonical representative from each equivalence class. By Theorem 2.2, this gives a bijection between these canonical sketches and the regions of the sub-arrangement.

The hyperplanes missing from missing from the type C arrangement are

$$2x_i = 0$$

for all $i \in [n]$. Hence a type D move, which we call a D move, is swapping adjacent $\overset{+}{i}$ and $\overset{-}{i}$ for any $i \in [n]$.

Example 2.5. $\overset{+}{4}\,\overset{+}{1}\,\overset{-}{3}\,\overset{+}{2} \mid \overset{-}{2}\,\overset{+}{3}\,\overset{-}{1}\,\overset{-}{4} \xrightarrow{\text{D move}} \overset{+}{4}\,\overset{+}{1}\,\overset{-}{3}\,\overset{-}{2} \mid \overset{+}{2}\,\overset{+}{3}\,\overset{-}{1}\,\overset{-}{4}$

In a sketch the only such pair is the last term of the first half and the first term of the second half. Hence D moves actually define an involution on the sketches. We could also choose a canonical sketch in each type D region to be the one where the first term of the second half is positive. Hence the number of regions of the type D arrangement is $2^{n-1}n!$.

2.2 Catalan deformation of type C

In this section we reprove, with a modification inspired by [4], the results of [45] about the regions of the type C Catalan arrangements.

Fix $n \geqslant 1$ throughout this section. The type C Catalan arrangement in $\mathbb{R}^n$ is the arrangement with hyperplanes

$$\begin{aligned} 2X_i &= -1, 0, 1 \\ X_i + X_j &= -1, 0, 1 \\ X_i - X_j &= -1, 0, 1 \end{aligned}$$

for all $1 \leqslant i < j \leqslant n$. In this case, instead of looking at this arrangement directly, we will study the arrangement obtained by performing the translation $X_i = x_i + \frac{1}{2}$ for all $i \in [n]$. It is easy to see that this does not change the combinatorics of the arrangement. The translated arrangement, which we call $\mathcal{C}_n$, has hyperplanes

$$\begin{aligned} 2x_i &= -2, -1, 0 \\ x_i + x_j &= -2, -1, 0 \\ x_i - x_j &= -1, 0, 1 \end{aligned} \tag{2.1}$$

for all $1 \leqslant i < j \leqslant n$. The arrangement $\mathcal{C}_n$ consists of all hyperplanes of the form $x_i + s = \pm(x_j + t)$ for $i, j \in [n]$ and $s, t \in \{0, 1\}$. This shows that the regions of $\mathcal{C}_n$ are given by valid total orders on

$$\{x_i + s \mid i \in [n],\ s \in \{0, 1\}\} \cup \{-x_i - s \mid i \in [n],\ s \in \{0, 1\}\}.$$

Such orders will be represented by using the symbol $\alpha_i^{(s)}$ for $x_i + s$ and $\alpha_{-i}^{(-s)}$ for $-x_i - s$ for all $i \in [n]$ and $s \in \{0, 1\}$. Let $C(n)$ be the set

$$\{\alpha_i^{(s)} \mid i \in [n],\ s \in \{0,1\}\} \cup \{\alpha_i^{(s)} \mid -i \in [n],\ s \in \{-1,0\}\}.$$

Hence, we use orders on the letters of $C(n)$ to represent regions of $\mathcal{C}_n$.

Example 2.6. The total order

$$x_1 < -x_2 - 1 < x_1 + 1 < x_2 < -x_2 < -x_1 - 1 < x_2 + 1 < -x_1$$

is represented as $\alpha_1^{(0)}\ \alpha_{-2}^{(-1)}\ \alpha_1^{(1)}\ \alpha_2^{(0)}\ \alpha_{-2}^{(0)}\ \alpha_{-1}^{(-1)}\ \alpha_2^{(1)}\ \alpha_{-1}^{(0)}$.

Considering $-x_i$ as x_{-i}, the letter $\alpha_i^{(s)}$ represents $x_i + s$ for any $\alpha_i^{(s)} \in C(n)$. For any $\alpha_i^{(s)} \in C(n)$, we use $\overline{\alpha_i^{(s)}}$ to represent the letter $\alpha_{-i}^{(-s)}$, which we call the *conjugate* of $\alpha_i^{(s)}$.

Definition 2.7. A *symmetric sketch* is an order on the letters in $C(n)$ such that the following hold for any $\alpha_i^{(s)}, \alpha_j^{(t)} \in C(n)$:

1. If $\alpha_i^{(s)}$ appears before $\alpha_j^{(t)}$, then $\overline{\alpha_j^{(t)}}$ appears before $\overline{\alpha_i^{(s)}}$.
2. If $\alpha_i^{(s-1)}$ appears before $\alpha_j^{(t-1)}$, then $\alpha_i^{(s)}$ appears before $\alpha_j^{(t)}$.
3. $\alpha_i^{(s-1)}$ appears before $\alpha_i^{(s)}$.

Proposition 2.8. *An order on the letters of $C(n)$ corresponds to a region of $\mathcal{C}_n$ if and only if it is a symmetric sketch.*

Proof. The idea of the proof is the same as that of [4, Lemma 5.2]. It is clear that any order that corresponds to a region must satisfy the properties in Definition 2.7 and hence be a symmetric sketch. For the converse, we show that there is a point in $\mathbb{R}^n$ satisfying the inequalities given by a symmetric sketch.

We prove this using induction on n, the case $n = 1$ being clear. Let $n \geqslant 2$ and w be a symmetric sketch. Without loss of generality, we can assume that the first letter of w is $\alpha_n^{(0)}$. Deleting the letters with subscript n and $-n$ from w gives a symmetric sketch w' in the letters $C(n-1)$. Using the induction hypothesis, we can choose a point $\mathbf{x}' \in \mathbb{R}^{n-1}$ satisfying the inequalities given by w'. Suppose the letter before $\alpha_n^{(1)}$ in w is $\alpha_i^{(s)}$ and the letter after it is $\alpha_j^{(t)}$. We choose $x_n \neq -1$ such that

$x'_i + s < x_n + 1 < x'_j + t$ in such a way that $x_n + 1$ is also in the correct position with respect 0 specified by w. This is possible since $\mathbf{x}'$ satisfies w'.

We show that $(x'_1, \ldots, x'_{n-1}, x_n)$ satisfies the inequalities given by w. We only have to check that x_n and $(x_n + 1)$ are in the correct relative position with respect to the other letters since property 1 of Definition 2.7 will then show that $-x_n$ and $-x_n - 1$ are also in the correct relative position. By the choice of x_n, we see that $x_n + 1$ in the correct position. We have to show that x_n is less than $\pm x'_i$ and $\pm(x'_i + 1)$ for all $i' \in [n-1]$. If $x_n > x'_1$, then $x_n + 1 > x'_1 + 1$ and since $x_n + 1$ satisfies the inequalities specified by w, $\alpha_1^{(1)}$ must be before $\alpha_n^{(1)}$ in w. But by property 2 of Definition 2.7, this means that $\alpha_1^{(0)}$ must be before $\alpha_n^{(0)}$ in w, which is a contradiction. The same logic can be used to show that x_n satisfies the other inequalities given by w. □

We now derive some properties of symmetric sketches. A symmetric sketch has $4n$ letters, so we call the word made by the first $2n$ letters its first half. Similarly we define its second half.

Lemma 2.9. *The second half of a symmetric sketch is completely specified by its first half. In fact, it is the 'mirror' of the first half, i.e., it is the reverse of the first half with each letter replaced with its conjugate.*

Proof. For any symmetric sketch, the letter $\alpha_i^{(s)}$ is in the first half if and only if the letter $\overline{\alpha_i^{(s)}}$ is in the second half. This property can be proved as follows: Suppose there is a pair of conjugates in the first half of a symmetric sketch. Since conjugate pairs partition $C(n)$, this means that there is a pair of conjugates in the second half as well. But this would contradict property 1 of a symmetric sketch in Definition 2.7.

Hence, the set of letters in the second half are the conjugates of the letters in the first half. The order in which they appear is forced by property 1 of Definition 2.7, that is, the conjugates appear in the opposite order as the corresponding letters in the first half. So if the first half of a symmetric sketch is $a_1 \cdots a_{2n}$ where $a_i \in C(n)$ for all $i \in [2n]$, the sketch is

$$a_1 \quad a_2 \quad \cdots \quad a_{2n} \quad \overline{a_{2n}} \quad \cdots \quad \overline{a_2} \quad \overline{a_1}.$$

□

We draw a vertical line between the $2n^{th}$ and $(2n+1)^{th}$ letter in a symmetric sketch to indicate both the mirroring and the change in sign (note that if the $2n^{th}$ letter is $\alpha_i^{(s)}$, we have $x_i + s < 0 < -x_i - s$ in the corresponding region).

Example 2.10. $\alpha_{-3}^{(-1)}\ \alpha_{-3}^{(0)}\ \alpha_{1}^{(0)}\ \alpha_{-2}^{(-1)}\ \alpha_{1}^{(1)}\ \alpha_{2}^{(0)} \mid \alpha_{-2}^{(0)}\ \alpha_{-1}^{(-1)}\ \alpha_{2}^{(1)}\ \alpha_{-1}^{(0)}\ \alpha_{3}^{(0)}\ \alpha_{3}^{(1)}$.

Similar to the convention used in [8], a letter in $C(n)$ is called an *α-letter* if it is of the form $\alpha_i^{(0)}$ or $\alpha_{-i}^{(-1)}$ where $i \in [n]$. The other letters are called *β-letters*. The β-letter 'corresponding' to an α-letter is the one with the same subscript. Hence, in a symmetric sketch, an α-letter always appears before its corresponding β-letter by property 3 in Definition 2.7. The order in which the subscripts of the α-letters appear is the same as the order in which the subscripts of the β-letters appear by property 2 of Definition 2.7. The proof of the following lemma is very similar to that of the previous lemma.

Lemma 2.11. *The order in which the subscripts of the α-letters in a symmetric sketch appear is of the form*

$$i_1 \quad i_2 \quad \cdots \quad i_n \quad -i_n \quad \cdots \quad -i_2 \quad -i_1$$

where $\{|i_1|, \ldots, |i_n|\} = [n]$.

Using Lemmas 2.9 and 2.11, to specify the sketch, we only need to specify the following:

1. The α, β-word corresponding to the first half.
2. The signed permutation given by the first n α-letters.

The α, β-word corresponding to the first half is a word of length $2n$ in the letters $\{\alpha, \beta\}$ such that the i^{th} letter is an α if and only if the i^{th} letter of the symmetric sketch is an α-letter.

There is at most one sketch corresponding to a pair of an α, β-word and a signed permutation. This is because the signed permutation tells us, by Lemma 2.11, the order in which the subscripts of the α-letters (and hence β-letters) appears. Using this and the α, β-word, we can construct the first half and, by Lemma 2.9, the entire sketch.

Example 2.12. To the symmetric sketch

$$\alpha_{-3}^{(-1)}\ \alpha_{-3}^{(0)}\ \alpha_{1}^{(0)}\ \alpha_{-2}^{(-1)}\ \alpha_{1}^{(1)}\ \alpha_{2}^{(0)}\ |\ \alpha_{-2}^{(0)}\ \alpha_{-1}^{(-1)}\ \alpha_{2}^{(1)}\ \alpha_{-1}^{(0)}\ \alpha_{3}^{(0)}\ \alpha_{3}^{(1)}$$

we associate the pair consisting of the following:

1. α, β-word: $\alpha\beta\alpha\alpha\beta\alpha$.

2. Signed permutation: $-3 \quad 1 \quad -2$.

If we are given the α, β-word and signed permutation above, the unique sketch corresponding to it is the one given above.

The next proposition characterizes the pairs of α, β-words and signed permutations that correspond to symmetric sketches.

Proposition 2.13. *A pair consisting of*

1. *an α, β-word of length $2n$ such that any prefix of the word has at least as many α-letters as β-letters and*

2. *any signed permutation*

corresponds to a symmetric sketch and all symmetric sketches correspond to such pairs.

Proof. By property 3 of Definition 2.7, any α, β-word corresponding to the first half of a sketch should have at least as many α-letters as β-letters in any prefix.

We now prove that given such a pair, there is a symmetric sketch corresponding to it. If the given α, β-word is $l_1 l_2 \cdots l_{2n}$ and the given signed permutation is $i_1 i_2 \cdots i_n$, we construct the symmetric sketch as follows:

1. Extend the α, β-word to the one of length $4n$ given by

$$l_1 \quad l_2 \quad \cdots \quad l_{2n} \quad \overline{l_{2n}} \quad \cdots \quad \overline{l_2} \quad \overline{l_1}$$

where $\overline{l_i} = \alpha$ if and only if $l_i = \beta$ for all $i \in [2n]$.

2. Extend the signed permutation to the sequence of length $2n$ given by

$$i_1 \quad i_2 \quad \cdots \quad i_n \quad -i_n \quad \cdots \quad -i_2 \quad -i_1.$$

3. Label the subscripts of the α-letters of the extended α, β-word in the order given by the extended signed permutation and similarly label the β-letters.

If we show that the word constructed is a symmetric sketch, it is clear that it will correspond to the given α, β-word and signed permutation. We have to check that the constructed word satisfies the properties in Definition 2.7.

The way the word was constructed, we see that it is of the form

$$a_1 \quad a_2 \quad \cdots \quad a_{2n} \quad \overline{a_{2n}} \quad \cdots \quad \overline{a_2} \quad \overline{a_1}$$

where $a_i \in C(n)$ for all $i \in [2n]$. Since the conjugate of the i^{th} α is the $(2n-i+1)^{\text{th}}$ β and vice-versa, the first half of the word cannot have a pair of conjugates. Hence the word has all letters of $C(n)$. This shows that property 1 of Definition 2.7 holds. Property 2 is taken care of since, by construction, the subscripts of the α-letters appear in the same order as those of the β-letters.

To show that property 3 holds, it suffices to show that any prefix of the word has at least as many α-letters as β-letters. This is already true for the first half. To show that this is true for the entire word, we consider α as $+1$ and β as -1. Hence, the condition is that any prefix has a non-negative sum. Since any prefix of size greater than $2n$ is of the form

$$l_1 \quad l_2 \quad \cdots \quad l_{2n} \quad \overline{l_{2n}} \quad \cdots \quad \overline{l_k}$$

for some $k \in [2n]$, the sum is $l_1 + \cdots + l_{k-1} \geqslant 0$. So property 3 holds as well and hence the constructed word is a symmetric sketch. □

We use this description to count symmetric sketches.

Lemma 2.14. *The number of α, β-words of length $2n$ having at least as many α-letters as β-letters in any prefix is $\binom{2n}{n}$.*

Proof. We consider these α, β-words as lattice paths. Using the step $U = (1, 1)$ for α and the step $D = (1, -1)$ for β, we have to count those lattice paths with each step U or D that start at the origin, have $2n$ steps, and never fall below the x-axis.

Using the reflection principle (for example, see [29]), we get that the number of such lattice paths that end at $(2n, 2k)$ for $k \in [0, n]$ is given by

$$\binom{2n}{n+k} - \binom{2n}{n+k+1}.$$

The (telescoping) sum over $k \in [0, n]$ gives the required result. □

The above lemma and Proposition 2.13 immediately give the following.

Theorem 2.15. *The number of symmetric sketches and hence regions of $\mathcal{C}_n$ is*

$$2^n n! \binom{2n}{n}.$$

In [4], Athanasiadis obtains bijections between several classes of non-nesting partitions and regions of certain arrangements. We will mention the one for the arrangement $\mathcal{C}_n$, which gives a bijection between the α, β-words associated to symmetric sketches and certain non-nesting partitions.

Definition 2.16. A *symmetric non-nesting partition* is a partition of $[-2n, 2n] \setminus \{0\}$ such that the following hold:

1. Each block is of size 2.
2. If $B = \{a, b\}$ is a block, so is $-B = \{-a, -b\}$.
3. If $\{a, b\}$ is a block and $c, d \in [-2n, 2n] \setminus \{0\}$ are such that $a < c < d < b$, then $\{c, d\}$ is not a block.

Remark 2.17. Partitions where each block is of size 2 are usually called *matchings*. Also, non-nesting matchings (those matchings that satisfy condition 3 above) are counted by the Catalan numbers (see [56, Exercise 64]). However, in Section 2.2.1, we will be considering partitions with larger part sizes and hence we use the term 'symmetric non-nesting partition'.

Symmetric non-nesting partitions are usually represented using arc-diagrams. This is done by using $4n$ dots to represent the numbers in $[-2n, 2n] \setminus \{0\}$ in order and joining dots in the same block using an arc. The properties of these partitions imply that there are no nesting arcs and that the diagram is symmetric, which we represent by drawing a line after $2n$ dots.

Example 2.18. The arc diagram associated to the symmetric non-nesting partition of $[-6,6] \setminus \{0\}$

$$\{-6,-3\},\{-5,-1\},\{-4,2\},\{-2,4\},\{1,5\},\{3,6\}$$

is given in Figure 2.2.

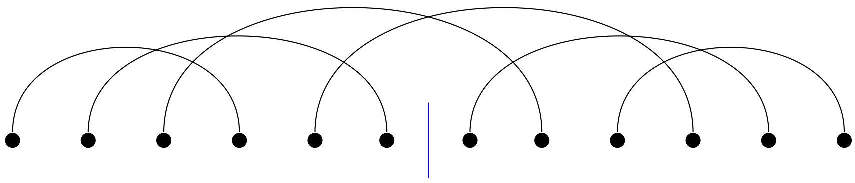


FIGURE 2.2: The symmetric non-nesting partition of Example 2.18.

It can also be seen that there are exactly n pairs of blocks of the form $\{B,-B\}$ with no block containing both a number and its negative. Also, the first n blocks, with blocks being read in order of the smallest element in it, do not have a pair of the form $\{B,-B\}$. Hence, we can label the first n blocks with a signed permutation and label the block $-B$ with the negative of the label of B to obtain a labeling of all blocks. We call such objects *labeled symmetric non-nesting partitions*. In the arc diagram, the labeling is done by replacing the dots representing the elements in a block with its label.

We can obtain a labeled symmetric non-nesting partition from a symmetric sketch by joining the letters $\alpha_i^{(0)}$ and $\alpha_i^{(1)}$ and similarly $\alpha_{-i}^{(-1)}$ and $\alpha_{-i}^{(0)}$ with arcs and replacing each letter in the sketch with its subscript. It can be shown that this construction is a bijection between symmetric sketches and labeled symmetric non-nesting partitions. In particular, the α, β-words associated with symmetric sketches are in bijection with symmetric non-nesting partitions.

Example 2.19. To the symmetric sketch

$$\alpha_3^{(0)}\alpha_2^{(0)}\alpha_{-1}^{(-1)}\alpha_3^{(1)}\alpha_1^{(0)}\alpha_2^{(1)}|\alpha_{-2}^{(-1)}\alpha_{-1}^{(0)}\alpha_{-3}^{(-1)}\alpha_1^{(1)}\alpha_{-2}^{(0)}\alpha_{-3}^{(0)}$$

we associate the labeled symmetric non-nesting partition in Figure 2.3.

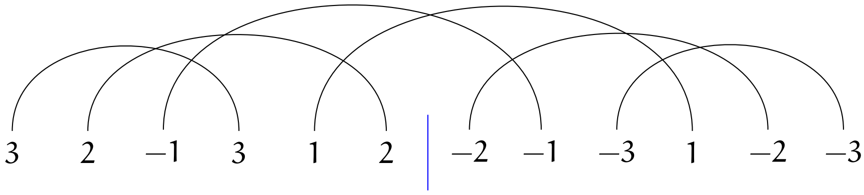


FIGURE 2.3: Arc diagram associated to the symmetric sketch in Example 2.19.

We now describe another way to represent the regions. We have already seen that a sketch corresponds to a pair consisting of an α, β-word and a signed permutation. We represent the α, β-word as a lattice path just as we did in the proof of Lemma 2.14. We specify the signed permutation by labeling the first n up-steps of the lattice path.

Example 2.20. The lattice path associated to the symmetric sketch

$$\alpha_{-3}^{(-1)}\ \alpha_{-3}^{(0)}\ \alpha_{1}^{(0)}\ \alpha_{-2}^{(-1)}\ \alpha_{1}^{(1)}\ \alpha_{2}^{(0)}\ |\ \alpha_{-2}^{(0)}\ \alpha_{-1}^{(-1)}\ \alpha_{2}^{(1)}\ \alpha_{-1}^{(0)}\ \alpha_{3}^{(0)}\ \alpha_{3}^{(1)}$$

is given in Figure 2.4.

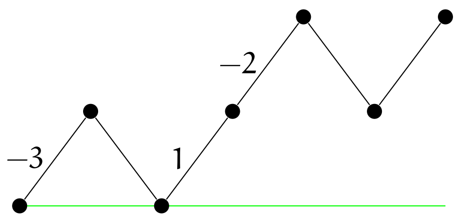


FIGURE 2.4: Lattice path associated to the symmetric sketch in Example 2.20.

These representations for the regions of $\mathcal{C}_n$ also allow us to determine and count which regions are bounded.

Theorem 2.21. *The number of bounded regions of the arrangement $\mathcal{C}_n$ is*

$$2^n n! \binom{2n-1}{n}.$$

Proof. First note that the arrangement $\mathcal{C}_n$ has rank n and is hence essential. From the bijection defined above, it can be seen that the arc diagram associated to any region R of $\mathcal{C}_n$ can be obtained by plotting a point $(x_1, \ldots, x_n) \in R$ on the real line. This is done by marking x_i and $x_i + 1$ on the real line using i for all $i \in [n]$ and then joining them with an arc and similarly marking $-x_i - 1$ and $-x_i$ using $-i$ and joining them with an arc.

This can be used to show that a region of $\mathcal{C}_n$ is bounded if and only if the arc diagram is 'interlinked'. For example, Figure 2.3 shows an arc diagram that is interlinked and Figure 2.5 shows one that is not. In terms of lattice paths, the bounded regions are those whose corresponding lattice path never touches the x-axis except at the origin.

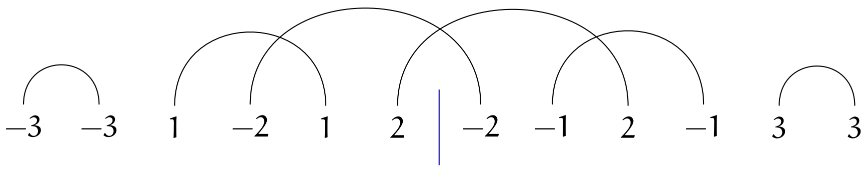


FIGURE 2.5: Arc diagram associated to the symmetric sketch of Example 2.12.

This shows that the number of bounded regions of $\mathcal{C}_n$ is $2^n n!$ times the number of unlabeled lattice paths of length $2n$ that never touch the x-axis apart from at the origin. Deleting the first step (which is necessarily an up-step) gives a bijection between such paths and those of length $2n-1$ that never fall below the x-axis. Using the same idea as in the proof of Lemma 2.14, it can be checked that the number of such paths is $\binom{2n-1}{n}$. This proves the required result. □

Remark 2.22. In [45], the authors study the type C Catalan arrangement directly, i.e., without using the translation $\mathcal{C}_n$ mentioned above. Hence, using the same logic, they use orders on the letters

$$\{\alpha_i^{(s)} \mid i \in [-n, n] \setminus \{0\},\ s \in \{0, 1\}\}$$

to represent the regions of the type C Catalan arrangement. They claim that these orders are those such that the following hold for any $i, j \in [-n, n] \setminus \{0\}$ and $s \in \{0, 1\}$:

1. If $\alpha_i^{(0)}$ appear before $\alpha_j^{(0)}$, then $\alpha_i^{(1)}$ appears before $\alpha_j^{(1)}$.
2. $\alpha_i^{(0)}$ appears before $\alpha_i^{(1)}$.
3. If $\alpha_i^{(0)}$ appears before $\alpha_j^{(s)}$, then $\alpha_{-j}^{(0)}$ appears before $\alpha_{-i}^{(s)}$.

Though this can be shown to be true, the method used in [45] to construct a point satisfying the inequalities given by such an order does not seem to work in general. We describe their method and then exhibit a case where it does not work.

Let $w = w_1 \cdots w_{4n}$ be an order satisfying the properties given above. Then construct $\boldsymbol{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$ as follows: Let $z_0 = 0$ (or pick z_0 arbitrarily). Then define z_p for $p = 1, 2, \ldots, 4n$ in order as follows: If $w_p = \alpha_i^{(0)}$ then set $z_p = z_{p-1} + \frac{1}{2n+1}$ and $x_i = z_p$, and if $w_p = \alpha_i^{(1)}$ then set $z_p = x_i + 1$. Here we consider $x_{-i} = -x_i$ for any $i \in [n]$. Then $\boldsymbol{x}$ satisfies the inequalities given by w.

The following example shows that this method does not always work; in fact $\boldsymbol{x}$ is not always well-defined. Consider the order $w = \alpha_{-2}^{(0)}\alpha_1^{(0)}\alpha_{-2}^{(1)}\alpha_1^{(1)}\alpha_{-1}^{(0)}\alpha_2^{(0)}\alpha_{-1}^{(1)}\alpha_2^{(1)}$. Following the above procedure, we would get that x_1 is both $\frac{2}{5}$ as well as $-1 - \frac{3}{5}$.

2.2.1 Extended type C Catalan

Fix $m, n \geqslant 1$. The type C m-Catalan arrangement in $\mathbb{R}^n$ has hyperplanes

$$\begin{aligned} 2X_i &= 0, \pm 1, \pm 2, \ldots, \pm m \\ X_i + X_j &= 0, \pm 1, \pm 2, \ldots, \pm m \\ X_i - X_j &= 0, \pm 1, \pm 2, \ldots, \pm m \end{aligned}$$

for all $1 \leqslant i < j \leqslant n$. We will study the arrangement obtained by performing the translation $X_i = x_i + \frac{m}{2}$ for all $i \in [n]$. The translated arrangement, which we call $\mathcal{C}_n^{(m)}$, has hyperplanes

$$\begin{aligned} 2x_i &= -2m, -2m+1, \ldots, 0 \\ x_i + x_j &= -2m, -2m+1, \ldots, 0 \\ x_i - x_j &= 0, \pm 1, \pm 2, \ldots, \pm m \end{aligned}$$

for all $1 \leqslant i < j \leqslant n$. Note that $\mathcal{C}_n = \mathcal{C}_n^{(1)}$. The regions of $\mathcal{C}_n^{(m)}$ are given by valid total orders on

$$\{x_i + s \mid i \in [n],\ s \in [0, m]\} \cup \{-x_i - s \mid i \in [n],\ s \in [0, m]\}.$$

Just as we did for $\mathcal{C}_n$, such orders will be represented by using the symbol $\alpha_i^{(s)}$ for $x_i + s$ and $\alpha_{-i}^{(-s)}$ for $-x_i - s$ for all $i \in [n]$ and $s \in [0, m]$. Let $C^{(m)}(n)$ be the set

$$\{\alpha_i^{(s)} \mid i \in [n],\ s \in [0, m]\} \cup \{\alpha_i^{(s)} \mid -i \in [n],\ s \in [-m, 0]\}.$$

For any $\alpha_i^{(s)} \in C^{(m)}(n)$, $\overline{\alpha_i^{(s)}}$ represents $\alpha_{-i}^{(-s)}$ and is called the conjugate of $\alpha_s^{(i)}$. Letters of the form $\alpha_i^{(0)}$ or $\alpha_{-i}^{(-m)}$ for any $i \in [n]$ are called α-letters. The others are called β-letters.

Definition 2.23. An order on the letters in $C^{(m)}(n)$ is called a *symmetric* m*-sketch* if the following hold for all $\alpha_i^{(s)}, \alpha_j^{(t)} \in C^{(m)}(n)$:

1. If $\alpha_i^{(s)}$ appears before $\alpha_j^{(t)}$, then $\overline{\alpha_j^{(t)}}$ appears before $\overline{\alpha_i^{(s)}}$.
2. If $\alpha_i^{(s-1)}$ appears before $\alpha_j^{(t-1)}$, then $\alpha_i^{(s)}$ appears before $\alpha_j^{(t)}$.
3. $\alpha_i^{(s-1)}$ appears before $\alpha_i^{(s)}$.

The following result can be proved just as Proposition 2.8.

Proposition 2.24. *An order on the letters in $C^{(m)}(n)$ corresponds to a region of $\mathcal{C}_n^{(m)}$ if and only if it is a symmetric m-sketch.*

Similar to Lemma 2.11, it can be shown that the order in which the subscripts of the α-letters appear in a symmetric m-sketch is of the form

$$i_1 \quad i_2 \quad \cdots \quad i_n \quad -i_n \quad \cdots \quad -i_2 \quad -i_1$$

where $\{|i_1|, \ldots, |i_n|\} = [n]$. Just as in the case of symmetric sketches, we associate an α, β-word and signed permutation to a symmetric m-sketch which completely determines it.

Example 2.25. To the symmetric 2-sketch

$$\alpha_2^{(0)}\alpha_{-1}^{(-2)}\alpha_2^{(1)}\alpha_{-1}^{(-1)}\alpha_1^{(0)}\alpha_{-2}^{(-2)} \mid \alpha_2^{(2)}\alpha_{-1}^{(0)}\alpha_1^{(1)}\alpha_{-2}^{(-1)}\alpha_1^{(2)}\alpha_{-2}^{(0)}$$

we associate the pair consisting of the following:

1. α, β-word: $\alpha\alpha\beta\beta\alpha\alpha$.
2. Signed permutation: $2 \quad -1$.

The set of α, β-words associated to symmetric m-sketches for $m > 1$ does not seem to have a simple characterization like those for symmetric sketches (see Proposition 2.13). However, looking at symmetric m-sketches as labeled non-nesting partitions as done in [4], we see that such objects have already been counted bijectively (refer [22]).

Definition 2.26. A *symmetric* m*-non-nesting partition* is a partition of $[-(m+1)n, (m+1)n] \setminus \{0\}$ such that the following hold:

1. Each block is of size $(m+1)$.
2. If B is a block, so is $-B$.
3. If a, b are in some block B, $a < b$ and there is no number $a < c < b$ such that $c \in B$, then if $a < c < d < b$, c and d are not in the same block.

Just as we did for the $m = 1$ case, we can obtain a labeled symmetric m-non-nesting partition from a symmetric m-sketch by joining the letters $\alpha_i^{(0)}, \alpha_i^{(1)}, \dots, \alpha_i^{(m)}$ and similarly $\alpha_{-i}^{(-m)}, \alpha_{-i}^{(-m+1)}, \dots, \alpha_{-i}^{(0)}$ with arcs and labeling each such chain with the subscript of the letters being joined.

Example 2.27. To the symmetric 2-sketch in Example 2.25, we associate the labeled 2-non-nesting partition of Figure 2.6.

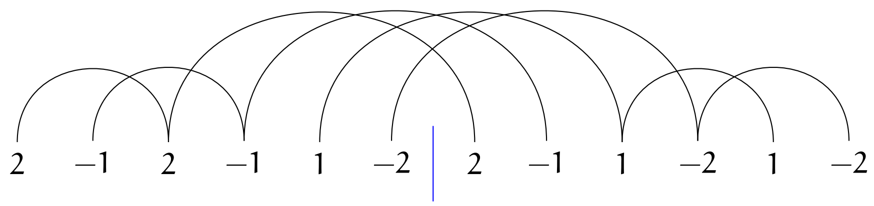


FIGURE 2.6: A labeled 2-non-nesting partition

The number of various classes of non-nesting partitions have been counted bijectively. In terms of [22] or [4], the symmetric m-non-nesting partitions defined above are called type C partitions of size $(m+1)n$ of type $(m+1, \dots, m+1)$ where this is an n-tuple representing the size of the (nonzero) block pairs $\{B, -B\}$. The number of such partitions is

$$\binom{(m+1)n}{n}.$$

Hence we get the following theorem.

Theorem 2.28. *The number of symmetric* m*-sketches, which is the number of regions of* $\mathcal{C}_n^{(m)}$ *is*

$$2^n n! \binom{(m+1)n}{n}.$$

2.3 Catalan deformations of other types

We will now use 'sketches and moves', as in [8], to count the regions of Catalan arrangements of other types. Depending on the context, we represent the regions of arrangements using sketches, arc diagrams, or lattice paths and frequently make use of the bijections identifying them. We usually use sketches to define moves and use arc diagrams and lattice paths to count regions as well as bounded regions.

2.3.1 Type D Catalan

Fix $n \geqslant 2$. The type D Catalan arrangement in $\mathbb{R}^n$ has hyperplanes

$$X_i + X_j = -1, 0, 1$$
$$X_i - X_j = -1, 0, 1$$

for $1 \leqslant i < j \leqslant n$. Translating this arrangement by setting $X_i = x_i + \frac{1}{2}$ for all $i \in [n]$, we get the arrangement $\mathcal{D}_n$ with hyperplanes

$$x_i + x_j = -2, -1, 0$$
$$x_i - x_j = -1, 0, 1$$

for $1 \leqslant i < j \leqslant n$. Figure 2.7 shows $\mathcal{D}_2$ as a sub-arrangement of $\mathcal{C}_2$. It also shows how the regions of $\mathcal{D}_2$ partition the regions of $\mathcal{C}_2$.

We use the idea of moves to count the regions of $\mathcal{D}_n$ by considering it as a sub-arrangement of $\mathcal{C}_n$. The hyperplanes from $\mathcal{C}_n$ that are missing in $\mathcal{D}_n$ are

$$2x_i = -2, -1, 0$$

for all $i \in [n]$. Hence, the type D Catalan moves on symmetric sketches (regions of $\mathcal{C}_n$), which we call $\mathcal{D}$ moves, are as follows:

1. Swapping the $2n^{\text{th}}$ and $(2n+1)^{\text{th}}$ letter.
2. Swapping the n^{th} and $(n+1)^{\text{th}}$ α-letters if they are adjacent, along with the n^{th} and $(n+1)^{\text{th}}$ β-letters.

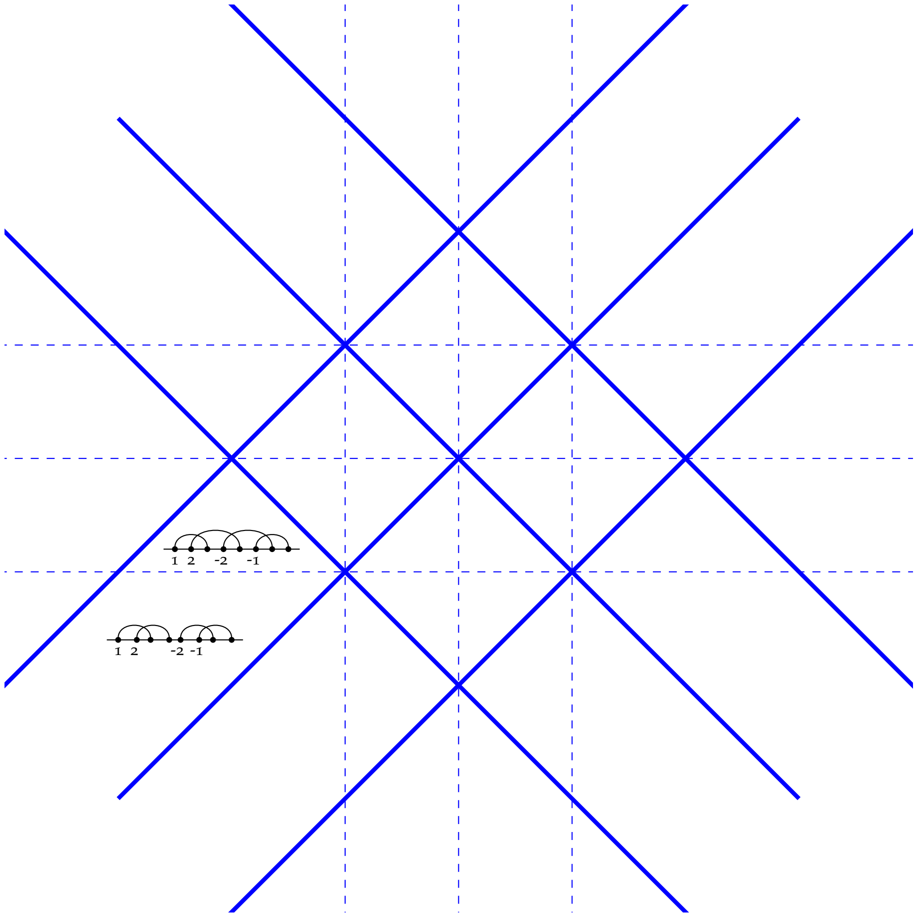


FIGURE 2.7: The arrangement $\mathcal{C}_2$ with the hyperplanes in $\mathcal{D}_2$ in bold. Two regions of $\mathcal{C}_2$ are labeled with their symmetric labeled non-nesting partition.

The first move covers the inequalities corresponding to the hyperplanes $x_i + 1 = -x_i - 1$ and $x_i = -x_i$ for all $i \in [n]$ since the only conjugates that are adjacent, by Lemma 2.9, are the $2n^{th}$ and $(2n+1)^{th}$ letter.

The second move covers the inequalities corresponding to the hyperplanes $x_i = -x_i - 1$ (equivalently, $x_i + 1 = -x_i$) for all $i \in [n]$. This is due to the fact that the only way $\alpha_i^{(0)}$ and $\alpha_{-i}^{(-1)}$ as well as $\alpha_i^{(1)}$ and $\alpha_{-i}^{(0)}$ can be adjacent is, by Lemma 2.11, when the n^{th} and $(n+1)^{th}$ α-letters are adjacent. Also, by Lemma 2.9, the n^{th} and $(n+1)^{th}$ α-letters are adjacent if and only if the n^{th} and $(n+1)^{th}$ β-letters are adjacent.

Example 2.29. A series of $\mathcal{D}$ moves applied to a symmetric sketch is given below:

$$\begin{aligned} &\alpha_{-1}^{(-1)}\alpha_{2}^{(0)}\alpha_{-2}^{(-1)}\alpha_{-1}^{(0)} \mid \alpha_{1}^{(0)}\alpha_{2}^{(1)}\alpha_{-2}^{(0)}\alpha_{1}^{(1)} \\ \xrightarrow{\mathcal{D}\text{ move}} &\alpha_{-1}^{(-1)}\alpha_{2}^{(0)}\alpha_{-2}^{(-1)}\alpha_{1}^{(0)} \mid \alpha_{-1}^{(0)}\alpha_{2}^{(1)}\alpha_{-2}^{(0)}\alpha_{1}^{(1)} \\ \xrightarrow{\mathcal{D}\text{ move}} &\alpha_{-1}^{(-1)}\alpha_{-2}^{(-1)}\alpha_{2}^{(0)}\alpha_{1}^{(0)} \mid \alpha_{-1}^{(0)}\alpha_{-2}^{(0)}\alpha_{2}^{(1)}\alpha_{1}^{(1)} \\ \xrightarrow{\mathcal{D}\text{ move}} &\alpha_{-1}^{(-1)}\alpha_{-2}^{(-1)}\alpha_{2}^{(0)}\alpha_{-1}^{(0)} \mid \alpha_{1}^{(0)}\alpha_{-2}^{(0)}\alpha_{2}^{(1)}\alpha_{1}^{(1)} \end{aligned}$$

To count the regions of $\mathcal{D}_n$, we have to count the number of equivalence classes of symmetric sketches where two sketches are equivalent if one can be obtained from the other via a series of $\mathcal{D}$ moves. In Figure 2.7, the two labeled regions of $\mathcal{C}_2$ are adjacent and lie in the same region of $\mathcal{D}_2$. They are related by swapping of the fourth and fifth letters of their sketches, which is a $\mathcal{D}$ move.

The fact about these moves that will help with the count is that a series of $\mathcal{D}$ moves do not change the sketch too much. Hence we can list the sketches that are $\mathcal{D}$ equivalent to a given sketch.

First, consider the case when the n^{th} α-letter of the symmetric sketch is not in the $(2n-1)^{\text{th}}$ position. In this case, the n^{th} α-letter is far enough from the $2n^{\text{th}}$ letter that a $\mathcal{D}$ move of the first kind (swapping the $2n^{\text{th}}$ and $(2n+1)^{\text{th}}$ letter) will not affect the letter after the n^{th} α-letter. Hence it does not change whether the n^{th} and $(n+1)^{\text{th}}$ α-letters are adjacent.

Let w be a sketch where the n^{th} α-letter is not in the $(2n-1)^{\text{th}}$ position. The number of sketches $\mathcal{D}$ equivalent to w is 4 when the n^{th} and $(n+1)^{\text{th}}$ α-letters are adjacent. They are illustrated below:

$$\begin{aligned} &\cdots\alpha_{-i}^{(-1)}\alpha_{i}^{(0)}\cdots\alpha_{j}^{(s)} \mid \alpha_{-j}^{(-s)}\cdots\alpha_{-i}^{(0)}\alpha_{i}^{(1)}\cdots \\ &\cdots\alpha_{-i}^{(-1)}\alpha_{i}^{(0)}\cdots\alpha_{-j}^{(-s)} \mid \alpha_{j}^{(s)}\cdots\alpha_{-i}^{(0)}\alpha_{i}^{(1)}\cdots \\ &\cdots\alpha_{i}^{(0)}\alpha_{-i}^{(-1)}\cdots\alpha_{j}^{(s)} \mid \alpha_{-j}^{(-s)}\cdots\alpha_{i}^{(1)}\alpha_{-i}^{(0)}\cdots \\ &\cdots\alpha_{i}^{(0)}\alpha_{-i}^{(-1)}\cdots\alpha_{-j}^{(-s)} \mid \alpha_{j}^{(s)}\cdots\alpha_{i}^{(1)}\alpha_{-i}^{(0)}\cdots \end{aligned}$$

The number of sketches $\mathcal{D}$ equivalent to w is 2 when the n^{th} and $(n+1)^{\text{th}}$ α-letter are not adjacent. They are illustrated below:

$$\cdots\alpha_{j}^{(s)} \mid \alpha_{-j}^{(-s)}\cdots \qquad \cdots\alpha_{-j}^{(-s)} \mid \alpha_{j}^{(s)}\cdots$$

Notice also that the equivalent sketches also satisfy the same properties (n^{th} α-letter not being in the $(2n-1)^{th}$ position and whether the n^{th} and $(n+1)^{th}$ α-letters are adjacent).

In case the n^{th} α-letter is in the $(2n-1)^{th}$ position of the symmetric sketch, it can be checked that it has exactly 4 equivalent sketches all of which also have the n^{th} α-letter in the $(2n-1)^{th}$ position:

$$\begin{gathered}
\cdots \alpha_i^{(0)} \alpha_i^{(1)} \mid \alpha_{-i}^{(-1)} \alpha_{-i}^{(0)} \cdots \\
\cdots \ \alpha_i^{(0)} \alpha_{-i}^{(-1)} \mid \alpha_i^{(1)} \alpha_{-i}^{(0)} \cdots \\
\cdots \alpha_{-i}^{(-1)} \alpha_i^{(0)} \mid \alpha_{-i}^{(0)} \alpha_i^{(1)} \cdots \\
\cdots \ \alpha_{-i}^{(-1)} \alpha_{-i}^{(0)} \mid \alpha_i^{(0)} \alpha_i^{(1)} \cdots
\end{gathered}$$

Figure 2.7 shows that each region of $\mathcal{D}_2$ contains exactly 2 or 4 regions of $\mathcal{C}_2$, as expected from the above observations.

Theorem 2.30. *The number of $\mathcal{D}$ equivalence classes on symmetric sketches and hence the number of regions of $\mathcal{D}_n$ is*

$$2^{n-1} \cdot \frac{(2n-2)!}{(n-1)!} \cdot (3n-2).$$

Proof. By the observations made above, the number of sketches equivalent to a given sketch only depends on its α, β-word (see Proposition 2.13). So, we need to count the number of α, β-words of length $2n$ with any prefix having at least as many α-letters as β-letters that are of the following types:

1. The n^{th} α-letter is not in the $(2n-1)^{th}$ position and
   (a) the letter after the n^{th} α-letter is an α.
   (b) the letter after the n^{th} α-letter is a β.
2. The n^{th} α-letter is in the $(2n-1)^{th}$ position.

We first count the second type of α, β-words. If the n^{th} α-letter is in the $(2n-1)^{th}$ position, the first $(2n-2)$ letters have $(n-1)$ α-letters and $(n-1)$ β-letters and hence form a ballot sequence. This means that there is no restriction on the $2n^{th}$ letter; it can be α or β. So, the total number of such α, β-words is

$$2 \cdot \frac{1}{n}\binom{2n-2}{n-1}.$$

The number of both the types 1(a) and 1(b) of α, β-words mentioned above are the same. This is because changing the letter after the n^{th} α-letter is an involution on the set of α, β-word of length $2n$ with any prefix having at least as many α-letters as β-letters. We have just counted such words that have the n^{th} α-letter in the $(2n-1)^{th}$ position. Hence, using Lemma 2.14, we get that the number of words of type 1(a) and 1(b) are both equal to

$$\frac{1}{2} \cdot \left[\binom{2n}{n} - \frac{2}{n}\binom{2n-2}{n-1} \right].$$

Combining the observations made above, we get that the number of regions of $\mathcal{D}_n$ is

$$2^n n! \cdot \left(\frac{1}{4} \cdot \left[\frac{2}{n}\binom{2n-2}{n-1} + \frac{1}{2} \cdot \left[\binom{2n}{n} - \frac{2}{n}\binom{2n-2}{n-1} \right] \right] + \frac{1}{2} \cdot \left[\frac{1}{2} \cdot \left[\binom{2n}{n} - \frac{2}{n}\binom{2n-2}{n-1} \right] \right] \right)$$

which simplifies to the required formula. □

Just as we did for $\mathcal{C}_n$, we can describe and count which regions of $\mathcal{D}_n$ are bounded.

Theorem 2.31. *The number of bounded regions of $\mathcal{D}_n$ is*

$$2^{n-1} \cdot \frac{(2n-3)!}{(n-2)!} \cdot (3n-4).$$

Proof. For $n \geqslant 2$, both $\mathcal{C}_n$ and $\mathcal{D}_n$ have rank n. Hence, a region of $\mathcal{D}_n$ is bounded exactly when all the regions of $\mathcal{C}_n$ it contains are bounded.

We have already seen in Theorem 2.21 that a region of $\mathcal{C}_n$ is bounded exactly when its corresponding lattice path does not touch the x-axis except at the origin. Such regions are not closed under $\mathcal{D}$ moves. However, if we include regions whose corresponding lattice paths touch the x-axis only at the origin and $(2n, 0)$, this set of regions, which we call S, is closed under the action of $\mathcal{D}$ moves because such lattice paths are closed under the action of changing the $2n^{th}$ step. Denote by $S_{\mathcal{D}}$ the set of equivalence classes that $\mathcal{D}$ moves partition S into, i.e., $S_{\mathcal{D}}$ is the set of regions of $\mathcal{D}_n$ that contain regions of S.

Just as in the proof of Theorem 2.30, one can check that the set S is closed under the action of changing the letter after the n^{th} α-letter. Also, note that the lattice paths in S do not touch the x-axis at $(2n-2, 0)$, and hence the n^{th} α-letter cannot be in the $(2n-1)^{\text{th}}$ position. Using the above observations and the same method to count regions of $\mathcal{D}_n$ as in the proof of Theorem 2.30, we get the number of regions in $S_{\mathcal{D}}$ is

$$2^n n! \cdot \frac{3}{8}\left(\binom{2n-1}{n} + \frac{1}{n}\binom{2n-2}{n-1}\right).$$

It can also be checked that each unbounded region in S is $\mathcal{D}$ equivalent to exactly one other region of S, and this region is bounded. This is because the lattice paths corresponding to these unbounded regions touch the x-axis at $(2n, 0)$. Hence, they cannot have the n^{th} and $(n+1)^{\text{th}}$ α-letters being adjacent and changing the $2n^{\text{th}}$ letter to an α gives a bounded region. Since the unbounded regions in S correspond to Dyck paths of length $(2n-2)$ (by deleting the first and last step), we get that the number of unbounded regions in $S_{\mathcal{D}}$ is

$$2^n n! \cdot \frac{1}{n}\binom{2n-2}{n-1}.$$

Combining the above results, we get that the number of bounded regions of $\mathcal{D}_n$ is

$$2^n n! \left(\frac{3}{8}\left(\binom{2n-1}{n} + \frac{1}{n}\binom{2n-2}{n-1}\right) - \frac{1}{n}\binom{2n-2}{n-1}\right).$$

This simplifies to give our required result. □

As mentioned earlier, we can choose a specific sketch from each $\mathcal{D}$ equivalence class to represent the regions of $\mathcal{D}_n$. It can be checked that symmetric sketches that satisfy the following are in bijection with regions of $\mathcal{D}_n$:

1. The last letter is a β-letter.
2. The n^{th} α-letter must have a negative label if the letter following it is an α-letter or the n^{th} β-letter.

We will call such sketches type D sketches. They will be used in Section 3.2 to interpret the coefficients of $\chi_{\mathcal{D}_n}$. Note that the type D sketches that correspond to bounded regions of $\mathcal{D}_n$ are those, when converted to a lattice path, do not touch the x-axis apart from at the origin.

2.3.2 Pointed type C Catalan

The type B and type BC Catalan arrangements we are going to consider now are not sub-arrangements of the type C Catalan arrangement. While it is possible to consider these arrangements as sub-arrangements of the type C 2-Catalan arrangement (see Section 2.2.1), this would add many extra hyperplanes. This would make defining moves and counting equivalence classes difficult. Also, we do not have a simple characterization of α, β-words associated to symmetric 2-sketches, as we do for symmetric sketches (see Proposition 2.13).

We instead consider them as a sub-arrangements of the arrangement $\mathcal{P}_n$ in $\mathbb{R}^n$ that has hyperplanes

$$\begin{aligned} x_i &= -\frac{5}{2}, -\frac{3}{2}, -1, -\frac{1}{2}, 0, \frac{1}{2}, \frac{3}{2} \\ x_i + x_j &= -2, -1, 0 \\ x_i - x_j &= -1, 0, 1 \end{aligned}$$

for all $1 \leqslant i < j \leqslant n$. It can be checked that the regions of $\mathcal{P}_n$ are given by valid total orders on

$$\{x_i + s \mid i \in [n],\ s \in \{0,1\}\} \cup \{-x_i - s \mid i \in [n],\ s \in \{0,1\}\} \cup \{-\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}\}.$$

Remark 2.32. The arrangement $\mathcal{P}_n$ is the arrangement $\mathcal{C}_n(\lambda)$ defined in [4, Equation (4)] with $\lambda_i = 2$ for all $i \in [n]$ and $m = 2$.

We now define sketches that represent such orders. Just as beofre, we represent $x_i + s$ as $\alpha_i^{(s)}$ and $-x_i - s$ as $\alpha_{-i}^{(-s)}$ for any $i \in [n]$ and $s \in \{0, 1\}$. The numbers $-\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}$ will be represented as $\alpha_-^{(-1.5)}$, $\alpha_-^{(-0.5)}$, $\alpha_+^{(0.5)}$, $\alpha_+^{(1.5)}$ respectively.

Example 2.33. The total order

$$-\frac{3}{2} < x_2 < -x_1 - 1 < -\frac{1}{2} < x_1 < x_2 + 1 < -x_2 - 1 < -x_1 < \frac{1}{2} < x_1 + 1 < -x_2 < \frac{3}{2}$$

is represented as $\alpha_-^{(-1.5)}\ \alpha_2^{(0)}\ \alpha_{-1}^{(-1)}\ \alpha_-^{(-0.5)}\ \alpha_1^{(0)}\ \alpha_2^{(1)}\ \alpha_{-2}^{(-1)}\ \alpha_{-1}^{(0)}\ \alpha_+^{(0.5)}\ \alpha_1^{(1)}\ \alpha_{-2}^{(0)}\ \alpha_+^{(1.5)}$.

Set $B(n)$ to be the set

$$\{\alpha_i^{(s)} \mid i \in [n],\ s \in \{0,1\}\} \cup \{\alpha_i^{(s)} \mid -i \in [n],\ s \in \{-1,0\}\} \cup \{\alpha_-^{(-1.5)}, \alpha_-^{(-0.5)}, \alpha_+^{(0.5)}, \alpha_+^{(1.5)}\}.$$

We define *pointed symmetric sketches* to be the words in $B(n)$ that correspond to regions of $\mathcal{P}_n$ (this terminology will become clear soon). Denote by $\overline{\alpha_x^{(s)}}$ the letter $\alpha_{-x}^{(-s)}$ for any $\alpha_x^{(s)} \in B(n)$. We have the following characterization of pointed symmetric sketches:

Proposition 2.34. *A word in the letters $B(n)$ is a pointed symmetric sketch if and only if the following hold for any $\alpha_x^{(s)}, \alpha_y^{(t)} \in B(n)$:*

1. *If $\alpha_x^{(s)}$ appears before $\alpha_y^{(t)}$ then $\overline{\alpha_y^{(t)}}$ appears before $\overline{\alpha_x^{(s)}}$.*
2. *If $\alpha_x^{(s-1)}$ appears before $\alpha_y^{(t-1)}$ then $\alpha_x^{(s)}$ appears before $\alpha_y^{(t)}$.*
3. *$\alpha_x^{(s-1)}$ appears before $\alpha_x^{(s)}$.*
4. *Each letter of $B(n)$ appears exactly once.*

Just as was done in the proof of Proposition 2.8, we can inductively construct a point in $\mathbb{R}^n$ satisfying the inequalities specified by a pointed sketch. Also, just as for type C sketches, it can be shown that these sketches are symmetric about the center. We also represent such sketches using arc diagrams in a similar manner. Note that in this case we also inlcude an arc between $\alpha_-^{(-0.5)}$ and $\alpha_+^{(0.5)}$.

Example 2.35. To the pointed sketch given below, we associate the arc diagram in Figure 2.8.

$$\alpha_-^{(-1.5)}\ \alpha_2^{(0)}\ \alpha_{-1}^{(-1)}\ \alpha_-^{(-0.5)}\ \alpha_1^{(0)}\ \alpha_2^{(1)}\ \mid\ \alpha_{-2}^{(-1)}\ \alpha_{-1}^{(0)}\ \alpha_+^{(0.5)}\ \alpha_1^{(1)}\ \alpha_{-2}^{(0)}\ \alpha_+^{(1.5)}$$

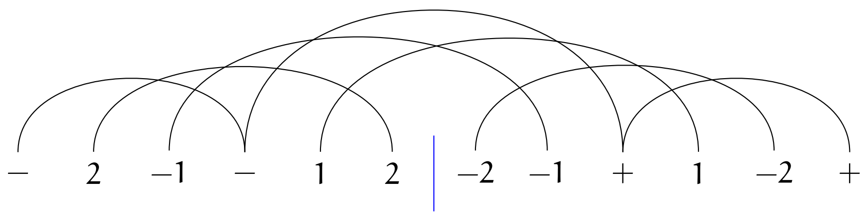


Figure 2.8: Arc diagram associated to the pointed symmetric sketch in Example 2.35.

To a pointed symmetric sketch, we can associate a pointed α, β-word of length $(2n+2)$ and a signed permutation as follows:

1. For the letters in the first half of the pointed sketch of the form $\alpha_i^{(0)}$, $\alpha_{-i}^{(-1)}$ or $\alpha_-^{(-1.5)}$, we write α and for the others we write β (α corresponds to 'openers' in the arc diagram and β to 'closers'). The β corresponding to $\alpha_-^{(0.5)}$ is pointed to.

2. The subscripts of the first n α-letters other than $\alpha_-^{(-1.5)}$ gives us the signed permutation.

Example 2.36. To the pointed sketch in Example 2.35, we associate the following pair:

1. Pointed α, β-word: $\alpha\alpha\alpha\beta\alpha\beta$.

2. Signed permutation: $2 \quad -1$.

As was done for symmetric sketches, we can see that the method given above to get a signed permutation does actually give a signed permutation. Also, such a pair has at most one pointed sketch associated to it. We now characterize the pointed α, β-words and signed permutations associated to pointed sketches.

Proposition 2.37. *A pair consisting of*

1. *a pointed α, β-word of length $(2n+2)$ satisfying the property that in any prefix, there are at least as many α-letters as β-letters and that the number of α-letters before the pointed β is $(n+1)$, and*

2. *any signed permutation*

corresponds to a pointed symmetric sketch and all pointed sketches correspond to such pairs.

Proof. Most of the proof is the same as that for type C sketches. The main difference is pointing to the β-letter corresponding to $\alpha_-^{(-0.5)}$. The property we have to take care of is that there is no nesting in the arc joining $\alpha_-^{(0.5)}$ to $\alpha_+^{(0.5)}$. This is the same as specifying when an arc drawn from a β-letter in the first half to its mirror image in the second half does not cause any nesting.

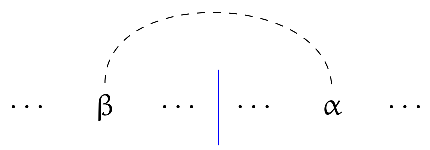


FIGURE 2.9: Arc from β to its mirror image.

Denote by $N_{\alpha,b}$ the number of α-letters before the β under consideration, $N_{\alpha,a}$ the number of α-letters in the first half after the β and similarly define $N_{\beta,b}$ and $N_{\beta,a}$. The condition that we do not want an arc inside the one joining the β to its mirror is given by

$$N_{\alpha,b} \geqslant N_{\beta,b} + 1 + N_{\beta,a} + N_{\alpha,a}.$$

This is because of the symmetry of the arc diagram and the fact that we want any β-letter between the pointed β and its mirror to have its corresponding α before the pointed β. Similarly, the condition that we do not want the arc joining the β to its mirror to be contained in any arc is given by

$$N_{\alpha,b} \leqslant N_{\beta,b} + 1 + N_{\beta,a} + N_{\alpha,a}.$$

This is because of the symmetry of the arc diagram and the fact that we want any α-letter before the pointed β to have its corresponding β before the mirror of the pointed β.

Combining the above observations, we get

$$N_{\alpha,b} = N_{\beta,b} + 1 + N_{\beta,a} + N_{\alpha,a}.$$

But this says that the number of α-letters before the pointed β should be equal to the number of remaining letters in the first half. Since the total number of letters in the first half is $(2n+2)$, we get that the arc joining a β in the first half to its mirror does not cause nesting problems if and only if the number of α-letters before it is $(n+1)$. □

Just as we used lattice paths for symmetric sketches, we use pointed lattice paths to represent pointed symmetric sketches. The one corresponding to the sketch in Example 2.35 is given in Figure 2.10.

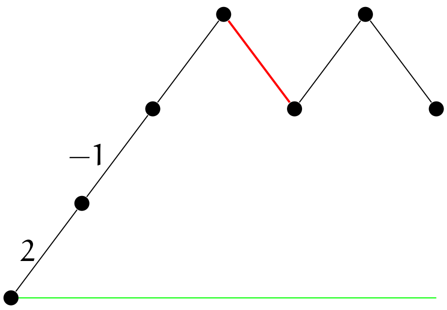


FIGURE 2.10: Pointed lattice path corresponding to the pointed sketch in Example 2.35.

Theorem 2.38. *The number of pointed symmetric sketches, which is the number of regions of $\mathcal{P}_n$, is*

$$2^n n! \binom{2n+2}{n}.$$

Proof. Since there is no condition on the signed permutations, we just have to count the α, β-words of the form mentioned in Proposition 2.37. We show that these words are in bijection with α, β-words of length $(2n+2)$ with any prefix having at least as many α-letters as β-letters that have at least $(n+2)$ α-letters. This means that their corresponding lattice paths do not end on the x-axis. This will prove the required result since the number of such words, using Lemma 2.14 and the fact that Catalan numbers count Dyck paths, is

$$\binom{2n+2}{n+1} - \frac{1}{n+2}\binom{2n+2}{n+1} = \binom{2n+2}{n}.$$

Given a pointed α, β-word, we replace the pointed β-letter with an α-letter to obtain an α, β-word of the type described above. Starting with an α, β-word with at least $(n+2)$ α-letters, changing the $(n+2)^{\text{th}}$ α-letter to a β and pointing to it gives a pointed α, β-word. This gives us the required bijection. □

Theorem 2.39. *The number of bounded regions of $\mathcal{P}_n$ is*

$$2^n n! \binom{2n+1}{n+1}.$$

Proof. Just as for type C regions, the region corresponding to a pointed sketch is bounded if and only if its arc diagram is interlinked. Also, the signed permutation does not play a role in determining if a region is bounded. Note that in this case, there is an arc joining a β-letter between the $(n+1)^{\text{th}}$ and $(n+2)^{\text{th}}$ α-letter to its mirror image. If the arc diagram obtained by deleting this arc from the pointed β-letter is interlinked, then clearly so was the initial arc diagram. However, even if the arc diagram consists of two interlinked pieces when the arc from the pointed β-letter is removed (one on either side of the reflecting line), the corresponding region would still be bounded. Examining the bijection between arc diagrams and lattice paths, it can be checked that this means that pointed lattice paths corresponding to bounded regions are those that never touch the x-axis after the origin except maybe at $(2n+2, 0)$.

Using the bijection mentioned in the proof of Theorem 2.38, we can see that the pointed α, β-words corresponding to bounded regions are in bijection α, β-words whose lattice paths never touch the x-axis after the origin. We have already counted such paths in Theorem 2.21 and their number is

$$\binom{2n+1}{n+1}.$$

This gives the required result. □

2.3.3 Type B Catalan

Fix $n \geqslant 1$. The type B Catalan arrangement in $\mathbb{R}^n$ has the hyperplanes

$$\begin{aligned} X_i &= -1, 0, 1 \\ X_i + X_j &= -1, 0, 1 \\ X_i - X_j &= -1, 0, 1 \end{aligned}$$

for all $1 \leqslant i < j \leqslant n$. Translating this arrangement by setting $X_i = x_i + \frac{1}{2}$, we get the arrangement $\mathcal{B}_n$ with hyperplanes

$$\begin{aligned} x_i &= -\frac{3}{2}, -\frac{1}{2}, \frac{1}{2} \\ x_i + x_j &= -2, -1, 0 \\ x_i - x_j &= -1, 0, 1 \end{aligned}$$

for all $1 \leqslant i < j \leqslant n$. We consider $\mathcal{B}_n$ as a sub-arrangement of $\mathcal{P}_n$. The hyperplanes missing from $\mathcal{P}_n$ are

$$x_i = -\frac{5}{2}, -1, 0, \frac{3}{2}$$

for all $i \in [n]$. Hence the moves on pointed sketches corresponding to changing one of the inequalities associated to these hyperplanes are as follows:

1. Corresponding to $x_i = 0$, $x_i = -1$: Swapping to $(2n+2)^{th}$ and $(2n+3)^{th}$ letter if they are not $\alpha_-^{(-0.5)}$ and $\alpha_+^{(0.5)}$.
2. Corresponding to $x_i = -\frac{5}{2}$, $x_i = \frac{3}{2}$: Swapping the pointed β, that is, $\alpha_-^{(-0.5)}$ and a β-letter immediately before or after it (and making the corresponding change in the second half).

We can see that such moves change the pointed α, β-word associated to a sketch by at most changing the last letter or changing which of the β-letters between the $(n+1)^{th}$ and $(n+2)^{th}$ α-letter is pointed to. So if we force that the last letter of the sketch has to be a β-letter and that the β-letter immediately after the $(n+1)^{th}$ α-letter has to be pointed to, we get a canonical sketch in each equivalence class. We will call such sketches type B sketches.

Theorem 2.40. *The number of type* B *sketches, which is the number of regions of* $\mathcal{B}_n$*, is*

$$2^n n! \binom{2n}{n}.$$

Proof. Since there is no condition on the signed permutation, we count the α, β-words associated to type B sketches. From Proposition 2.37, we can see that the α, β-words we need to count are those that satisfy the following properties:

1. Length of the word is $(2n+2)$.
2. In any prefix, there are at least as many α-letters as β-letters.
3. The letter immediately after the $(n+1)^{th}$ α-letter is a β (pointed β).
4. The last letter is a β.

We exhibit a bijection between these words and α, β-words of length $2n$ that satisfy property 2. We already know, from Lemma 2.14, that the number of such words is $\binom{2n}{n}$ and so this will prove the required result.

If the $(n+1)^{\text{th}}$ α-letter is at the $(2n+1)^{\text{th}}$ position, deleting the last two letters gives us an α, β-word of length $2n$ with n α-letters that satisfies property 2. If the $(n+1)^{\text{th}}$ α-letter is not at the $(2n+1)^{\text{th}}$ position, we delete the β-letter after it as well as the last letter of the word. This gives us an α, β-word of length $2n$ with more than n α-letters that satisfies property 2. The process described gives us the required bijection. □

Theorem 2.41. *The number of bounded regions of $\mathcal{B}_n$ is*

$$2^n n! \binom{2n-1}{n}.$$

Proof. Both $\mathcal{B}_n$ and $\mathcal{P}_n$ have rank n. Hence a region of $\mathcal{B}_n$ if bounded if and only if all regions of $\mathcal{P}_n$ that it contains are bounded.

In the proof of Theorem 2.39 we have characterized the pointed α, β-words associated to bounded regions of $\mathcal{P}_n$. These are the pointed lattice paths of length $(2n+2)$ that satisfy the following properties (irrespective of the position of the pointed β):

1. The step after the $(n+1)^{\text{th}}$ up-step is a down step (for there to exist a pointed β).
2. The path never touches the x-axis after the origin expect maybe at $(2n+2, 0)$.

We noted in Theorem 2.31 that lattice paths satisfying property 2 are closed under action of changing the letter after the $(n+1)^{\text{th}}$ up-step as well as the action of changing the last step. This shows that the regions of $\mathcal{P}_n$ that lie inside a region of $\mathcal{B}_n$ are either all bounded or all unbounded. Hence the number of bounded regions of $\mathcal{B}_n$ is just the number of type B sketches whose corresponding lattice path satify property 1 and 2, which is

$$2^n n! \cdot \frac{1}{4} \cdot \left(\binom{2n+1}{n+1} + \frac{1}{n+1} \binom{2n}{n} \right).$$

This simplifies to give the required result. □

2.3.4 Type BC Catalan

The type BC Catalan arrangement in $\mathbb{R}^n$ has hyperplanes

$$\begin{aligned} X_i &= -1, 0, 1 \\ 2X_i &= -1, 0, 1 \\ X_i + X_j &= -1, 0, 1 \\ X_i - X_j &= -1, 0, 1 \end{aligned}$$

for all $1 \leqslant i < j \leqslant n$. Translating this arrangement by setting $X_i = x_i + \frac{1}{2}$, we get the arrangement $\mathcal{BC}_n$ with hyperplanes

$$\begin{aligned} x_i &= -\frac{3}{2}, -1, -\frac{1}{2}, 0, \frac{1}{2} \\ x_i + x_j &= -2, -1, 0 \\ x_i - x_j &= -1, 0, 1 \end{aligned}$$

for all $1 \leqslant i < j \leqslant n$. Again, we consider this arrangement as a sub-arrangement of $\mathcal{P}_n$. To define moves on pointed sketches, note that the hyperplanes missing from $\mathcal{P}_n$ are

$$x_i = -\frac{5}{2}, \frac{3}{2}$$

for all $i \in [n]$. Hence, the moves on pointed sketches corresponding to changing the inequalities associated to these hyperplanes are of the following form: Swapping the pointed β, that is, $\alpha_-^{(-0.5)}$ and a β-letter immediately before or after it (and making the corresponding change in the second half).

We can see that such moves change the pointed α, β-word associated to a sketch by at most changing which of the β-letters between the $(n+1)^{\text{th}}$ and $(n+2)^{\text{th}}$ α-letter is pointed to. So if we force that the β-letter immediately after the $(n+1)^{\text{th}}$ α-letter has to be pointed to, we get a canonical sketch in each equivalence class. We will call such sketches type BC sketches.

Theorem 2.42. *The number of type* BC *sketches, which is the number of regions of* $\mathcal{BC}_n$, *is*

$$2^{n-1} n! \binom{2n+2}{n+1}.$$

Proof. Since there is no condition on the signed permutation for type BC sketches, we count the number of α, β-words that satisfy the following properties:

1. Length of the word is $(2n+2)$.
2. In any prefix, there are at least as many α-letters as β-letters.
3. The letter immediately after the $(n+1)^{\text{th}}$ α-letter is a β (pointed β).

Using the involution on the set of words satisfying properties 1 and 2 of changing the letter immediately after the $(n+1)^{\text{th}}$ α-letter and the fact that there are $\binom{2n+2}{n+1}$ words satisfying properties 1 and 2, we get that the number of words satisfying the required properties is

$$\frac{1}{2} \cdot \binom{2n+2}{n+1}.$$

This gives the required result. □

Theorem 2.43. *The number of bounded regions of $\mathcal{BC}_n$ is*

$$2^n n! \binom{2n}{n}.$$

Proof. The proof of this result is very similar to that of Theorem 2.41. Since type BC sketches don't have the condition that the $2n^{\text{th}}$ letter should be a β-letter, the number of bounded regions of $\mathcal{BC}_n$ is

$$2^n n! \cdot \frac{1}{2} \cdot \left(\binom{2n+1}{n+1} + \frac{1}{n+1} \binom{2n}{n} \right).$$

This simplifies to give the required result. □

Chapter 3

Statistics on regions

Recall that the 'sketches and moves' idea mentioned in the previous chapter was first used by Bernardi [8, Section 8] to describe trees that correspond to regions of certain braid deformations. In this chapter, we first describe statistics on these trees whose distribution is given by the coefficients of the characteristic polynomial of the corresponding arrangements. We then use similar ideas to describe statistics on the objects corresponding to regions of Catalan deformations of reflection arrangements mentioned in the previous chapter.

We obtain these statistics by giving combinatorial meaning to the exponential generating functions of the characteristic polynomials of these arrangements. The results in Section 3.1 are from [16] and those in Section 3.2 are from [17, Section 5], which are both joint work with Priyavrat Deshpande.

3.1 Deformations of the braid arrangement

As mentioned in Chapter 1, the characteristic polynomial of an arrangement $\mathcal{A}$ in $\mathbb{R}^n$ is of the form

$$\chi_{\mathcal{A}}(t) = \sum_{i=0}^{n} (-1)^{n-i} c_i t^i$$

where c_i is a non-negative integer for all $0 \leqslant i \leqslant n$ and Zaslavsky's theorem tells us that

$$\begin{aligned} r(\mathcal{A}) &= (-1)^n \chi_{\mathcal{A}}(-1) \\ &= \sum_{i=0}^{n} c_i. \end{aligned}$$

We now give combinatorial interpretations to the coefficients of the characteristic polynomials of certain deformations of the braid arrangement.

For any finite set of integers S, we associate a deformation of the braid arrangement $\mathcal{A}_S(n)$ in $\mathbb{R}^n$ with hyperplanes

$$\{x_i - x_j = k \mid k \in S,\ 1 \leqslant i < j \leqslant n\}.$$

Important examples of such arrangements are the Catalan, Shi, Linial and semiorder arrangements. These correspond to $S = \{-1, 0, 1\}$, $\{0, 1\}$, $\{1\}$, and $\{-1, 1\}$ respectively. For any $m \geqslant 1$, the extended Catalan arrangement, or m-Catalan arrangement, in $\mathbb{R}^n$ is $\mathcal{A}_S(n)$ where $S = \{-m, \ldots, m\}$. Similarly, the extended Shi, Linial, and semiorder arrangements correspond to $S = \{-m+1, \ldots, m\}$, $\{-m+1, \ldots, m\} \setminus \{0\}$, and $\{-m, \ldots, m\} \setminus \{0\}$ respectively.

We now recall results from [8] about trees that correspond to regions of such arrangements and also some results from [57].

3.1.1 Trees and exponential structures

A *tree* is a graph with no cycles. A *rooted tree* is a tree with a distinguished vertex called the root. We will draw rooted trees with their root at the bottom. Children of a vertex v in a rooted tree are those vertices w that are adjacent to v and such that the unique path from the root to w passes through v. Similarly, we can define the parent of a vertex v to be the vertex w for which v is the child of w. Any non-root vertex has a unique parent. All the vertices that have at least one child are called *nodes* and those that do not are called *leaves*.

A *rooted plane tree* is a rooted tree with a specified ordering for the children of each node. When drawing a rooted plane tree, the children of any node will be ordered from left to right. The *left siblings* of a vertex v are the vertices that are also children of the parent of v but are to the left of v. We denote the number of left siblings of v as $\text{lsib}(v)$.

Definition 3.1. An $(m+1)$-*ary tree* is a rooted plane tree where each node has exactly $(m+1)$ children. We will denote by $\mathcal{T}^{(m)}(n)$ the set of all $(m+1)$-ary trees with n nodes labeled with distinct elements from $[n]$.

For trees in $\mathcal{T}^{(m)}(n)$, we will denote the node having label $i \in [n]$ by just i.

Definition 3.2. If a node i in a tree $T \in \mathcal{T}^{(m)}(n)$ has at least one child that is a node, the *cadet* of i is the rightmost such child, which we denote by $\text{cadet}(i)$.

Example 3.3. Figure 3.1 shows an element of $\mathcal{T}^{(1)}(4)$ where

- 4 is the root,
- $\text{lsib}(2) = 0$, $\text{lsib}(3) = 0$, $\text{lsib}(1) = 1$,
- $\text{cadet}(4) = 2$, and $\text{cadet}(2) = 1$.

Definition 3.4. For any finite set of integers S with $m = \max\{|s| \mid s \in S\}$, define $\mathcal{T}_S(n)$ to be the set of trees in $\mathcal{T}^{(m)}(n)$, such that if $\text{cadet}(i) = j$:

- $\text{lsib}(j) \notin S \cup \{0\} \Rightarrow i < j$.
- $-\text{lsib}(j) \notin S \Rightarrow i > j$.

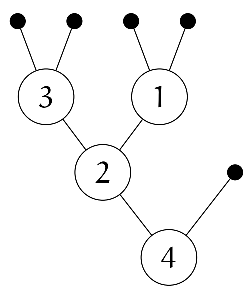


FIGURE 3.1: A tree in $\mathcal{T}_{\{0,1\}}(4)$

Definition 3.5. A finite set of integers S is said to be *transitive* if for any $s, t \notin S$,

- $st > 0 \Rightarrow s + t \notin S$.
- $s > 0$ and $t \leqslant 0 \Rightarrow s - t \notin S$ and $t - s \notin S$.

Example 3.6. For any $m \geqslant 1$, the sets $\{-m, \ldots, m\}$, $\{-m+1, \ldots, m\}$, $\{-m, \ldots, m\} \setminus \{0\}$, and $\{-m+1, \ldots, m\} \setminus \{0\}$ are all transitive.

Recall that for any finite set of integers S, we defined the arrangement $\mathcal{A}_S(n)$ as the deformation of the braid arrangement in $\mathbb{R}^n$ with hyperplanes

$$\{x_i - x_j = k \mid k \in S,\ 1 \leqslant i < j \leqslant n\}.$$

We can now state the result for arrangements $\mathcal{A}_S(n)$ where S is transitive. Though Bernardi [8] derived results for more general deformations, we will only be focused on these.

Theorem 3.7. *[8, Theorem 3.8] For any transitive set of integers S, the regions of the arrangement $\mathcal{A}_S(n)$ are in bijection with the trees in $\mathcal{T}_S(n)$.*

Before looking at the characteristic polynomials of such arrangements, we recall a few results from [57]. Suppose that $c : \mathbb{N} \to \mathbb{N}$ is a function and for each $n, j \in \mathbb{N}$, we define

$$c_j(n) = \sum_{\{B_1,\ldots,B_j\} \in \Pi_n} c(|B_1|) \cdots c(|B_j|)$$

where Π_n is the set of partitions of $[n]$. Define for each $n \in \mathbb{N}$,

$$h(n) = \sum_{j=0}^{n} c_j(n).$$

From [57, Example 5.2.2], we know that in such a situation,

$$\sum_{n,j \geqslant 0} c_j(n) t^j \frac{x^n}{n!} = \left(\sum_{n \geqslant 0} h(n) \frac{x^n}{n!} \right)^t.$$

Informally, we consider $h(n)$ to be the number of “structures” that can be placed on an n-set where each structure can be uniquely broken up into a disjoint union of “connected sub-structures”. Here $c(n)$ denotes the number of connected structures on an n-set and $c_j(n)$ denotes the number of structures on an n-set with exactly j connected sub-structures. We will call such structures *exponential structures*.

We now consider the characteristic polynomials of arrangements of the form $\mathcal{A}_S(n)$. For a fixed set S, the sequence of arrangements $(\mathcal{A}_S(1), \mathcal{A}_S(2), \ldots)$ forms what is called an *exponential sequence of arrangements* (ESA).

Definition 3.8. [54, Definition 5.14] A sequence of arrangements $(\mathcal{A}_1, \mathcal{A}_2, \ldots)$ is called an ESA if

- $\mathcal{A}_n$ is a braid deformation in $\mathbb{R}^n$.
- For any k-subset I of $[n]$, the arrangement

$$\mathcal{A}_n^I = \{H \in \mathcal{A}_n \mid H \text{ is of the form } x_i - x_j = s \text{ for some } i, j \in I\}$$

  satisfies $L(\mathcal{A}_n^I) \cong L(\mathcal{A}_k)$ (isomorphic as posets).

The result on ESAs that we will need is the following.

Theorem 3.9. *[54, Theorem 5.17] If $(\mathcal{A}_1, \mathcal{A}_2, \ldots)$ is an ESA, then*

$$\sum_{n \geqslant 0} \chi_{\mathcal{A}_n}(t) \frac{x^n}{n!} = \left(\sum_{n \geqslant 0} (-1)^n r(\mathcal{A}_n) \frac{x^n}{n!} \right)^{-t}.$$

Remark 3.10. Proposition 3.21, which is a more general version of the above theorem, is proved in the following section. We also note that this theorem is a special case of [8, Theorem 5.2].

Using this result and the above discussion on exponential structures, we note that interpreting the coefficients of the polynomial $\chi_{\mathcal{A}_S(n)}(t)$ is equivalent to defining a notion of "connected structures" for trees in $\mathcal{T}_S(n)$. We do this in the next subsection.

3.1.2 A branch statistic

A *label set* is a finite set of positive integers. For any label set V, we define $\mathcal{T}^{(m)}(V)$ to be the set of $(m+1)$-ary trees with $|V|$ nodes labeled distinctly using V. Note that $\mathcal{T}^{(m)}([n]) = \mathcal{T}^{(m)}(n)$.

We now describe the method we use to break up a tree in $\mathcal{T}^{(m)}(V)$ into "connected sub-structures", which we call *branches*.

Definition 3.11. The *trunk* of a tree in $\mathcal{T}^{(m)}(V)$ is the path from the root to the leftmost leaf. The nodes on the trunk of the tree break up the tree into sub-trees, which we call *twigs* (see Figure 3.2).

Let the nodes on the trunk of a tree be $v_1, v_2, \dots, v_k$, where v_1 is the root and v_{i+1} is the leftmost child of v_i for any $i \in [k-1]$. If $v_i = \max\{v_1, \dots, v_k\}$, then the first branch of the tree consists of the twigs corresponding to the nodes $v_1, \dots, v_i$. If $v_j = \max\{v_{i+1}, \dots, v_k\}$, then the second branch of the tree consists of the twigs corresponding to the nodes $v_{i+1}, \dots, v_j$. Continuing this way, we break up the tree into branches.

Note that the number of branches of the tree is just the number of right-to-left maxima of the sequence $v_1, v_2, \dots, v_k$ of nodes on the trunk, *i.e.*, the number of v_i such that $v_i > v_j$ for all $j > i$. We will call such v_i the *branch nodes* of the trunk.

Example 3.12. The tree in Figure 3.2 has 3 twigs and 2 branches. The first branch consists of just the first twig since 6 is the largest node in the trunk. The second branch consists of the second and third twigs since 5 is larger than 4. Here 6 and 5 are the branch nodes.

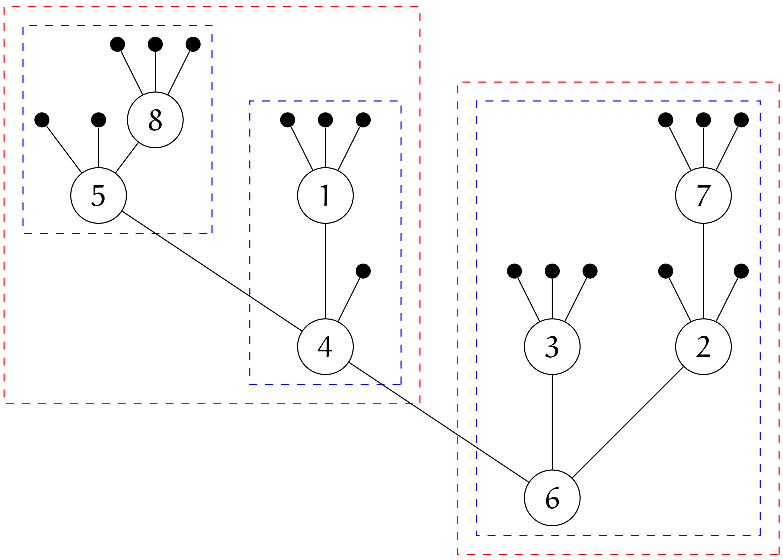


FIGURE 3.2: A labeled 3-ary tree with twigs and branches specified.

We use the notation $\mathcal{T}_j^{(m)}(V)$ to denote the trees in $\mathcal{T}^{(m)}(V)$ having j branches. To prove that branches give trees an exponential structure, we have to prove that

$$|\mathcal{T}_j^{(m)}(V)| = \sum_{\{B_1,\dots,B_j\}\in\Pi_V} |\mathcal{T}_1^{(m)}(B_1)| \cdots |\mathcal{T}_1^{(m)}(B_j)|. \tag{3.1}$$

Hence, "connected" trees are those with exactly one branch, *i.e.*, trees where the last node of the trunk is the one with the largest label. Similarly, the connected components associated to a given tree are the branches of the tree.

Example 3.13. The connected components associated to the tree in Figure 3.2 are given in Figure 3.3.

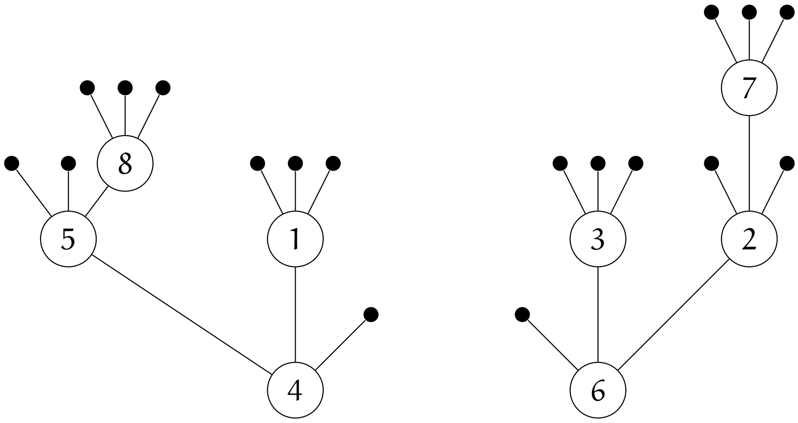


FIGURE 3.3: Connected components of the tree in Figure 3.2.

A collection of connected trees (with disjoint label sets) can be put together in exactly one way to form a tree for which they form the branches. This is done as follows: Find the largest label among those on the trunks of the connected trees. The connected tree T_1 with this label is made the first branch of the tree we are building. Again, find the largest label among those on the trunks of the remaining connected trees. The connected tree T_2 with this label is made the second branch of the tree we are building by gluing it to T_1. This is done by deleting the leftmost leaf of T_1 and fixing the root of T_2 in its position. This process is repeated to until all the connected trees are glued together.

The observations above show that branches give the trees in $\mathcal{T}^{(m)}(V)$ an exponential structure.

Example 3.14. The tree associated to the collection of connected trees in Figure 3.4 is given in Figure 3.5.

Recall that for a finite set of integers S with $m = \max\{|s| : s \in S\}$, the set $\mathcal{T}_S(V)$, for some label set V, is the set of trees in $\mathcal{T}^{(m)}(V)$ such that if $\text{cadet}(u) = v$, then

- if $\text{lsib}(v) \notin S \cup \{0\}$, we must have $u < v$, and
- if $-\text{lsib}(v) \notin S$, we must have $u > v$.

We call this set of conditions "Condition S".

We set $\mathcal{T}_S := \bigcup_V \mathcal{T}_S(V)$ where the union is over all label sets V. We now show that

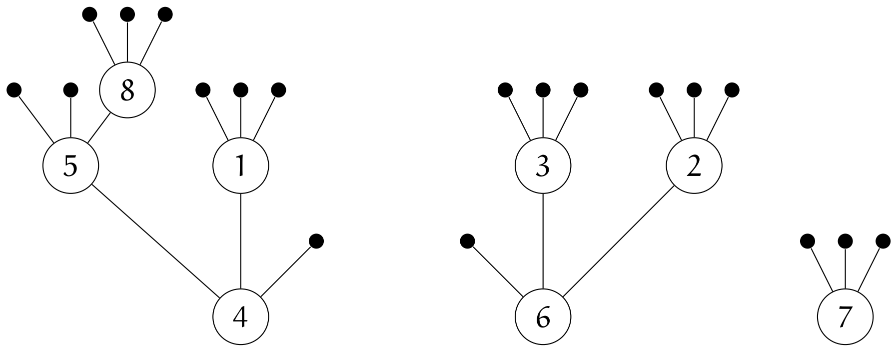


FIGURE 3.4: A collection of connected trees.

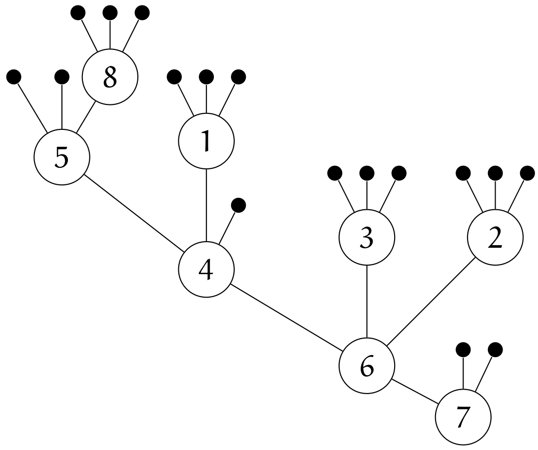


FIGURE 3.5: The tree associated to the collection of connected trees in Figure 3.4.

1. the connected components of any tree in $\mathcal{T}_S$ are also in $\mathcal{T}_S$, and
2. trees that are built using connected trees in $\mathcal{T}_S$ are also in $\mathcal{T}_S$.

We first note that statement 1 follows since the condition for a tree to be in $\mathcal{T}_S$ is a local condition. This is also because, if T' is a connected component of the tree T, the cadet of any node in T' (if it exists) is the same as its cadet when considered as a node of T.

To prove statement 2, we only have to check that Condition S is satisfied for the branch nodes of a tree built using connected trees in $\mathcal{T}_S$. If a branch node does not have a cadet, Condition S is trivially satisfied. If a branch node u has a cadet v, we consider two cases:

- If the cadet is not the first child, then Condition S is satisfied since it is satisfied by the connected components of the tree.

- If the cadet is the first child, then we must have $u > v$ since u is a branch node. This makes sure that Condition S is satisfied since we have $\text{lsib}(v) = 0$ and hence $\text{lsib}(v) \in S \cup \{0\}$.

From the preceding, we get an equation analogous to (3.1) for the trees $\mathcal{T}_S$. This shows that branches give the trees in $\mathcal{T}_S$ an exponential structure. Hence, from the discussion in Section 3.1.1, we get the following result.

Theorem 3.15. *For a transitive set of integers S, the absolute value of the coefficient of t^j in $\chi_{\mathcal{A}_S(n)}(t)$ is the number of trees in $\mathcal{T}_S(n)$ with j branches.*

Example 3.16. When $S = \{0\}$, we obtain the braid arrangement. Here, $\mathcal{T}_{\{0\}}(n)$ corresponds to permutations of $[n]$ and Theorem 3.15 states that the absolute value of the coefficient of t^j in $\chi_{\mathcal{A}_{\{0\}}(n)}(t)$ is the number of permutations of $[n]$ with j right-to-left maxima. By [55, Corollary 1.3.11], this agrees with the fact that the coefficients are the Stirling numbers of the first kind [54, Corollary 2.2].

Example 3.17. The Shi arrangement $\mathcal{S}_n$ in $\mathbb{R}^n$ is the deformation $\mathcal{A}_{\{0,1\}}(n)$. The trees in $\mathcal{T}_{\{0,1\}}(n)$, called Shi trees, are those labeled binary trees where any right node has a label less than that of its parent. The Shi trees for $n = 3$ are given in Figures 3.6 and 3.7. Counting the branches in these trees, we get $\chi_{\mathcal{S}_3}(t) = t^3 - 6t^2 + 9t$, which agrees with the known formula for the characteristic polynomial (for example, see [3, Theorem 3.3]).

Remark 3.18. The Shi trees $\mathcal{T}_{\{0,1\}}(n)$ are in bijection with Cayley trees on $n+1$ vertices. Using a decomposition of Cayley trees, one can show that the coefficient of t^j in $\chi_{\mathcal{S}_n}(t)$ is the number of such Cayley trees where the vertex $n+1$ has degree j.

Example 3.19. The Linial arrangement $\mathcal{L}_n$ in $\mathbb{R}^n$ is the deformation $\mathcal{A}_{\{1\}}(n)$. The trees in $\mathcal{T}_{\{1\}}(n)$, called Linial trees, are those Shi trees that also satisfy the property that any left node whose sibling is a leaf has smaller label than that of its parent. The Linial trees for $n = 3$ are given in Figure 3.7. Counting the branches in these trees, we get $\chi_{\mathcal{L}_3}(t) = t^3 - 3t^2 + 3t$, which agrees with the known formula for the characteristic polynomial (for example, see [3, Theorem 4.2]).

3.2 Catalan deformations of reflection arrangements

In this section, for each arrangement we have studied in Chapter 2, we first define a statistic on the objects that we have seen correspond to its regions. We then show

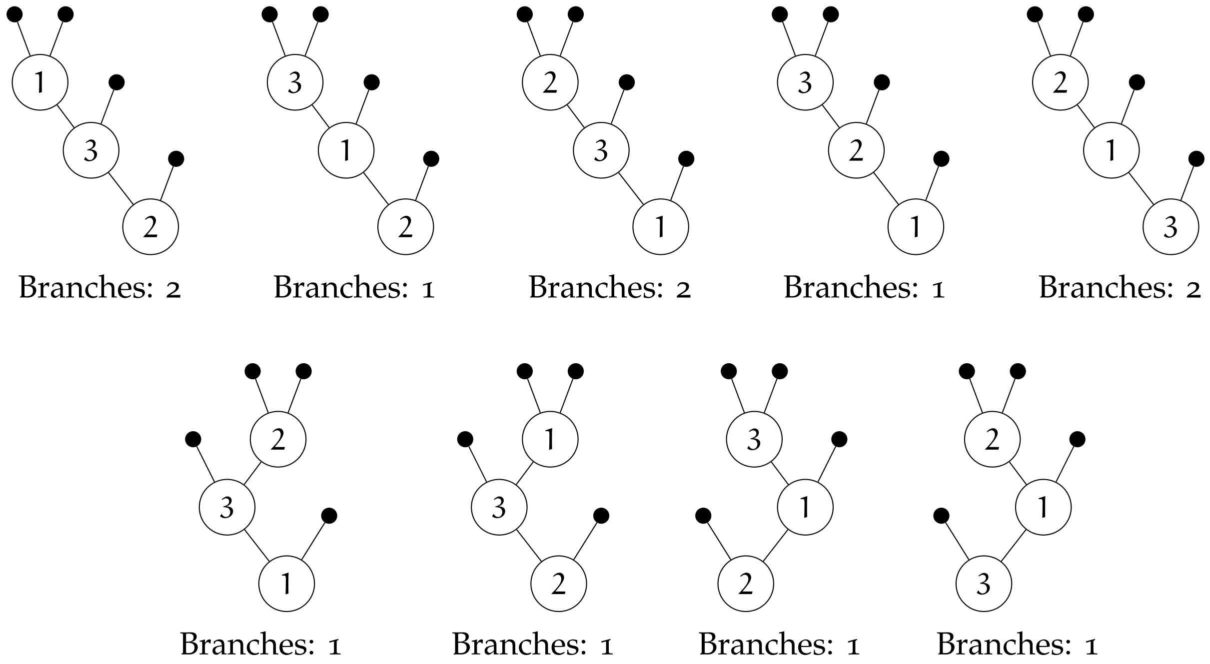


Figure 3.6: Shi trees for $n = 3$ that are not Linial.

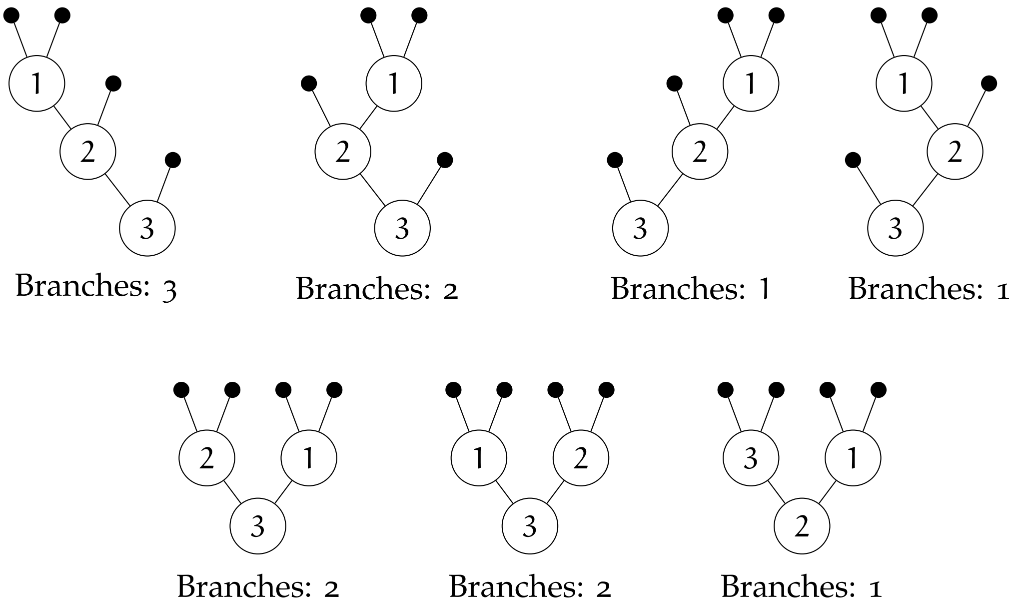


Figure 3.7: Linial trees for $n = 3$.

that the distribution of this statistic is given by the coefficients of the characteristic polynomial.

Just as in the previous section, we do this by giving combinatorial meaning to the exponential generating functions for the characteristic polynomials of the arrangements we have studied. To obtain these generating functions, we use [54, Exercise 5.10], which we state and prove for convenience.

Definition 3.20. A sequence of arrangements $(\mathcal{A}_1, \mathcal{A}_2, \dots)$ is called a *Generalized Exponential Sequence of Arrangements* (GESA) if

- $\mathcal{A}_n$ is an arrangement in $\mathbb{R}^n$ such that every hyperplane is parallel to one of the form $x_i = cx_j$ for some $c \in \mathbb{R}$.
- For any k-subset I of $[n]$, the arrangement

  $$\mathcal{A}_n^I = \{H \in \mathcal{A}_n \mid H \text{ is parallel to } x_i = cx_j \text{ for some } i, j \in I \text{ and some } c \in \mathbb{R}\}$$

  satisfies $L(\mathcal{A}_n^I) \cong L(\mathcal{A}_k)$ (isomorphic as posets).

Note that all the arrangements we have studied are GESAs.

Proposition 3.21. *[54, Exercise 5.10] Let $(\mathcal{A}_1, \mathcal{A}_2, \dots)$ be a GESA, and define*

$$F(x) = \sum_{n \geqslant 0} (-1)^n r(\mathcal{A}_n) \frac{x^n}{n!}$$
$$G(x) = \sum_{n \geqslant 0} (-1)^{\operatorname{rank}(\mathcal{A}_n)} b(\mathcal{A}_n) \frac{x^n}{n!}.$$

Then, we have

$$\sum_{n \geqslant 0} \chi_{\mathcal{A}_n}(t) \frac{x^n}{n!} = \frac{G(x)^{(t+1)/2}}{F(x)^{(t-1)/2}}.$$

Proof. The idea of the proof is the same as that of [54, Theorem 5.17]. By Whitney's Theorem [54, Theorem 2.4], we have for all n,

$$\chi_{\mathcal{A}_n}(t) = \sum_{\mathcal{B} \subseteq \mathcal{A},\, \bigcap \mathcal{B} \neq \phi} (-1)^{\#\mathcal{B}} t^{n - \operatorname{rank}(\mathcal{B})}.$$

To each $\mathcal{B} \subseteq \mathcal{A}_n$, such that $\bigcap \mathcal{B} \neq \phi$, we associate a graph $G(\mathcal{B})$ on the vertex set $[n]$ where there is an edge between the vertices i and j if there is a hyperplane in $\mathcal{B}$ parallel to a hyperplane of the form $x_i = cx_j$ for some $c \in \mathbb{R}$.

Using [57, Corollary 5.1.6], we get

$$\sum_{n\geqslant 0} \chi_{\mathcal{A}_n}(t)\frac{x^n}{n!} = \exp \sum_{n\geqslant 1} \tilde{\chi}_{\mathcal{A}_n}(t)\frac{x^n}{n!}$$

where for any n we define

$$\tilde{\chi}_{\mathcal{A}_n}(t) = \sum_{\substack{\mathcal{B}\subseteq\mathcal{A},\ \bigcap\mathcal{B}\neq\phi \\ G(\mathcal{B}) \text{ connected}}} (-1)^{\#\mathcal{B}} t^{n-\text{rank}(\mathcal{B})}.$$

Note that if $G(\mathcal{B})$ is connected, then any point in $\bigcap\mathcal{B}$ is determined by any one of its coordinates, say x_1. This is because any path from the vertex 1 to a vertex i in $G(\mathcal{B})$ can be used to determine x_i. This shows us that $\text{rank}(\mathcal{B})$ is either n or $n-1$. Hence, $\chi_{\tilde{\mathcal{A}}_n}(t) = c_n t + d_n$ for some $c_n, d_n \in \mathbb{Z}$. Setting

$$\exp \sum_{n\geqslant 1} c_n \frac{x^n}{n!} = \sum_{n\geqslant 0} b_n \frac{x^n}{n!}$$
$$\exp \sum_{n\geqslant 1} d_n \frac{x^n}{n!} = \sum_{n\geqslant 0} a_n \frac{x^n}{n!}$$

we get

$$\sum_{n\geqslant 0} \chi_{\mathcal{A}_n}(t)\frac{x^n}{n!} = \left(\sum_{n\geqslant 0} b_n \frac{x^n}{n!}\right)^t \left(\sum_{n\geqslant 0} a_n \frac{x^n}{n!}\right).$$

Substituting $t = 1$ and $t = -1$ and using Theorem 1.14, we obtain expressions for the exponential generating functions of $\{b_n\}$ and $\{c_n\}$ and this gives us the required result. □

Recall that in the previous section, we have seen how to interpret the generating function equality

$$\sum_{n,j\geqslant 0} c_j(n) t^j \frac{x^n}{n!} = \left(\sum_{n\geqslant 0} h(n)\frac{x^n}{n!}\right)^t.$$

That is, we consider $h(n)$ to be the number of "structures" that can be placed on an n-set where each structure can be uniquely broken up into a disjoint union of "connected sub-structures". Here $c_1(n)$ denotes the number of connected structures on an n-set and $c_j(n)$ denotes the number of structures on an n-set with exactly j connected sub-structures. We called such structures *exponential structures*.

In fact, in most of the computations below, we will be dealing with generating functions of the form

$$\left(\sum_{n \geqslant 0} h(n) \frac{x^n}{n!}\right)^{\frac{t+1}{2}}. \tag{3.2}$$

We can interpret such a generating function as follows. Suppose that there are two types of connected structures, say positive and negative connected structures. Also, suppose that the number of positive connected structures on $[n]$ is the same as the number of negative ones, i.e., $c_1(n)/2$. Then the coefficient of $t^j \frac{x^n}{n!}$ in the generating function given above is the number of structures on $[n]$ that have j positive connected sub-structures.

Also, note that since the coefficients of the characteristic polynomial alternate in sign, the distribution of any appropriate statistic we define would be

$$\sum_{n \geqslant 0} \chi_{\mathcal{A}_n}(-t) \frac{(-x)^n}{n!}.$$

3.2.1 Simple examples

Before defining statistics for the Catalan arrangements, we first do so for the reflection arrangements we studied in Section 2.1.1. We also note that the following results can be proved by directly looking at the coefficients of the characteristic polynomials as done in Section 1.4.

The type C arrangement. We have seen that the regions of the type C arrangement in $\mathbb{R}^n$ correspond to sketches (Section 2.1.1) of length $2n$. We use the second half of the sketch to represent the regions, and call them signed permutations on $[n]$.

A statistic on signed permutations whose distribution is given by the coefficients of the characteristic polynomial is given in [19, Section 2]. We define a similar statistic. First break the signed permutation into *compartments* using right-to-left minima as follows: Ignoring the signs, draw a line before the permutation and then repeatedly draw a line immediately following the least number after the last line drawn. This is repeated until a line is drawn at the end of the permutation. It can be checked that compartments give signed permutations an exponential structure. A *positive compartment* of a signed permutations is one where the last term is positive.

Example 3.22. The signed permutation given by

$$\overset{+}{3}\,\overset{+}{1}\,\overset{-}{6}\,\overset{-}{7}\,\overset{-}{5}\,\overset{+}{2}\,\overset{-}{4}$$

is split into compartments as

$$|\,\overset{+}{3}\,\overset{+}{1}\,|\,\overset{-}{6}\,\overset{-}{7}\,\overset{-}{5}\,\overset{+}{2}\,|\,\overset{-}{4}\,|$$

and hence has 3 compartments, 2 of which are positive.

By the combinatorial interpretation of (3.2), the distribution of the statistic 'number of positive compartments' on signed permutations is given by

$$\left(\frac{1}{1-2x}\right)^{\frac{t+1}{2}}.$$

Note that for the type C arrangement, in terms of Proposition 3.21, we have

$$\begin{aligned} F(x) &= \left(\frac{1}{1+2x}\right), \\ G(x) &= 1. \end{aligned}$$

Hence, we get that the distribution of the statistic 'number of positive compartments' on signed permutations is given by the coefficients of the characteristic polynomial.

The type D arrangement. From Section 2.1.1, we can see that the regions of the type D arrangement in $\mathbb{R}^n$ correspond to signed permutations on $[n]$ where the first sign is positive. We will show that 'number of positive compartments' is a statistic that works for this situation as well.

Given $i \in [n]$ and a signed permutation σ of $[n] \setminus \{i\}$, the signed permutation of $[n]$ obtained by appending $\bar{i}$ to the start of σ has the same number of positive compartments as σ. This shows that the distribution of the statistic on signed permutations whose first term is positive is

$$(1-x)\left(\frac{1}{1-2x}\right)^{\frac{t+1}{2}}.$$

This agrees with the generating function for the characteristic polynomial we get from Proposition 3.21 since we have

$$F(x) = \left(\frac{1+x}{1+2x}\right),$$
$$G(x) = 1 + x.$$

Note that the expression for $G(x)$ is due to the fact that the type D arrangement in $\mathbb{R}^1$ is empty.

3.2.2 Statistics on symmetric partitions

We start with defining a statistic for the extended type C Catalan arrangements. Using Proposition 3.21, we then show that the generating function for the statistic and the characteristic polynomials match.

Fix $m \geqslant 1$. We define a statistic on labeled symmetric non-nesting partitions and show that its distribution is given by the characteristic polynomial. To do this, we first recall some definitions and results about the type A extended Catalan arrangement.

Definition 3.23. An m-*non-nesting partition of size* n is a partition of $[(m+1)n]$ such that the following hold:

1. Each block is of size $(m+1)$.
2. If a, b are in the same block B and $[a, b] \cap B = \{a, b\}$, then for any c, d such that $a < c < d < b$, c and d are not in the same block.

Just as before, such partitions can be represented using arc diagrams.

Example 3.24. The arc diagram corresponding to the 2-non-nesting partition of size 3

$$\{1, 2, 4\}, \{3, 5, 6\}, \{7, 8, 9\}$$

is given in Figure 3.8.

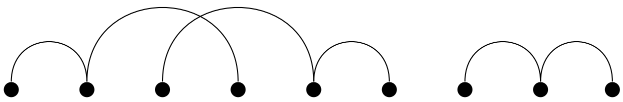


FIGURE 3.8: Arc diagram corresponding to the 2-non-nesting partition in Example 3.24.

It is known (for example, see [4, Theorem 2.2]) that the number of m-non-nesting partitions of size n is

$$\frac{1}{mn+1}\binom{(m+1)n}{n}.$$

These numbers are called the Fuss-Catalan numbers or generalized Catalan numbers. Setting $m=1$ gives us the usual Catalan numbers. Labeling the n blocks distinctly using $[n]$ gives us labeled m-non-nesting partitions. These objects correspond to the regions of the type A m-Catalan arrangement in $\mathbb{R}^n$ whose hyperplanes are

$$x_i - x_j = 0, \pm 1, \pm 2, \ldots, \pm m$$

for all $1 \leqslant i < j \leqslant n$ (for example, see [8, Section 8.1]).

We now define a statistic on labeled non-nesting partitions similar to the one defined in [16, Section 4]. The statistic defined in [16] is for labeled m-Dyck paths but these objects are in bijection with labeled m-non-nesting partitions.

A labeled non-nesting partition can be broken up into interlinked pieces, say $P_1, P_2, \ldots, P_k$. We group these pieces into *compartments* as follows. If the label 1 is in the r^{th} interlinked piece, then the interlinked pieces $P_1, P_2, \ldots, P_r$ form the first compartment. Let j be the smallest number in $[n] \setminus A$ where A is the set of labels in first compartment. If j is in the s^{th} interlinked piece then interlinked pieces $P_{r+1}, P_{r+2}, \ldots, P_s$ form the second compartment. Continuing this way, we break up a labeled non-nesting partition into compartments.

Example 3.25. The labeled non-nesting partition in Figure 3.9 has 3 interlinked pieces. The first compartment consists of just the first interlinked piece since it contains the label 1. The smallest label in the rest of the diagram is 3 which is in the last interlinked piece. Hence, this labeled non-nesting partition has 2 compartments.

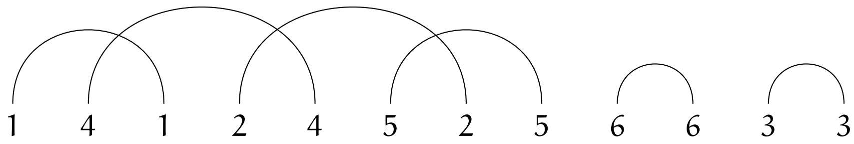


FIGURE 3.9: A labeled non-nesting partition with 3 interlinked pieces and 2 compartments.

A non-nesting partition labeled with distinct integers (not necessarily of the form $[n]$) can be broken up into compartments in the same way. Here the first compartment consists of the interlinked pieces up to the one containing the smallest label.

It can be checked that compartments give labeled non-nesting partitions an exponential structure. This is because the order in which they appear can be determined by their labels. A labeled non-nesting partition is said to be *connected* if it has only one compartment.

We now define a similar statistic for labeled symmetric non-nesting partitions. To a symmetric non-nesting partition we can associate a pair consisting of

1. an interlinked symmetric non-nesting partition, which we call the *bounded part* and
2. a non-nesting partition, which we call the *unbounded part*.

This is easy to do using arc diagrams, as illustrated in the following example. The terminology becomes clear when one considers the boundedness of the coordinates in the region corresponding to a labeled symmetric non-nesting partition.

Example 3.26. To the symmetric 2-non-nesting partition in Figure 3.10 we associate

1. the interlinked symmetric 2-non-nesting partition marked A and
2. the 2-non-nesting partition marked B.

Here A is the bounded part and B is the unbounded part. We can obtain the original arc diagram back from A and B by placing a copy of B on either side of A.

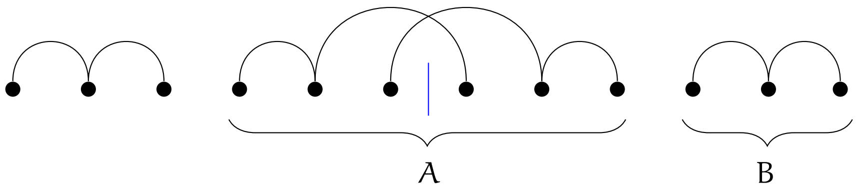


FIGURE 3.10: Break up of a symmetric 2-non-nesting partition.

This is a bijection between symmetric non-nesting partitions and such pairs. Given a labeled symmetric non-nesting partition, we define the statistic using just the unbounded part. Ignoring the signs, we break the unbounded part into compartments just as we did for non-nesting partitions. A *positive compartment* is one whose last element has a positive label.

Example 3.27. Suppose the arc diagram in Figure 3.11 is the unbounded part of some symmetric non-nesting partition. Notice that ignoring the signs, this arc diagrams breaks up into compartments just as Figure 3.9. But only the first compartment is positive since its last element has label 6 which is positive.

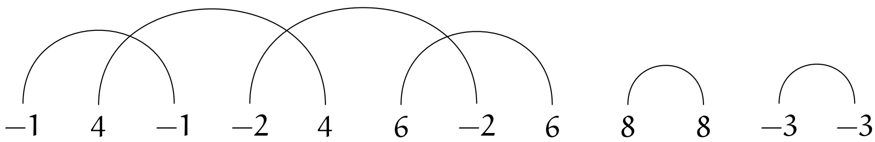


FIGURE 3.11: The unbounded part of a symmetric non-nesting partition that has 1 positive compartment.

We claim that the statistic 'number of positive compartments' meets our requirements. To prove that the distribution of this statistic is given by the characteristic polynomial, we apply Proposition 3.21 to the sequence of arrangements $\{\mathcal{C}_n^{(m)}\}$. Using the bijection between labeled symmetric m-non-nesting partitions and regions of $\mathcal{C}_n^{(m)}$, we note that those arc diagrams that are interlinked are the ones that correspond to bounded regions. Hence, using the notations form Proposition 3.21, and [57, Proposition 5.1.1], we have

$$F(-x) = G(-x) \cdot \left(\sum_{n \geqslant 0} \frac{2^n n!}{mn+1} \binom{(m+1)n}{n} \frac{x^n}{n!} \right). \tag{3.3}$$

Note that $\mathrm{rank}(\mathcal{C}_n^{(m)}) = n$. This gives us

$$\sum_{n\geqslant 0} \chi_{\mathcal{A}_n}(-t)\frac{(-x)^n}{n!} = G(-x)\cdot\left(\sum_{n\geqslant 0}\frac{2^n n!}{mn+1}\binom{(m+1)n}{n}\frac{x^n}{n!}\right)^{\frac{t+1}{2}}.$$

Using the combinatorial interpretation of (3.2), we see that the right hand side of the above equation is the generating function for the distribution of the statistic.

We also obtain corresponding statistics on symmetric sketches using the bijection in Section 2.2.1. This gives us the following result.

Theorem 3.28. *The absolute value of the coefficient of t^j in $\chi_{\mathcal{C}_n^{(m)}}(t)$ is the number of symmetric m-sketches of size n that have j positive compartments.*

For the arrangements $\mathcal{D}_n$, $\mathcal{P}_n$, $\mathcal{B}_n$, and $\mathcal{BC}_n$ as well, the analogue of (3.3) holds. That is, for each of these arrangements, using the notation of Proposition 3.21, we have

$$F(-x) = G(-x)\cdot\left(\sum_{n\geqslant 0}\frac{2^n n!}{n+1}\binom{2n}{n}\frac{x^n}{n!}\right).$$

This can be proved using the definitions of type D, pointed, type B, and type BC sketches and the description of which sketches correspond to bounded regions.

There is a slight difference in the proof for the sequence of arrangements $\{\mathcal{D}_n\}$. The arrangement $\mathcal{D}_1$ is empty and hence

$$G(-x) = 1 - x + \sum_{n\geqslant 2} b(\mathcal{D}_n)\frac{x^n}{n!}.$$

However, from the definition of type D sketches, we see that we must not allow those symmetric non-nesting partitions where the bounded part is empty and the first interlinked piece of the unbounded part is of size 1 with negative label. Hence, we still get the required expression for $F(-x)$.

Just as we did for the extended type C Catalan arrangements, we define positive compartments for the arc diagrams corresponding to the regions of these arrangements, which gives corresponding statistics on the sketches.

Example 3.29. The arc diagram in Figure 3.12 corresponds to a pointed sketch with 2 positive compartments.

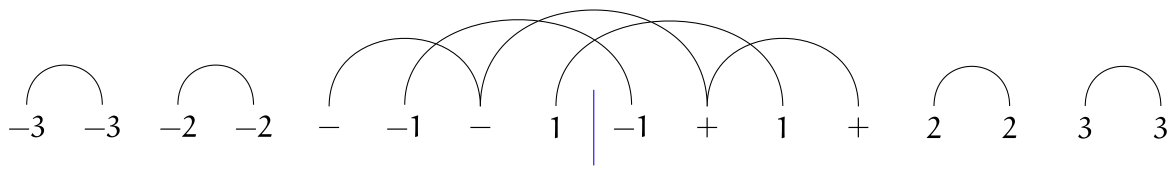


FIGURE 3.12: Arc diagram corresponding to a pointed sketch with 2 positive compartments.

The following result can be proved just as before.

Theorem 3.30. *The absolute value of the coefficient of* t^j *in* $\chi_{\mathcal{A}}(t)$ *for* $\mathcal{A} = \mathcal{D}_n$ *(respectively* $\mathcal{P}_n$, $\mathcal{B}_n$, $\mathcal{BC}_n$*) is the number of type* D *(respectively pointed, type* B*, type* BC*) sketches of size* n *that have* j *positive compartments.*

Chapter 4

Threshold deformations

The *threshold arrangement* in $\mathbb{R}^n$ consists of the hyperplanes $x_i + x_j = 0$ for $1 \leqslant i < j \leqslant n$. The definition and some properties of this arrangement are given in [54, Exercise 5.25]. Although these are not reflection arrangements, they are of interest because their regions correspond to certain labeled graphs called *threshold graphs* which have been extensively studied (see [39]). In this section, we study this arrangement and some of its deformations using similar techniques as in previous chapters.

The results in this chapter are form [17, Section 6], which is joint work with Priyavrat Deshpande.

4.1 Sketches and moves

We use the sketches and moves idea to study the regions of the threshold arrangement by considering it as a sub-arrangement of the type C arrangement (Section 2.2). Before doing that, we first study the arrangement obtained by adding the coordinate hyperplanes to the threshold arrangement.

4.1.1 Fubini arrangement

We define the *Fubini arrangement* in $\mathbb{R}^n$ to be the one with hyperplanes

$$\begin{aligned} 2x_i &= 0 \\ x_i + x_j &= 0 \end{aligned}$$

for all $1 \leqslant i < j \leqslant n$. We name this arrangement the 'Fubini arrangement' since its regions are counted by twice the Fubini numbers (defined below). The hyperplanes missing from the type C arrangement are

$$x_i - x_j = 0$$

for all $1 \leqslant i < j \leqslant n$. Hence a Fubini move, which we call an F move, is swapping adjacent $\overset{+}{i}$ and $\overset{+}{j}$ as well as $\overset{-}{j}$ and $\overset{-}{i}$ for distinct $i, j \in [n]$.

Example 4.1. We can use a series of F moves on a sketch as follows:

$$\overset{-}{3}\,\overset{-}{6}\,\overset{-}{2}\,\overset{+}{1}\,\overset{+}{4}\,\overset{-}{5} \mid \overset{+}{5}\,\overset{-}{4}\,\overset{-}{1}\,\overset{+}{2}\,\overset{+}{6}\,\overset{+}{3} \longrightarrow \overset{-}{6}\,\overset{-}{3}\,\overset{-}{2}\,\overset{+}{1}\,\overset{+}{4}\,\overset{-}{5} \mid \overset{+}{5}\,\overset{-}{4}\,\overset{-}{1}\,\overset{+}{2}\,\overset{+}{3}\,\overset{+}{6} \longrightarrow \overset{-}{6}\,\overset{-}{3}\,\overset{-}{2}\,\overset{+}{4}\,\overset{+}{1}\,\overset{-}{5} \mid \overset{+}{5}\,\overset{-}{1}\,\overset{-}{4}\,\overset{+}{2}\,\overset{+}{3}\,\overset{+}{6}$$

We define a *block* to be the set of absolute values in a maximal string of contiguous terms in the second half of a sketch that have the same sign. The blocks of the initial sketch in Example 4.1 are $\{5\}, \{1, 4\}, \{2, 3, 6\}$ (these blocks appear in this order with the first one being positive). It can be checked that F moves do not change the sequence of signs (above the numbers) and that they can only be used to reorder the elements in a block. Hence, each equivalence class has a unique sketch where the numbers in each block appear in ascending order. The last sketch in Example 4.1 is the unique such sketch in its equivalence class.

The number of such sketches is equal to the number of ways of choosing an ordered partition of $[n]$ (which correspond to the blocks of the sketch in order) and then choosing a sign for the first block. Hence the number of regions of the Fubini arrangement is $2 \cdot a(n)$ where $a(n)$ is the n^{th} Fubini number, which is the number of ordered partitions of $[n]$ listed as A000670 in the OEIS [53].

4.1.2 Threshold arrangement

The threshold arrangement in $\mathbb{R}^n$ has the hyperplanes

$$x_i + x_j = 0$$

for all $1 \leqslant i < j \leqslant n$. The hyperplanes missing from the type C arrangement are

$$\begin{aligned} 2x_i &= 0 \\ x_i - x_j &= 0 \end{aligned}$$

for all $1 \leqslant i < j \leqslant n$. Hence the threshold moves, which we call T moves, are as follows:

1. (D move) Swapping adjacent $\overset{+}{i}$ and $\overset{-}{i}$ for any $i \in [n]$.
2. (F move) Swapping adjacent $\overset{+}{i}$ and $\overset{+}{j}$ as well as $\overset{-}{j}$ and $\overset{-}{i}$ for distinct $i, j \in [n]$.

For any sketch, there is a T equivalent sketch for which the first block has more than 1 element. This is because, if the sketch has first block of size 1, applying a D move (swapping the n^{th} and $(n+1)^{\text{th}}$ term), will result in a sketch where the first block has size greater than 1 (first step in Example 4.2).

Example 4.2. We can use a series of T moves on a sketch as follows:

$$\overset{+}{5}\,\overset{-}{4}\,\overset{-}{1}\,\overset{+}{2}\,\overset{+}{6}\,\overset{-}{3} \mid \overset{+}{3}\,\overset{-}{6}\,\overset{-}{2}\,\overset{+}{1}\,\overset{+}{4}\,\overset{-}{5} \xrightarrow{\text{D move}} \overset{+}{5}\,\overset{-}{4}\,\overset{-}{1}\,\overset{+}{2}\,\overset{+}{6}\,\overset{+}{3} \mid \overset{-}{3}\,\overset{-}{6}\,\overset{-}{2}\,\overset{+}{1}\,\overset{+}{4}\,\overset{-}{5} \xrightarrow{\text{F moves}} \overset{+}{5}\,\overset{-}{4}\,\overset{-}{1}\,\overset{+}{6}\,\overset{+}{3}\,\overset{+}{2} \mid \overset{-}{2}\,\overset{-}{3}\,\overset{-}{6}\,\overset{+}{1}\,\overset{+}{4}\,\overset{-}{5}$$

To obtain a canonical sketch for each threshold region, we will need a small lemma.

Lemma 4.3. *Two T equivalent sketches that have their first block of size greater than 1 have the same blocks which appear in the same order with the same signs.*

Proof. Looking at what the T moves do to the sequence of signs (above the numbers), we can see that they at most swap the n^{th} and $(n+1)^{\text{th}}$ sign (D move). Hence, if we require the first blocks to have size greater than 1, both the sketches have the same number of blocks and the number of elements in the corresponding blocks are the same. An F move can only reorder elements in the same block of a sketch. A D

move changes the sign of the first element of the second half. So if there are $k > 1$ elements in the first block of a T equivalent sketch, then the set of absolute values of the first k elements of the second half remains the same in all T equivalent sketches. This gives us the required result. □

Using the above lemma, we can see that for any sketch there is a unique T equivalent sketch where the size of the first block is greater than 1 and the elements of each block are in ascending order. The last sketch in Example 4.2 is the unique such sketch in its equivalence class. Similar to the count for Fubini regions, we get that the number of regions of the threshold arrangement is

$$2 \cdot (a(n) - n \cdot a(n-1))$$

where, as before, $a(n)$ is the n^{th} Fubini number. The number of regions of the threshold arrangement is listed as A005840 in the OEIS [53].

Remark 4.4. The regions of the threshold arrangement in $\mathbb{R}^n$ are known to be in bijection with labeled threshold graphs on n vertices (see [54, Exercise 5.25]). Labeled threshold graphs on n vertices are inductively constructed starting from the empty graph. Vertices labeled $1, \ldots, n$ are added in a specified order. At each step, the vertex added is either 'dominant' or 'recessive'. A dominant vertex is one that is adjacent to all vertices added before it and a recessive vertex is one that is isolated from all vertices added before it. It is not difficult to see that the canonical sketches described above are in bijection with threshold graphs.

4.2 Statistics

The characteristic polynomial of the threshold arrangement and a statistic on its regions whose distribution is given by the characteristic polynomial has been studied in [19]. This is done by directly looking at the coefficients of the characteristic polynomial. In fact, even the coefficients of the characteristic polynomial of the Fubini arrangement (Section 4.1.1) have already been combinatorially interpreted in [19, Section 4.1]. This can be used to define an appropriate statistic on the regions of the Fubini arrangement. Here, just as in Section 3.2, we use Proposition 3.21 to combinatorially interpret the generating functions of the characteristic polynomials for the Fubini and threshold arrangements. Just as before, we will show that the statistic 'number of positive compartments' works for our purposes.

4.2.1 Fubini arrangement

We will use the second half of the canonical sketches described in Section 4.1.1 to represent the regions. We define blocks for signed permutations just as we did for sketches. Hence, the regions of the Fubini arrangement in $\mathbb{R}^n$ correspond to signed permutations on $[n]$ where each block is increasing.

In this special class of signed permutations as well, compartments give them an exponential structure. This is because there is no condition relating the signs of the last element of a compartment and the first element of the compartment following it. This is because the last element of a compartment is necessarily smaller in absolute value than the element following it. Also, suppose we are given a signed permutation such that each block is increasing. It can be checked that the signed permutation obtained by changing all the signs also satisfies this property.

Using the above observations and the combinatorial interpretation of (3.2), we get that

$$\left(\frac{e^x}{2-e^x}\right)^{\frac{t+1}{2}}$$

is the exponential generating function for signed permutations where each block is increasing where t keeps track of the number of positive compartments. This agrees with the generating function for the characteristic polynomial we get from Proposition 3.21 since we have

$$\begin{aligned} F(x) &= \left(\frac{1}{2e^x-1}\right), \\ G(x) &= 1. \end{aligned}$$

4.2.2 Threshold arrangement

From Section 4.1.2, we can see that the regions of the threshold arrangement in $\mathbb{R}^n$ correspond to signed permutations on $[n]$ where each block is increasing and the first block has size greater than 1. If such a permutation starts with $\overline{1}$, we instead use the signed permutation obtained by changing $\overset{-}{1}$ to $\overset{+}{1}$ to represent the region. Similar to how we obtained the generating function for the statistic for type D from the one for type C, we obtain our generating function from the one we have for the Fubini arrangement.

Suppose that we are given $i \in [n]$ and a signed permutation σ on $[n] \setminus \{i\}$ whose blocks are increasing. If $i = 1$ we construct the signed permutation on $[n]$ obtained by appending $\overset{-}{1}$ to the front of σ. If $i > 1$, and the first element of σ is $\overset{\pm}{j}$. We construct the signed permutation on $[n]$ obtained by appending $\overset{\mp}{i}$ to the start of σ. In both cases, it can be checked that the number of positive compartment of the new signed permutation constructed is the same as that for σ.

This shows that the distribution of the statistic 'number of positive compartments' on the signed permutations that correspond to regions of the threshold arrangement is

$$(1-x)\left(\frac{e^x}{2-e^x}\right)^{\frac{t+1}{2}}.$$

This agrees with the generating function for the characteristic polynomial we get from Proposition 3.21 since we have

$$\begin{aligned} F(x) &= \left(\frac{1+x}{2e^x-1}\right), \\ G(x) &= 1+x. \end{aligned}$$

4.3 Some deformations

Deformations of the threshold arrangement have not been as well-studied as those of the braid arrangement. However, the finite field method has been used to compute the characteristic polynomial for some deformations. In [50, 51], Seo computed the characteristic polynomials of the so called Shi and Catalan threshold arrangements. Expressions for the characteristic polynomials of more general deformations have been computed in [7].

In this section, we use the sketches and moves technique to obtain certain non-nesting partitions that are in bijection with the regions of the Catalan and Shi threshold arrangements. We do this by considering these arrangements as sub-arrangements of the type C Catalan arrangement (Section 2.2). Unfortunately, we were not able to directly count the non-nesting partitions we obtained since their description is not as simple as the ones we have seen before.

Fix $n \geqslant 2$ throughout this section. Recall that we studied the type C Catalan arrangement by considering a translation of it called $\mathcal{C}_n$ whose hyperplane are given by (2.1) and whose regions correspond to symmetric sketches of size n (see Definition 2.7). Symmetric sketches can also be viewed as labeled symmetric non-nesting partitions (see Example 2.19).

4.3.1 Catalan threshold

The Catalan threshold arrangement in $\mathbb{R}^n$ consists of the hyperplanes

$$X_i + X_j = -1, 0, 1$$

for all $1 \leqslant i < j \leqslant n$. The translated arrangement by setting $X_i = x_i + \frac{1}{2}$, which we call $\mathcal{CT}_n$, has hyperplanes

$$x_i + x_j = -2, -1, 0$$

for all $1 \leqslant i < j \leqslant n$. We consider this arrangement as a sub-arrangement of $\mathcal{C}_n$. Using Bernardi's idea of moves, we can define an equivalence on the symmetric sketches such that two sketches are equivalent if they lie in the same region of $\mathcal{CT}_n$.

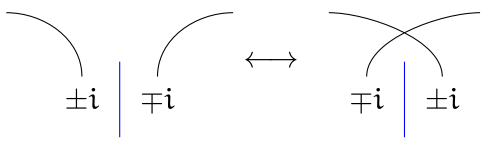
An α_+ letter is an α-letter whose subscript is positive. We similarly define α_-, β_+ and β_- letters. The 'mod-value' of a letter $\alpha_i^{(s)}$ is $|i|$.

The hyperplanes in $\mathcal{C}_n$ that are not in $\mathcal{CT}_n$ are

$$\begin{aligned} 2x_i &= -2, -1, 0 \\ x_i - x_j &= -1, 0, 1 \end{aligned}$$

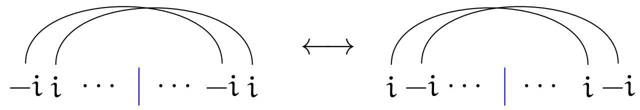
where $1 \leqslant i < j \leqslant n$. Changing the inequality corresponding to exactly one of these hyperplanes is given by the following moves on a sketch, which we call $\mathcal{CT}$ moves.

(a) Swapping the $2n^{\text{th}}$ and $(2n+1)^{\text{th}}$ letter.



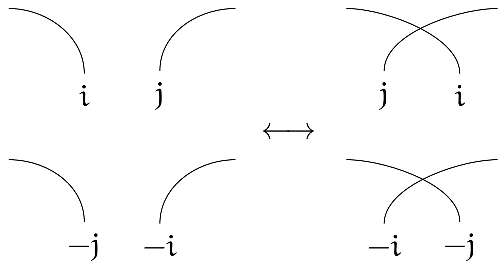
This corresponds to changing the inequality corresponding to a hyperplane of the form $2x_i = -2$ or $2x_i = 0$.

(b) Swapping the n^{th} and $(n+1)^{th}$ α-letter if they are consecutive (along with the n^{th} and $(n+1)^{th}$ β).



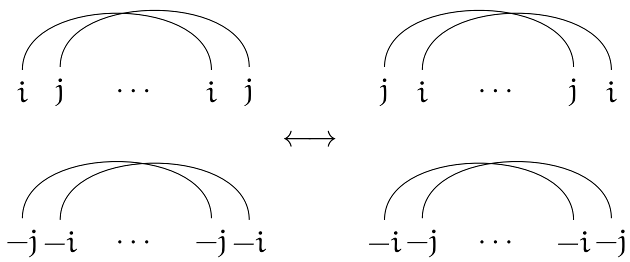
This corresponds to changing the inequality corresponding to a hyperplane of the form $2x_i = -1$.

(c) Swapping consecutive α_+ and β_+ letters (along with their negatives).



This corresponds to changing the inequality corresponding to a hyperplane of the form $x_i - x_j = 1$.

(d) Swapping $\{\alpha_i^{(0)}, \alpha_j^{(0)}\}$ as well as $\{\alpha_i^{(1)}, \alpha_j^{(1)}\}$ if both pairs are consecutive (as well as their negatives) where $i, j \in [n]$ are distinct.



This corresponds to changing the inequality corresponding to the hyperplane $x_i - x_j = 1$.

Two sketches are in the same region of $\mathcal{CT}_n$ if and only if they are related by a series of $\mathcal{CT}$ moves. We call such sketches $\mathcal{CT}$ equivalent.

Consider the sketches to be ordered in the lexicographic order induced by the following order on the letters.

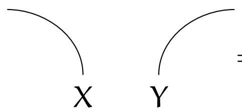
$$\alpha_n^{(0)} \succ \cdots \succ \alpha_1^{(0)} \succ \alpha_{-1}^{(-1)} \succ \cdots \succ \alpha_{-n}^{(-1)} \succ \alpha_n^{(1)} \succ \cdots \succ \alpha_1^{(1)} \succ \alpha_{-1}^{(0)} \succ \cdots \succ \alpha_{-n}^{(0)}$$

In other words, the α-letters are greater than the β-letters and for letters of the same type, the order is given by comparing the subscripts.

A sketch is called $\mathcal{CT}$ maximal if it is greater (in the lexicographic order) than all sketches to which it is $\mathcal{CT}$ equivalent. Hence the regions of $\mathcal{CT}_n$ are in bijection with the $\mathcal{CT}$ maximal sketches.

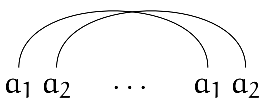
Theorem 4.5. *A symmetric sketch is $\mathcal{CT}$ maximal if and only if the following hold.*

1. *If a β-letter is followed by an α-letter, they should be of opposite signs and different mod-values.*



$\Longrightarrow$ *X and Y of opposite sign and different mod value.*

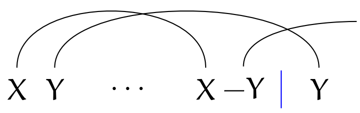
2. *If two α-letters and their corresponding β-letters are both consecutive and of the same sign then the subscript of the first one should be greater.*



and a_1, a_2 *same sign* $\Longrightarrow a_1 > a_2$.

3. *If the n^{th} and $(n+1)^{th}$ α-letters are consecutive, then so are the $(n-1)^{th}$ and n^{th} with the n^{th} α-letter being positive. In such a situation, if the $(n-1)^{th}$ α-letter is negative and the $(n-1)^{th}$ and n^{th} β-letters are consecutive, the $(n-1)^{th}$ α-letter should have a subscript greater than that of the $(n+1)^{th}$ α.*

4. *If the $(2n-1)^{th}$ and $(2n+1)^{th}$ letters are both β-letters of the same sign and their corresponding α-letters are consecutive, the subscript of the $(2n-1)^{th}$ letter should be greater than that of the $(2n+1)^{th}$.*



and X, Y *same sign* $\Longrightarrow X > Y$.

Hence the regions of $\mathcal{CT}_n$ are in bijection with sketches of the form described above.

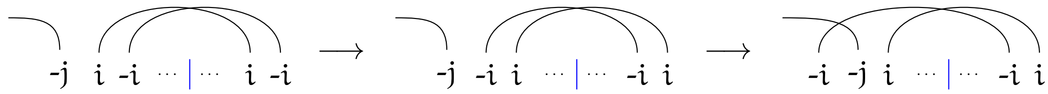
Remark 4.6. The idea of ordering sketches and choosing the maximal sketch in each region of $\mathcal{CT}_n$ to represent it is the same one used by Bernardi [8] to study certain deformations of the braid arrangement. In fact, [8, Lemma 8.13] shows that in this case, any sketch that is locally maximal (greater than any sketch that can be obtained by applying a single move) is maximal. Note that the sketches described in the above theorem are precisely the *2-locally maximal* sketches. That is, these are the sketches that can neither be converted into a greater sketch by applying a single $\mathcal{CT}$ move nor by applying two $\mathcal{CT}$ moves. It is clear that any $\mathcal{CT}$ maximal sketch is 2-locally maximal. The theorem states the converse is true as well.

Proof of Theorem 4.5. We first show that these conditions are required for a sketch to be $\mathcal{CT}$ maximal.

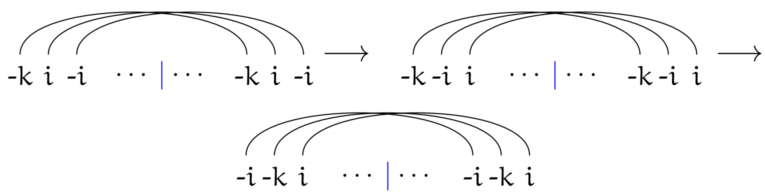
1. The first condition is necessary since the $\mathcal{CT}$ moves of type (a) or (c) would result in a greater sketch if it were false.

2. The second condition corresponds to $\mathcal{CT}$ moves of type (d).

3. The part about the n^{th} α-letter being positive if the n^{th} and $(n+1)^{th}$ α-letters are consecutive is due to $\mathcal{CT}$ moves of type (c). Suppose the letter before the n^{th} α-letter is a β-letter. Then it can't be positive since we have already seen that condition (1) of the theorem statement must be satisfied. But if it is negative, we can do the following to obtain a larger $\mathcal{CT}$ equivalent sketch:

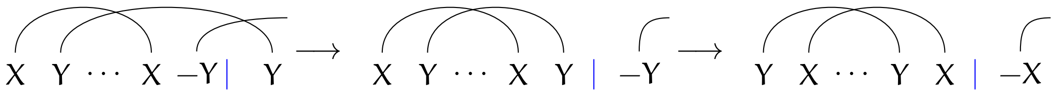


Hence the letter before the n^{th} α-letter has to be an α-letter. Now, suppose that the $(n-1)^{th}$ α-letter is negative and the $(n-1)^{th}$ and n^{th} β-letters are consecutive. Let the subscript of the $(n-1)^{th}$ α-letter be $-k$ and that of the $(n+1)^{th}$ α-letter be $-i$ for some $k, i \in [n]$. If $-k < -i$, we can do the following to obtain a larger $\mathcal{CT}$ equivalent sketch:



Hence we must have $-k > -i$ in this case.

4. Suppose the $(2n-1)^{\text{th}}$ and $(2n+1)^{\text{th}}$ letters are both β-letters of the same sign and their corresponding α-letters are consecutive but the subscript X of the $(2n-1)^{\text{th}}$ letter is less than the subscript Y of the $(2n+1)^{\text{th}}$ letter. We can do the following to obtain a larger $\mathcal{CT}$ equivalent sketch:



We now have to prove that these conditions are sufficient for a sketch to be $\mathcal{CT}$ maximal. Suppose w is a symmetric sketch that satisfies the four properties mentioned in the statement of the theorem. Suppose there is a sketch w' which is $\mathcal{CT}$ equivalent to w but larger in the lexicographic order. This means that if $w = w_1 \cdots w_{4n}$ and $w' = w'_1 \cdots w'_{4n}$, there is some $p \in [4n]$ such that

$$w_i = w'_i \text{ for } i \in [p-1] \text{ and } w_p \prec w'_p.$$

The possible ways in which this can happen are listed below.

1. w_p is a β_+ letter and w'_p is an α_+ letter.
2. w_p is a β_- letter and w'_p is an α_- letter.
3. w_p is a β_+ letter and w'_p is an α_- letter.
4. w_p is a β_- letter and w'_p is an α_+ letter.
5. w_p and w'_p are both α_+ letters.
6. w_p and w'_p are both α_- letters.
7. w_p is an α_- letter and w'_p is an α_+ letter.

The case of both w_p and w'_p being β-letters is not possible since, by the properties of a sketch, this would mean $w_p = w'_p$. Since $\alpha_- \prec \alpha_+$ we cannot have w_p being an α_+ letter and w'_p being an α_- letter. We will now show that each case leads to a contradiction, which will complete the proof of the theorem.

Before going forward, we formulate the meaning of w and w' being $\mathcal{CT}$ equivalent in terms of sketches. Since they have to be in the same region of $\mathcal{CT}_n$, the inequalities corresponding to the hyperplanes

$$x_i + x_j = -2, -1, 0$$

for all $1 \leqslant i < j \leqslant n$ are the same in both sketches. This means that the relationship between the pairs of the form

$$\{\alpha_i^{(1)}, \alpha_{-j}^{(-1)}\},\ \{\alpha_i^{(1)}, \alpha_{-j}^{(0)}\},\ \{\alpha_i^{(0)}, \alpha_{-j}^{(-1)}\},\ \text{and } \{\alpha_i^{(0)}, \alpha_{-j}^{(0)}\}$$

for any distinct $i, j \in [n]$ are the same in both w and w'. This can be written as follows:

$$\text{The relationship between letters of opposite sign and different mod value have to be the same in both } w \text{ and } w'. \tag{4.1}$$

<u>**Case 1:** w_p is a β_+ letter and w'_p is an α_+ letter.</u>

In this case w and w' are of the form

$$\begin{aligned} w &= w_1 \cdots w_{p-1} \alpha_k^{(1)} \cdots \\ w' &= w'_1 \cdots w'_{p-1} \alpha_l^{(0)} \cdots \end{aligned}$$

for some $k, l \in [n]$. Hence, $\alpha_l^{(0)}$ appears after $\alpha_k^{(1)}$ in w. By (4.1), every letter between $\alpha_k^{(1)}$ and $\alpha_l^{(0)}$ in w should be positive or one of $\alpha_{-l}^{(-1)}$ and $\alpha_{-l}^{(0)}$. If all the letters are positive, since $\alpha_k^{(1)}$ is a β_+ letter and $\alpha_l^{(0)}$ is an α_+ letter, there would be a consecutive pair of the form $\beta_+\alpha_+$ in w, which is a contradiction to property 1.

Now suppose $\alpha_{-l}^{(0)}$ is between $\alpha_k^{(1)}$ and $\alpha_l^{(0)}$ in w. It cannot be immediately before $\alpha_l^{(0)}$ since this would contradict property 1. But if it is not immediately before $\alpha_l^{(0)}$, since $\alpha_{-l}^{(0)}$ and $\alpha_l^{(0)}$ are negatives of each other, there should be some negative letter between them. But this letter cannot be $\alpha_{-l}^{(-1)}$ (since this should be before $\alpha_{-l}^{(0)}$). This is a contradiction to (4.1). Hence $\alpha_{-l}^{(0)}$ cannot be between $\alpha_k^{(1)}$ and $\alpha_l^{(0)}$.

So we must have $\alpha_{-l}^{(-1)}$ between $\alpha_k^{(1)}$ and $\alpha_l^{(0)}$ in w. Again, $\alpha_{-l}^{(-1)}$ cannot be immediately before $\alpha_l^{(0)}$ since this would contradict property 3. This means that there is at least one letter between $\alpha_{-l}^{(-1)}$ and $\alpha_l^{(0)}$ and all such letters are positive. If one of them is a β_+ letter, since $\alpha_l^{(0)}$ is an α_+ letter, there would be a consecutive pair of the form $\beta_+\alpha_+$, which is a contradiction to property 1. Hence all the letters between $\alpha_{-l}^{(-1)}$ and $\alpha_l^{(0)}$ are α_+ letters. But this is impossible by Lemma 2.11.

Case 2: w_p is a β_- letter and w'_p is an α_- letter.

In this case w and w' are of the form

$$w = w_1 \cdots w_{p-1}\alpha_{-k}^{(0)} \cdots$$
$$w' = w'_1 \cdots w'_{p-1}\alpha_{-l}^{(-1)} \cdots$$

for some $k, l \in [n]$. Hence, $\alpha_{-l}^{(-1)}$ appears after $\alpha_{-k}^{(0)}$ in w. By (4.1), each letter between $\alpha_{-k}^{(0)}$ and $\alpha_{-l}^{(-1)}$ in w has to be negative or one of $\alpha_l^{(0)}$ and $\alpha_l^{(1)}$. Just as before, all letters between $\alpha_{-k}^{(0)}$ and $\alpha_{-l}^{(-1)}$ cannot be negative. The fact that $\alpha_l^{(1)}$ cannot be between $\alpha_{-k}^{(0)}$ and $\alpha_{-l}^{(-1)}$ also has a similar proof as in the last case.

So we must have $\alpha_l^{(0)}$ between $\alpha_{-k}^{(0)}$ and $\alpha_{-l}^{(-1)}$. All the letters between $\alpha_l^{(0)}$ and $\alpha_{-l}^{(-1)}$ have to be negative. There are no β_- letters between them, otherwise there would be consecutive letters of the form $\beta_-\alpha_-$, which contradicts property 1. So if there are letters between $\alpha_l^{(0)}$ and $\alpha_{-l}^{(-1)}$ they should all be α_- letters, but this cannot happen by Lemma 2.11. So $\alpha_l^{(0)}$ and $\alpha_{-l}^{(-1)}$ are consecutive. By property 3, the letter before $\alpha_l^{(0)}$ should be an α-letter. And by (4.1), it is an α_- letter. But since $\alpha_{-k}^{(0)}$ is a β_- letter and all letters between $\alpha_{-k}^{(0)}$ and $\alpha_l^{(0)}$ are negative, there will be a consecutive pair of the form $\beta_-\alpha_-$, which is a contradiction to property 1.

Case 3: w_p is a β_+ letter and w'_p is an α_- letter.

In this case w and w' are of the form

$$w = w_1 \cdots w_{p-1}\alpha_{k}^{(1)} \cdots$$
$$w' = w'_1 \cdots w'_{p-1}\alpha_{-l}^{(-1)} \cdots$$

for some $k, l \in [n]$. If $k \neq l$, this will contradict (4.1) since $\alpha_k^{(1)}$ will be before $\alpha_{-l}^{(-1)}$ in w but not in w'. So $\alpha_{-k}^{(-1)}$ appears after $\alpha_k^{(1)}$ in w and all letters between them are negative by (4.1) (note that $\alpha_k^{(0)}$ is before $\alpha_k^{(1)}$). Again, $\alpha_{-k}^{(-1)}$ cannot be immediately after $\alpha_k^{(1)}$ since this would contradict property 1 and if there were some letters between $\alpha_k^{(1)}$ and $\alpha_{-k}^{(-1)}$, at least one of them would be negative, which contradicts (4.1).

Case 4: w_p is a β_- letter and w'_p is an α_+ letter.

Arriving at a contradiction in this case follows using the same method as in the last case.

Case 5: w_p and w'_p are both α_+ letters.

In this case w and w' are of the form

$$w = w_1 \cdots w_{p-1} \alpha_k^{(0)} \cdots$$
$$w' = w'_1 \cdots w'_{p-1} \alpha_l^{(0)} \cdots$$

for some $1 \leqslant k < l \leqslant n$. We split this case into two possibilities depending on whether or not $\alpha_l^{(0)}$ is before $\alpha_k^{(1)}$.

Case 5(a): $\alpha_l^{(0)}$ is before $\alpha_k^{(1)}$ in w.

In this case w and w' are of the form

$$w = w_1 \cdots w_{p-1} \alpha_k^{(0)} \cdots \alpha_l^{(0)} \cdots \alpha_k^{(1)} \cdots \alpha_l^{(1)} \cdots$$
$$w' = w'_1 \cdots w'_{p-1} \alpha_l^{(0)} \cdots .$$

By (4.1), each letter between $\alpha_k^{(0)}$ and $\alpha_l^{(0)}$ in w is positive or one of $\alpha_{-l}^{(-1)}$ or $\alpha_{-l}^{(0)}$. Just as in the **Case 1**, we can prove that $\alpha_{-l}^{(-1)}$ and $\alpha_{-l}^{(0)}$ cannot between $\alpha_k^{(0)}$ and $\alpha_l^{(0)}$. Hence all the letters between $\alpha_k^{(0)}$ and $\alpha_l^{(0)}$ are positive. In fact, they all have to be α-letters. Otherwise we would be a consecutive pair of the form $\beta_+\alpha_+$, which contradicts property 1.

Each letter between $\alpha_k^{(1)}$ and $\alpha_l^{(1)}$ is positive or one of $\alpha_{-k}^{(-1)}$, $\alpha_{-l}^{(-1)}$, $\alpha_{-k}^{(0)}$ or $\alpha_{-l}^{(0)}$. Neither $\alpha_{-k}^{(0)}$ nor $\alpha_{-l}^{(0)}$ can be between $\alpha_k^{(1)}$ and $\alpha_l^{(1)}$, since this would mean that $\alpha_{-k}^{(-1)}$ or $\alpha_{-l}^{(-1)}$ is between $\alpha_k^{(0)}$ and $\alpha_l^{(0)}$, which cannot happen since we have already seen that there are only positive α-letters between them.

If $\alpha_{-k}^{(-1)}$ were between $\alpha_k^{(1)}$ and $\alpha_l^{(1)}$, it could not be immediately after $\alpha_k^{(1)}$ since this would contradict property 1. If there were some letters between $\alpha_k^{(1)}$ and $\alpha_{-k}^{(-1)}$, at least one of them would be a negative letter other than $\alpha_{-l}^{(-1)}$, which contradicts (4.1) (since $\alpha_l^{(1)}$ is after $\alpha_{-k}^{(-1)}$).

So the only negative letter that can be between $\alpha_k^{(1)}$ and $\alpha_l^{(1)}$ is $\alpha_{-l}^{(-1)}$. First, suppose that all letters between $\alpha_k^{(1)}$ and $\alpha_l^{(1)}$ are positive. Then all of them would have to be β_+ letters (otherwise there would be consecutive $\beta_+\alpha_+$ which contradicts property 1). Then we would have that all letters between $\alpha_k^{(0)}$ and $\alpha_l^{(0)}$ are α_+ letters and all letters between $\alpha_k^{(1)}$ and $\alpha_l^{(1)}$ are β_+ letters and repeated application of property 2 would give $k > l$, which is a contradiction.

Next, suppose $\alpha_{-l}^{(-1)}$ is between $\alpha_k^{(1)}$ and $\alpha_l^{(1)}$. If $\alpha_{-l}^{(-1)}$ is not immediately before $\alpha_l^{(1)}$, there will be some negative letter other than $\alpha_{-l}^{(-1)}$ between $\alpha_k^{(1)}$ and $\alpha_l^{(1)}$, which we have already shown is not possible. So $\alpha_{-l}^{(-1)}$ is immediately before $\alpha_l^{(1)}$ and all the letters between $\alpha_k^{(1)}$ and $\alpha_{-l}^{(-1)}$ are positive and they have to all be β_+ letters (otherwise there would be a consecutive pair of the form $\beta_+\alpha_+$). If $\alpha_{k'}^{(1)}$ is the β_+ letter before $\alpha_{-l}^{(-1)}$ (k' could be k), then $\alpha_{k'}^{(0)}$ is the letter before $\alpha_l^{(0)}$ and hence we get that the letters between $\alpha_k^{(0)}$ and $\alpha_{k'}^{(0)}$ are all α_+ letters and their corresponding β-letters are consecutive and so by property 2, $k \geqslant k'$. But property 4 tells us that $k' > l$. So we get $k > l$, which is a contradiction.

Case 5(b): $\alpha_l^{(0)}$ is after $\alpha_k^{(1)}$ in w.

In this case w and w' are of the form

$$w = w_1 \cdots w_{p-1}\alpha_k^{(0)} \cdots \alpha_k^{(1)} \cdots \alpha_l^{(0)} \cdots$$
$$w' = w'_1 \cdots w'_{p-1}\alpha_l^{(0)} \cdots .$$

By (4.1), each letter between $\alpha_k^{(0)}$ and $\alpha_l^{(0)}$ in w is positive or one of $\alpha_{-l}^{(-1)}$ or $\alpha_{-l}^{(0)}$. Just as in **Case 1**, we can prove that $\alpha_{-l}^{(-1)}$ and $\alpha_{-l}^{(0)}$ cannot between $\alpha_k^{(0)}$ and $\alpha_l^{(0)}$. Hence all the letters between $\alpha_k^{(0)}$ and $\alpha_l^{(0)}$ are positive. Since $\alpha_k^{(1)}$ is a β_+ letter and $\alpha_l^{(0)}$ is an α_+ letter and all letters in between are positive, there is a consecutive pair of the form $\beta_+\alpha_+$, which is a contradiction to property 1.

Case 6: w_p and w'_p are both α_- letters.

In this case w and w' are of the form

$$w = w_1 \cdots w_{p-1}\alpha_{-k}^{(-1)} \cdots$$
$$w' = w'_1 \cdots w'_{p-1}\alpha_{-l}^{(-1)} \cdots$$

for some $1 \leqslant l < k \leqslant n$. We split this case into two possibilities depending on whether or not $\alpha_{-l}^{(-1)}$ is before $\alpha_{-k}^{(0)}$.

Case 6(a): $\alpha_{-l}^{(-1)}$ is before $\alpha_{-k}^{(0)}$ in w.

In this case w and w' are of the form

$$\begin{aligned} w &= w_1 \cdots w_{p-1} \alpha_{-k}^{(-1)} \cdots \alpha_{-l}^{(-1)} \cdots \alpha_{-k}^{(0)} \cdots \alpha_{-l}^{(0)} \cdots \\ w' &= w_1' \cdots w_{p-1}' \alpha_{-l}^{(-1)} \cdots . \end{aligned}$$

By (4.1), each letter between $\alpha_{-k}^{(-1)}$ and $\alpha_{-l}^{(-1)}$ is negative or one of $\alpha_l^{(0)}$ or $\alpha_l^{(1)}$. If $\alpha_l^{(1)}$ is between $\alpha_{-k}^{(-1)}$ and $\alpha_{-l}^{(-1)}$, it should not be immediately before $\alpha_{-l}^{(-1)}$ since this would contradict property 1. But then there would be some positive letter other than $\alpha_l^{(0)}$ between $\alpha_{-l}^{(-1)}$ and $\alpha_l^{(1)}$ which would contradict (4.1).

First, suppose $\alpha_l^{(0)}$ is between $\alpha_{-k}^{(-1)}$ and $\alpha_{-l}^{(-1)}$. Just as before, using property 1 and Lemma 2.11, we can show that $\alpha_l^{(0)}$ has to be immediately before $\alpha_{-l}^{(-1)}$. Also, all the letters between $\alpha_{-k}^{(-1)}$ and $\alpha_l^{(0)}$ have to be negative by (4.1). By property 3, the letter before $\alpha_l^{(0)}$ has to be an α-letter and hence here it is an α_- letter. Hence, the letters between $\alpha_{-k}^{(-1)}$ and $\alpha_l^{(0)}$ have to be α_- letters since otherwise there be a consecutive pair of the form $\beta_-\alpha_-$.

By (4.1), each letter between $\alpha_{-k}^{(0)}$ and $\alpha_{-l}^{(0)}$ is negative or one of $\alpha_k^{(0)}$, $\alpha_l^{(0)}$, $\alpha_k^{(1)}$ or $\alpha_l^{(1)}$. Now, $\alpha_k^{(1)}$ cannot be between $\alpha_{-k}^{(0)}$ and $\alpha_{-l}^{(0)}$ since this would mean $\alpha_k^{(0)}$ is between $\alpha_{-k}^{(-1)}$ and $\alpha_{-l}^{(-1)}$, which we have already shown is not possible. We have already assumed $\alpha_l^{(0)}$ is between $\alpha_{-k}^{(-1)}$ and $\alpha_{-l}^{(-1)}$ and hence it cannot also be between $\alpha_{-k}^{(0)}$ and $\alpha_{-l}^{(0)}$. If $\alpha_k^{(0)}$ were between $\alpha_{-k}^{(0)}$ and $\alpha_{-l}^{(0)}$, it could not have been immediately after $\alpha_{-k}^{(0)}$ since this would contradict property 1. But then there would be some positive letter other than $\alpha_l^{(1)}$ between $\alpha_{-k}^{(0)}$ and $\alpha_k^{(0)}$ (since $\alpha_{-l}^{(-1)}$ is before $\alpha_{-k}^{(0)}$ and hence $\alpha_l^{(1)}$ is after $\alpha_k^{(0)}$), which is a contradiction to (4.1). This means that the only positive letter between $\alpha_{-k}^{(0)}$ and $\alpha_{-l}^{(0)}$ is $\alpha_l^{(1)}$ which is between them since $\alpha_l^{(0)}$ is between $\alpha_{-k}^{(-1)}$ and $\alpha_{-l}^{(-1)}$. Since $\alpha_l^{(0)}$ and $\alpha_{-l}^{(-1)}$ are consecutive, so are $\alpha_l^{(1)}$ and $\alpha_{-l}^{(0)}$. The letters between $\alpha_{-k}^{(0)}$ and $\alpha_l^{(1)}$ are all negative and should be β_- letters or else it would cause a contradiction to property 1.

Hence, the situation in the case that $\alpha_l^{(0)}$ is between $\alpha_{-k}^{(-1)}$ and $\alpha_{-l}^{(-1)}$ is the following: There is a string of consecutive α_- letters starting with $\alpha_{-k}^{(-1)}$ ending before $\alpha_l^{(0)}$ which is immediately before $\alpha_{-l}^{(-1)}$ and the corresponding β-letters for all these α-letters are consecutive. If $\alpha_{-k'}^{(-1)}$ is the α_- letter immediately before $\alpha_l^{(0)}$ (k' could be k), then property 3 gives that $-k' > -l$ and property 2 gives that $-k \geqslant -k'$ and hence we get $-k > -l$, which is a contradiction.

Next, suppose that all the letters between $\alpha_{-k}^{(-1)}$ and $\alpha_{-l}^{(-1)}$ are negative. All of them should be α_- letters by property 1. It can be shown, just as before, that the only possible positive letter between $\alpha_{-k}^{(0)}$ and $\alpha_{-l}^{(0)}$ is $\alpha_l^{(0)}$. If $\alpha_l^{(0)}$ is not between $\alpha_{-k}^{(0)}$ and $\alpha_{-l}^{(0)}$, property 2 leads to a contradiction just as in **Case 5(a)**. If $\alpha_l^{(0)}$ is between $\alpha_{-k}^{(0)}$ and $\alpha_{-l}^{(0)}$, it should be immediately before $\alpha_{-l}^{(0)}$ and again, following a method similar to **Case 5(a)**, this leads to a contradiction using property 4.

<u>**Case 6(b):** $\alpha_{-l}^{(-1)}$ is after $\alpha_{-k}^{(0)}$ in w.</u>

In this case w and w' are of the form

$$\begin{aligned} w &= w_1 \cdots w_{p-1} \alpha_{-k}^{(-1)} \cdots \alpha_{-k}^{(0)} \cdots \alpha_{-l}^{(-1)} \cdots \alpha_{-l}^{(0)} \cdots \\ w' &= w'_1 \cdots w'_{p-1} \alpha_{-l}^{(-1)} \cdots . \end{aligned}$$

By (4.1), each letter between $\alpha_{-k}^{(-1)}$ and $\alpha_{-l}^{(-1)}$ is negative or one of $\alpha_l^{(0)}$ or $\alpha_l^{(1)}$. Just as before, $\alpha_l^{(1)}$ cannot be between $\alpha_{-k}^{(-1)}$ and $\alpha_{-l}^{(-1)}$. If $\alpha_l^{(0)}$ is not between $\alpha_{-k}^{(-1)}$ and $\alpha_{-l}^{(-1)}$, then all the letters between them are negative and there is a β_- letter, namely $\alpha_{-k}^{(0)}$, between them and this would result in a consecutive pair of the form $\beta_-\alpha_-$, which contradicts property 1.

So $\alpha_l^{(0)}$ is the only positive letter between $\alpha_{-k}^{(-1)}$ and $\alpha_{-l}^{(-1)}$. If $\alpha_l^{(0)}$ is before $\alpha_{-k}^{(0)}$, we would get a consecutive pair of the form $\beta_-\alpha_-$ between $\alpha_{-k}^{(0)}$ and $\alpha_{-l}^{(-1)}$ which contradicts property 1. So $\alpha_l^{(0)}$ is between $\alpha_{-k}^{(0)}$ and $\alpha_{-l}^{(-1)}$. If $\alpha_l^{(0)}$ and $\alpha_{-l}^{(-1)}$ were not consecutive, we would get a contradiction to property 1 if there were some β_- letter between them and if all were α_- letters, this would contradict Lemma 2.11. So $\alpha_l^{(0)}$ and $\alpha_{-l}^{(-1)}$ are consecutive, and by property 3, the letter before $\alpha_l^{(0)}$ should be an α-letter and in this case an α_- letter, say $\alpha_{-k'}^{(-1)}$. But then we would get a consecutive pair of the form $\beta_-\alpha_-$ between $\alpha_{-k}^{(0)}$ and $\alpha_{-k'}^{(-1)}$ which contradicts property 1.

<u>**Case 7:** w_p is a α_- letter and w'_p is an α_+ letter.</u>

In this case w and w' are of the form

$$w = w_1 \cdots w_{p-1} \alpha_{-k}^{(-1)} \cdots$$
$$w' = w'_1 \cdots w'_{p-1} \alpha_l^{(0)} \cdots$$

for some $k, l \in [n]$. If $k \neq l$, we would get a contradiction to (4.1) since $\alpha_{-k}^{(-1)}$ is before $\alpha_l^{(0)}$ is w but not in w'. So $\alpha_k^{(0)}$ appears after $\alpha_{-k}^{(-1)}$ in w and each letter between them is positive or $\alpha_{-k}^{(0)}$. Just as before $\alpha_{-k}^{(0)}$ being between $\alpha_{-k}^{(-1)}$ and $\alpha_k^{(0)}$ would either contradict property 1 or (4.1). So all letters between $\alpha_{-k}^{(-1)}$ and $\alpha_k^{(0)}$ are positive. If there is some β_+ letter between them, there will be a consecutive pair of the form $\beta_+\alpha_+$, which would contradict property 1. Hence, all letters between $\alpha_{-k}^{(-1)}$ and $\alpha_k^{(0)}$ are α_+ letters. But this contradicts Lemma 2.11. □

4.3.2 Shi threshold

The Shi threshold arrangement in $\mathbb{R}^n$ consists of the hyperplanes

$$X_i + X_j = 0, 1$$

for all $1 \leqslant i < j \leqslant n$. The translated arrangement by setting $X_i = x_i + \frac{1}{2}$, which we call $\mathcal{ST}_n$, has hyperplanes

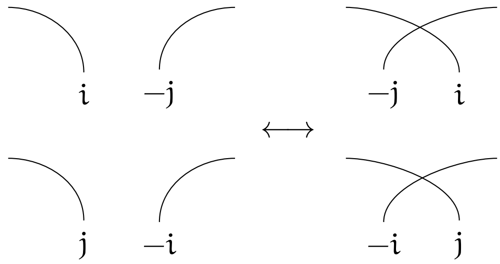
$$x_i + x_j = -1, 0$$

for all $1 \leqslant i < j \leqslant n$. We use the same method as before to study the regions of this arrangement by considering $\mathcal{ST}_n$ as a sub-arrangement of $\mathcal{C}_n$.

The hypeplanes in $\mathcal{C}_n$ that are not in $\mathcal{ST}_n$ are

$$2x_i = -2, -1, 0$$
$$x_i + x_j = -2$$
$$x_i - x_j = -1, 0, 1$$

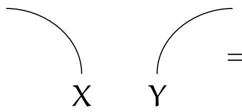
where $1 \leqslant i < j \leqslant n$. Changing the inequality corresponding to exactly one of these hyperplanes are given by the $\mathcal{CT}$ moves as well as the move corresponding to $x_i + x_j = -2$ where $i \neq j$ are in $[n]$: Swapping consecutive β_+ and α_- letters (along with their negatives).



Two sketches are in the same region of $\mathcal{ST}_n$ if and only if they are related by a series of such moves and we call such sketches $\mathcal{ST}$ equivalent. A sketch is called $\mathcal{ST}$ maximal if it is greater (in the lexicographic order) than all sketches to which it is $\mathcal{ST}$ equivalent. Hence the regions of $\mathcal{ST}_n$ are in bijection with the $\mathcal{ST}$ maximal sketches. The following result can be proved just as Theorem 4.5.

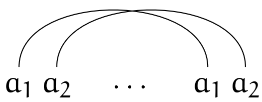
Theorem 4.7. *A symmetric sketch is $\mathcal{ST}$ maximal if and only if the following hold.*

1. *If a β-letter is followed by an α-letter, the β-letter should be negative and the α-letter should be positive with different mod-values.*



$\Longrightarrow$ *X negative and Y positive and different mod value.*

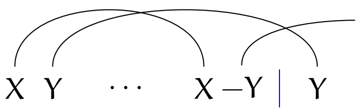
2. *If two α-letters and their corresponding β-letters are both consecutive and of the same sign then the subscript of the first one should be greater.*



and a_1, a_2 *same sign* $\Longrightarrow a_1 > a_2$.

3. *If the n^{th} and $(n+1)^{\text{th}}$ α-letters are consecutive, then so are the $(n-1)^{\text{th}}$ and n^{th} with the n^{th} α-letter being positive. In such a situation, if the $(n-1)^{\text{th}}$ α-letter is negative and the $(n-1)^{\text{th}}$ and n^{th} β-letters are consecutive, the $(n-1)^{\text{th}}$ α-letter should have a subscript greater than that of the $(n+1)^{\text{th}}$ α-letter.*

4. *If the $(2n-1)^{\text{th}}$ and $(2n+1)^{\text{th}}$ letters are both negative β-letters and their corresponding α-letters are consecutive, the subscript of the $(2n-1)^{\text{th}}$ letter should be greater than that of the $(2n+1)^{\text{th}}$.*



and X, Y *negative* $\Longrightarrow X > Y$.

Hence the regions of $\mathcal{ST}_n$ are in bijection with sketches of the form described above.

Chapter 5

Future directions

Properties of coefficients. The main question we tackle in this thesis is to give a combinatorial interpretation to these coefficients. One could also try to use this interpretation to prove various properties satisfied by the coefficients. For example, in [16, Section 4], for several deformations of the braid arrangement, we use this method to prove that the coefficients of the characteristic polynomials form an increasing sequence. It is known that for any arrangement $\mathcal{A}$, the coefficients of $\chi_{\mathcal{A}}(t)$ are log-concave and hence unimodal (see [30]). For the arrangements we have studied in this thesis, one could ask for a combinatorial proof of these facts using the statistics we've defined.

Refinement to Möbius values. We noted that for any arrangement $\mathcal{A}$ in $\mathbb{R}^n$, $r(\mathcal{A})$ is the sum of the absolute values of the coefficients of $\chi_{\mathcal{A}}(t)$. This is a consequence of the stronger result that $r(\mathcal{A})$ is the sum of the absolute values of the Möbius values of $L_{\mathcal{A}}$, that is,

$$r(\mathcal{A}) = \sum_{x \in L_{\mathcal{A}}} |\mu(x)|.$$

Hence, one could ask for a statistic on the regions of $\mathcal{A}$ graded by elements of $L_{\mathcal{A}}$ instead of the numbers in $[0, n]$. Note that the absolute value of the coefficient of t^i in $\chi_{\mathcal{A}}(t)$ is the sum of the absolute values of the Möbius values of the element of rank i in $L_{\mathcal{A}}$. This means that the new statistic would be a refinement of the one given by the coefficients of $\chi_{\mathcal{A}}(t)$.

Geometric statistic. For any arrangement $\mathcal{A}$, one can define a geometric statistic `gstat` whose distribution is given by the coefficients of $\chi_{\mathcal{A}}(t)$. We given an idea of how this statistic is defined (see [33] for details): Select a point p in $\mathbb{R}^n$ that does not lie on any of the hyperplanes of $\mathcal{A}$ (this point should also satisfy some other conditions, but they are not too restrictive). To each region R of $\mathcal{A}$, find the point q in the closure of R that is closest to the point p. If the dimension of the face of $\mathcal{A}$ that contains q is i, then we set $\texttt{gstat}(R) = i$. Then the absolute value of the coefficient of t^i in $\chi_{\mathcal{A}}(t)$ is the number of regions R such that $\texttt{gstat}(R) = i$.

For the arrangements we've studied in this thesis, one could check if the statistic on regions we've defined coincides with a geometric statistic obtained using the procedure mentioned above. If not, one could ask for a bijection on the regions of the arrangement that proves that both statistics are equidistributed.

Part II

Pattern Avoidance

Chapter 6

Background

Pattern avoidance is a relatively recent topic in combinatorics which has been garnering a lot of attention. For a class of combinatorial objects, we first define what it means for one object to be *contained* in another. When an object A contains an object B, we usually refer to B as a *pattern* and say that A contains the pattern B. If an object A does not contain the pattern B, we say that A *avoids* B. The usual question in pattern avoidance is: Given a pattern (or set of patterns), describe or count the objects that avoid them.

6.1 Permutations

The most popular class of combinatorial objects for which pattern avoidance is studied is permutations. For $n \geqslant 1$, let $\sigma = \sigma_1 \cdots \sigma_n$ be the one-line representation of a permutation of the set $[n]$. For $n \geqslant m \geqslant 1$, a permutation $\sigma = \sigma_1 \cdots \sigma_n$ *contains* a permutation (or pattern) $\pi = \pi_1 \cdots \pi_m$ if there exists a subsequence $1 \leqslant h(1) < h(2) < \cdots < h(m) \leqslant n$ such that for any $i, j \in [m]$, $\sigma_{h(i)} < \sigma_{h(j)}$ if and only if $\pi_i < \pi_j$. In this case, $\sigma_{h(1)} \cdots \sigma_{h(m)}$ is said to be *order isomorphic* to π.

Informally, σ contains π if it has a subsequence that 'looks' like π. A nice way to see this is via permutation diagrams, where we represent a permutation σ of $[n]$ in $\mathbb{Z}^2$ by plotting (i, σ_i) for all $i \in [n]$. For example, Figure 6.1 shows one way in which the permutation 425613 contains the pattern 132.

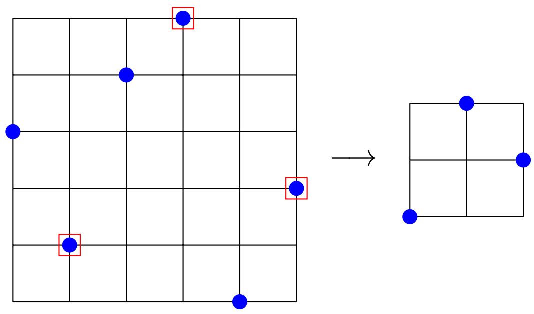


FIGURE 6.1: Permutation diagrams showing that 425613 contains 132.

We say that the permutation σ *avoids* π if it does not contain π. We denote the set of all permutations of $[n]$ by $\mathfrak{S}_n$. We denote the set of permutations in $\mathfrak{S}_n$ that avoid π by $\mathrm{Av}_n(\pi)$. For example, it is easy to see that $\mathrm{Av}_n(21)$ contains only the increasing permutation $\iota_n := 12\cdots n$ and $\mathrm{Av}_n(12)$ contains only the decreasing permutation $\delta_n := n\cdots 21$. As a non-trivial example, we study the permutations in $\mathrm{Av}_n(312)$.

For any $n \geqslant 0$, we prove that $\#\mathrm{Av}_n(312) = \frac{1}{n+1}\binom{2n}{n}$, the Catalan numbers. We first present a recursive proof. Let $\sigma \in \mathrm{Av}_n(312)$ and suppose $\sigma_j = 1$ where $j \in [n]$. Writing $\sigma = \rho\ 1\ \tau$, we see that

- ρ is a permutation of $[2, j]$ that avoids 312, and
- τ is a permutation of $[j, n]$ that avoids 312.

One can also check that any permutation of this form avoids 312. This shows that

$$\mathrm{Av}_n(312) = \sum_{j=1}^{n} \mathrm{Av}_{j-1}(312)\mathrm{Av}_{n-j}(312).$$

Verifying the initial conditions, this shows that $(\mathrm{Av}_n(312))_{n\geqslant 0}$ is the sequence of Catalan numbers.

We also mention a bijection between the permutations of $\mathrm{Av}_n(312)$ and Dyck paths length $2n$. Starting with a Dyck path of length $2n$, we label the up steps using the numbers $1, 2, \ldots, n$ from left to right in order. We then label the down steps as follows: For any down step, find the closest up step to its left that is at the same height. Label the down step using the label of this up step. The permutation in $\mathrm{Av}_n(312)$ corresponding to this Dyck path is the one obtained by reading the labels on the down steps from left to right.

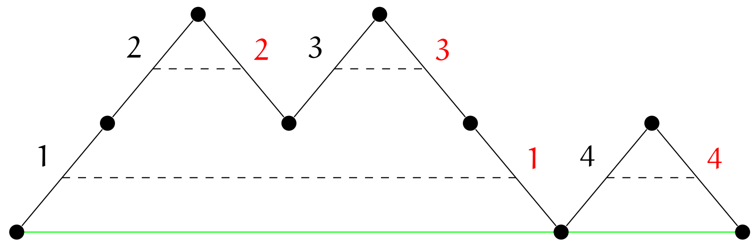


Figure 6.2: The permutation in $\mathrm{Av}_4(312)$ corresponding to this Dyck path is 2314.

One can prove that this is a bijection recursively. Note that the usual way to show that Dyck paths satisfy the recursion for Catalan numbers is by finding the first place it touches the x-axis. One can check that, under the map described above, this recursion matches the one for 312-avoiding permutations mentioned above.

In fact, it is easy to see that $\#\mathrm{Av}_n(312) = \#\mathrm{Av}_n(\pi)$ for all $\pi \in \{132, 213, 231, 312\}$ and $n \geqslant 0$. This is because these patterns can be obtained from one another via symmetries of the square (symmetries of permutation diagrams). Similarly, one can show that $\mathrm{Av}_n(123) = \mathrm{Av}_n(321)$ for all $n \geqslant 0$.

It is well-known that for all patterns π of length 3, we have $\#\mathrm{Av}_n(\pi) = \frac{1}{n+1}\binom{2n}{n}$ (for example, see [49, Section 1.12]). Since we have already proved this for $\pi = 312$, it is only left to prove this for the pattern $\pi = 123$ (or equivalently 321). We leave this as an exercise for the reader.

However, the situation already becomes quite complicated for patterns of length 4. For example, there is no known closed-form expression for $\#\mathrm{Av}_n(1324)$ (see [14]).

The study of pattern avoidance in permutations was initiated by Knuth [36], and the work of Simion and Schmidt [52] was the first one to focus solely on enumerative results. Since then, the topic of pattern avoidance has seen a strong growth in the area of enumerative combinatorics because of its connections to algebraic geometry (see, for example [23, 59]) and computer science (see, for example [36, 47]). For more about pattern avoidance in permutations, see the books of Bóna [11], Kitaev [34], or Sagan [49].

Pattern avoidance has also been studied in various other combinatorial objects. This includes the study of pattern avoidance in binary trees [15, 48], rooted forests [2, 24], Dyck paths [9], set partitions [25, 35, 41], and compositions [28]. In this thesis, we will focus on pattern avoidance in *circular permutations*.

6.2 Circular permutations

A *circular permutation* $[\pi]$ is the set of all rotations of a permutation $\pi = \pi_1 \cdots \pi_n$, *i.e.*,

$$[\pi] = \{\pi_1 \cdots \pi_n, \pi_2 \cdots \pi_n \pi_1, \ldots, \pi_n \pi_1 \cdots \pi_{n-1}\}.$$

We make the convention of using the rotation starting with 1 to represent a circular permutation. As in [21], we denote the set of all circular permutations of $[n]$ by $[\mathfrak{S}_n]$. For example, $[\mathfrak{S}_3] = \{[123], [132]\}$. Observe that the cardinality of the set $[\mathfrak{S}_n]$ is $(n-1)!$. We say that a circular permutation $[\sigma]$ *contains* a circular permutation (or pattern) $[\pi]$ if there exists a rotation σ' of σ such that σ' contains π as usual permutations. If there is no rotation of σ containing π, we say that $[\sigma]$ *avoids* $[\pi]$. For instance, $[14523]$ contains $[1234]$ because the permutation 23145 (which is a rotation of 14523) has the subsequence 2345 which is order isomorphic to 1234.

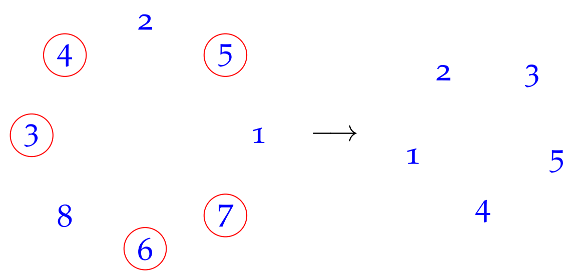


Figure 6.3: The pattern $[12354]$ in the circular permutation $[17683425]$.

Clearly, if $[\sigma]$ contains $[\pi]$ where $[\sigma] \in [\mathfrak{S}_m]$ and $[\pi] \in [\mathfrak{S}_n]$, then $m \geqslant n$. The set of all elements of $[\mathfrak{S}_n]$ avoiding a fixed pattern $[\pi]$ is denoted by $\mathrm{Av}_n[\pi]$, *i.e.*,

$$\mathrm{Av}_n[\pi] = \{[\sigma] \in [\mathfrak{S}_n] \mid [\sigma] \text{ avoids } [\pi]\}.$$

Also, $\mathrm{Av}[\pi]$ will denote the set of *all* circular permutations avoiding $[\pi]$.

Callan [12] and Vella [58] independently studied circular permutations avoiding a fixed pattern of size 4. Gray, Lanning and Wang continued work in this direction and studied other notions of pattern avoidance in circular permutations (see [26, 27]). Very recently, Domagalski et al. [21] studied circular pattern avoidance for multiple patterns of size 4. Vincular pattern avoidance (where one forces that a pattern must have certain specified indices to be consecutive) in circular permutations has also been studied by Li [38] as well as Mansour and Shattuck [40].

For a given set $\{[\pi_1], \ldots, [\pi_k]\}$ of circular permutations, we say that $[\sigma]$ avoids $\{[\pi_1], \ldots, [\pi_k]\}$ if $[\sigma]$ avoids $[\pi_i]$ for each $i \in [k]$. For simplicity, we use $[\pi_1, \ldots, \pi_k]$ to denote this set of patterns. Just as before, set of elements of $[\mathfrak{S}_n]$ that avoid $[\pi_1, \ldots, \pi_k]$ is denoted by $\mathrm{Av}_n[\pi_1, \ldots, \pi_k]$, *i.e.*,

$$\mathrm{Av}_n[\pi_1, \ldots, \pi_k] = \{[\sigma] \in [\mathfrak{S}_n] \mid [\sigma] \text{ avoids } [\pi_i] \text{ for each } i \in [k]\}.$$

If $[\pi_i]$ contains $[\pi_j]$ for some distinct $i, j \in [k]$, then omitting $[\pi_i]$ from the sets of patterns does not affect the avoidance class. Hence, we can assume that the permutations in any set of patterns avoid each other.

An important notion in the study of pattern avoidance is the Wilf equivalence on sets of patterns. Two sets $[\pi_1, \ldots, \pi_k]$ and $[\tau_1, \ldots, \tau_\ell]$ of circular permutations are called *(circular) Wilf equivalent*, denoted by $[\pi_1, \ldots, \pi_k] \equiv [\tau_1, \ldots, \tau_\ell]$, if $\#\mathrm{Av}_n[\pi_1, \ldots, \pi_k] = \#\mathrm{Av}_n[\tau_1, \ldots, \tau_\ell]$ for each $n \geqslant 1$. For $[\pi] = [\pi_1 \cdots \pi_n]$, the *trivial Wilf equivalences* are those of the form

$$[\pi] \equiv [\pi^r] \equiv [\pi^c] \equiv [\pi^{rc}]$$

where $[\pi^r] = [\pi_n \cdots \pi_1]$ is the *reversal* of $[\pi]$, $[\pi^c] = [(n+1-\pi_1) \cdots (n+1-\pi_n)]$ is the *complement* of $[\pi]$ and $[\pi^{rc}] = [(n+1-\pi_n) \cdots (n+1-\pi_1)]$ is the *reverse complement* of $[\pi]$. Similarly, we have trivial Wilf equivalences on sets of patterns. For example, $[1342, 12345] \equiv [1342^r, 12345^r] = [1243, 15432]$ is a trivial Wilf equivalence.

Motivated by the study of pattern avoidance of $(3, k)$-pairs in set partitions done in [32], we study circular permutations avoiding two patterns $\{[\sigma], [\tau]\}$, where $[\sigma]$ is of size 4 and $[\tau]$ is of size k. For simplicity, we say that such pairs of patterns are $[4, k]$-pairs. Observe that, using trivial Wilf equivalences among circular permutations of size 4, it is enough to study those pairs where the pattern of size 4 is [1342], [1324], or [1432]. In the next chapter, we study avoidance of $[4, k]$-pairs and split our results into three sections bases on the pattern of size 4. We also use these results to obtain a complete characterization of Wilf equivalence and counts for avoidance classes of $[4, 5]$-pairs.

Chapter 7

Avoiding size 4 patterns in circular permutations

In this chapter, we study avoidance of a single pattern of size 4 in circular permutations. We do this by using well-studied combinatorial objects to represent circular permutations avoiding a single pattern of size 4. We use these objects to study avoidance of $[4, k]$-pairs in the next chapter.

The results in this chapter are from [42], which is joint work with Anurag Singh.

7.1 Avoiding $[1342]$: Binary words

In this section, we obtain a convenient representation of the permutations in $\mathrm{Av}[1342]$ by relating these cyclic permutations with the linear permutations in $\mathrm{Av}(213, 231)$.

Definition 7.1. A *binary word* is a finite sequence whose terms are in $\{0, 1\}$. A *run* in a binary word $w_1 w_2 \cdots w_n$ is a subsequence $w_i w_{i+1} \cdots w_{i+k}$ of consecutive terms such that

1. all terms are equal,
2. either $i = 1$ or $w_{i-1} \neq w_i$, and
3. either $i + k = n$ or $w_{i+k+1} \neq w_{i+k}$.

Hence, a run is a maximal subsequence of consecutive terms that are all equal.

Example 7.2. The binary word 000110100 is of length 9 and has 5 runs. This word can be written more compactly as $0^3 1^2 010^2$.

To the binary word $w = w_1 w_2 \cdots w_{n-1}$, we associate the permutation $\sigma(w) = \sigma_1 \sigma_2 \cdots \sigma_n$ as follows:

- If $w_1 = 0$, then set $\sigma_1 = 1$, or else set $\sigma_1 = n$.
- For any $k \in [n-2]$, if $\sigma_1, \ldots, \sigma_k$ are defined, then set $\sigma_{k+1} = \min([n] \setminus \{\sigma_1, \ldots, \sigma_k\})$ if $w_{k+1} = 0$, or else set $\sigma_{k+1} = \max([n] \setminus \{\sigma_1, \ldots, \sigma_k\})$.
- Set σ_n to be the unique number in $[n] \setminus \{\sigma_1, \ldots, \sigma_{n-1}\}$.

Example 7.3. For the binary word $w = 0^3 101^2$, we have $\sigma(w) = 12384765$. This association may become clearer when the permutation is represented pictorially (see Figure 7.1).

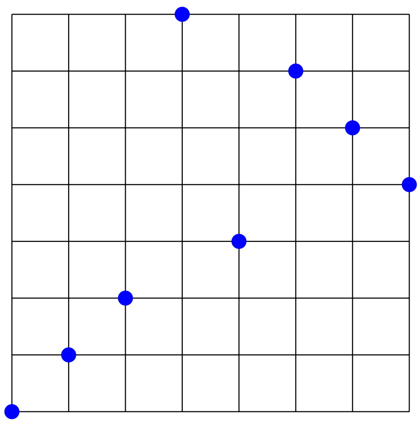


FIGURE 7.1: Permutation associated to the binary word $0^3 101^2$.

We have the following result from [12].

Theorem 7.4 ([12, Proposition 1]). *For any $n \geqslant 2$, the set $\mathrm{Av}_n(213, 231)$ consists of the permutations of the form $\sigma(w)$ where w is a binary word of length $n-1$.*

The following theorem can be obtained by studying the proof of [12, Theorem 1].

Theorem 7.5. *For any $n \geqslant 2$, we have*

$$\mathrm{Av}_n[1342] = \{[\sigma] : \sigma \in \mathrm{Av}_n(213, 231)\}.$$

Proof. We first recall from [21, Lemma 3.3] that if $[\sigma] \in [\mathfrak{S}_n]$ is written as $\sigma = 1\ \rho\ n\ \tau$, then $[\sigma] \in \mathrm{Av}_n[1342]$ if and only if

1. $\rho, \tau \in \mathrm{Av}(213, 231)$,
2. $\max \rho < \min \tau$, and
3. there is not both a descent in ρ and an ascent in τ.

Suppose $[\sigma] \in \mathrm{Av}_n[1342]$ is expressed as above. If ρ has no descents, then σ is already in the form of a permutation associated to a binary word. If ρ has a descent, then by the third condition, τ is the decreasing permutation. Hence, cyclically shifting σ to $n\ \tau\ 1\ \rho$ gives a permutation associated to a binary word.

Similarly, if σ is a permutation associated to a binary word, it either starts with 1 or n. If it starts with 1, it is of the form $1\ \rho\ n\ \tau$ where

1. ρ is increasing,
2. $\max \rho < \min \tau$, and
3. $\tau \in \mathrm{Av}(213, 231)$.

If σ starts with n, it is of the form $n\ \tau\ 1\ \rho$ where

1. τ is decreasing,
2. $\max \rho < \min \tau$, and
3. $\rho \in \mathrm{Av}(213, 231)$.

In either case, we see that $[\sigma]$ is in $\mathrm{Av}_n[1342]$. □

The above theorem tells us that the cyclic permutations in $\mathrm{Av}_n[1342]$ are those of the form $[\sigma(w)]$ for some binary word w of length $n-1$. We now describe when such cyclic permutations are equal.

Theorem 7.6. *For any two binary words w_1, w_2, we have $[\sigma(w_1)] = [\sigma(w_2)]$ if and only if*

1. *$w_1 = w_2$, or*

2. $w_1 = 0^{a+1}1^b$ *and* $w_2 = 1^{b+1}0^a$ *for some* $a, b \geqslant 0$.

Proof. Since any permutation associated to a binary word of length $n-1$ should start with either 1 or n, there are at most two ways that a cyclic permutation can be written in the form $[\sigma(w)]$ for some binary word w. It can be checked that for any $a, b \geqslant 0$, $0^{a+1}1^b$ and $1^{b+1}0^a$ have the same corresponding cyclic permutation. To prove the result, we have to show that if w is a binary word with more than 2 runs, then no other binary word has $[\sigma(w)]$ as its corresponding cyclic permutation.

Let $w = 0^a1^b0^c \cdots$ be a binary word of length $(n-1)$ starting with 0 which has more than 2 runs, i.e., $a, b, c \geqslant 1$. Since $\sigma(w)$ starts with 1, we have to show that the cyclic shift of $\sigma(w)$ that starts with n does not correspond to a binary word. To do this we note that the cyclic shift of $\sigma(w)$ starting with n starts as

$$n\ (n-1)\ \cdots\ (n-b+1)\ (a+1)\ \cdots.$$

For this permutation to correspond to a binary word, we must have either $a+1 = n-b$ or $a+1 = 1$. Since $a \geqslant 1$, we cannot have $a+1 = 1$. Also, $a+1 = n-b$ implies that $a+b = n-1$ and hence that $c = 0$, which is false. A similar argument works if the binary word w starts with 1. □

We now prove some known results about Av[1342] using their binary word representation described above.

Example 7.7. The three theorems mentioned immediately give [12, Theorem 2], which states that for any $n \geqslant 2$,

$$\#\mathrm{Av}_n[1342] = 2^{n-1} - (n-1).$$

Definition 7.8. For a permutation $\sigma = \sigma_1\sigma_2\cdots\sigma_n$ of $[n]$, the *cyclic descent number* of $[\sigma]$ is given by

$$\mathrm{cdes}[\sigma] = \#\{i \in [n] : \sigma_i > \sigma_{i+1}\}$$

where we take subscripts modulo n, that is, we consider $n+1$ to be 1.

Cyclic descents were introduced by Cellini [13] and have been studied by many others. For example, see [1] and the references therein.

As in [21], we use the notation

$$D_n([\pi_1,\dots,\pi_k];q) = \sum_{[\sigma]\in Av_n[\pi_1,\dots,\pi_k]} q^{cdes[\sigma]}.$$

Example 7.9. From [21, Theorem 3.5], we know that

$$D_n([1342];q) = 2q(1+q)^{n-2} - \frac{q(1-q^{n-1})}{1-q}.$$

We prove this using binary words.

For any binary word $w = w_1 w_2 \cdots w_{n-1}$, we have

$$cdes[\sigma(w)] = 1 + \#\{i \in \{2,\dots,n-1\} : w_i = 1\}. \tag{7.1}$$

Here, the second term on the right-hand side counts all the descents in $\sigma(w)$ having index in $\{2,\dots,n-1\}$. If $w_1 = 1$, then the index 1 is a descent for $\sigma(w)$ but there is no cyclic descent at index n. If $w_1 = 0$, then the index 1 is not a descent for $\sigma(w)$ but there is a cyclic descent at index n. This proves the equality in (7.1).

Let B_{n-1} be the set of binary words of length $n-1$ and $E_{n-1} \subseteq B_{n-1}$ be the set of words $\{0^{n-1}, 0^{n-2}1, \dots, 01^{n-2}\}$. Using Theorem 7.6, we have,

$$D_n([1342];q) = \sum_{w\in B_{n-1}} q^{cdes[\sigma(w)]} - \sum_{w\in E_{n-1}} q^{cdes[\sigma(w)]}.$$

This is because, for each $w \in E_{n-1}$ there is exactly one other word not in E_{n-1} that represented the same permutation in $Av_n[1342]$. Using (7.1), it is straightforward to verify that

$$\sum_{w\in B_{n-1}} q^{cdes[\sigma(w)]} = 2q(1+q)^{n-2} \quad \text{and} \quad \sum_{w\in E_{n-1}} q^{cdes[\sigma(w)]} = \frac{q(1-q^{n-1})}{1-q}.$$

7.2 Avoiding [1324]: Circled compositions

We now develop a convenient representation of the permutations in Av[1324]. We first recall the following definition from [58].

Definition 7.10. A *run* in a permutation $\sigma : [n] \to [n]$ is a maximal interval $T \subseteq [n]$ such that σ restricted to T is increasing. A run $T = [a, b]$ is *contiguous* if $\sigma(b) - \sigma(a) = b - a$, *i.e.*, σ maps T to an interval.

Combining [58, Theorem 2.4] and [58, Proposition 3.4], we get the following result.

Theorem 7.11 ([58]). *Let* $[\sigma] \in [\mathfrak{S}_n]$ *be a permutation written so that* $\sigma = \rho\ 1\ \tau\ n$. *Then* $[\sigma]$ *avoids* $[1324]$ *if and only if*

1. τ *is increasing, and*

2. *all runs in* ρ *are contiguous.*

Lemma 7.12. *Permutations in* $\mathfrak{S}_n$ *that have all runs contiguous are in one-to-one correspondence with compositions of* n.

Proof. Let $\sigma \in \mathfrak{S}_n$ be a permutations with all runs contiguous. Suppose its runs are $T_1, \dots, T_k$. We claim that the composition $(\#T_k, \dots, \#T_1)$ of n determines σ.

Note that since all runs are contiguous, 1 has to be the smallest number in $\sigma(T_k)$. Otherwise, if $1 \in \sigma(T_i)$ for $i \neq k$, and $T_i = [a, b]$, then since T_i is contiguous, we get $\sigma(b+1) > \sigma(b)$. This contradicts the fact the T_i is a run. Hence we get that

$$\sigma(n - i) = \#T_k - i \text{ for all } i \in [0, \#T_k - 1].$$

Similarly, the smallest number in $\sigma(T_{k-1})$ is $\#T_k + 1$, and we can obtain $\sigma(i)$ for $i \in T_{k-1}$. Continuing this way, we can determine σ using the composition $(\#T_k, \dots, \#T_1)$.

It can be checked that this is indeed a bijection between compositions of n and permutations of $\mathfrak{S}_n$ with all runs contiguous. □

Remark 7.13. In the above proof, we use the composition $(\#T_k, \dots, \#T_1)$ instead of its reverse since, under the bijection, the terms of the composition correspond to the numbers $1, 2, \dots, n$ in order. That is the first term, $\#T_k$, corresponds to the numbers $1, 2, \dots, \#T_k$, the second term, $\#T_{k-1}$, corresponds to the numbers $\#T_k + 1, \dots, \#T_k + \#T_{k-1}$, and so on. We should also note that the permutations described in the lemma above are the reverses of *layered permutations*. These permutations have been extensively studied from the perspective of pattern avoidance (for example, see [10]).

Example 7.14. The claim and bijection given in the above lemma might be more clear when the permutation is represented pictorially. For example, the composition $(3,1,3,2)$ corresponds to the permutation 895674123 (see Figure 7.2).

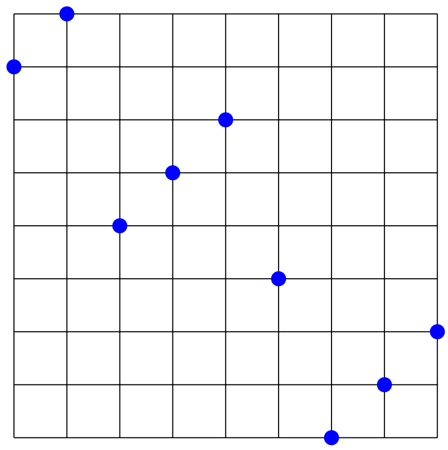


FIGURE 7.2: Permutation with contiguous runs associated to the composition $(3,1,3,2)$.

Definition 7.15. A *circled composition* of n is a pair (a, C) where $a = (a_1, \ldots, a_k)$ is a composition of n with k parts and C is a subset of $[k]$ such that

1. the elements 1 and k are contained in C, and
2. for any $i \in C$, we have $a_i = 1$.

We represent a circled composition (a, C) as the composition a with the parts with indices in C circled.

Example 7.16. The circled composition $((1,1,6,1,2,1,1,1,3,1,1),\{1,2,6,7,8,11\})$ of 19 is represented as

$$\textcircled{1}\quad \textcircled{1}\quad 6\quad 1\quad 2\quad \textcircled{1}\quad \textcircled{1}\quad \textcircled{1}\quad 3\quad 1\quad \textcircled{1}$$

where we omit the brackets and commas for convenience. This can be written more compactly as

$$\textcircled{1}^2\quad 6\quad 1\quad 2\quad \textcircled{1}^3\quad 3\quad 1\quad \textcircled{1}.$$

Theorem 7.17. *The circular permutations in* $\mathrm{Av}_n[1324]$ *are in one-to-one correspondence with circled compositions of* n*.*

Proof. From Theorem 7.11, we know that if a permutation $[\sigma] \in \mathrm{Av}_n[1324]$ is written so that $\sigma = \rho\ 1\ \tau\ n$, then

1. τ is increasing, and
2. all runs in ρ are contiguous.

Let the runs in ρ be $T_1, \ldots, T_k$. Just as in Lemma 7.12, we consider the composition $(\#T_k, \ldots, \#T_1)$. Since $1\ \tau\ n$ is increasing, we can obtain σ from $(\#T_k, \ldots, \#T_1)$ by specifying the number of elements in $1\ \tau\ n$ that are:

1. less than the elements of $\rho(T_k)$,
2. greater than the elements of $\rho(T_i)$ but less than those in $\rho(T_{i-1})$ for each $i \in [2, k]$, and
3. greater than the elements of $\rho(T_1)$.

This information can be represented as a circled composition by inserting m circled 1s into $(\#T_k, \ldots, \#T_1)$ before $\#T_k$ if there are m elements of $1\ \tau\ n$ less than the elements of $\rho(T_k)$. Similarly, we place m circled 1s into $(\#T_k, \ldots, \#T_1)$ between $\#T_i$ and $\#T_{i-1}$ if there are m elements of $1\ \tau\ n$ greater than the elements of $\rho(T_i)$ but less than those in $\rho(T_{i-1})$ for each $i \in [2, k]$. Finally, we place m circled 1s into $(\#T_k, \ldots, \#T_1)$ after $\#T_1$ if there are m elements of $1\ \tau\ n$ greater than the elements of $\rho(T_1)$.

Note that this is a circled composition since 1 is less than the elements of $\rho(T_k)$ and n is greater than the elements of $\rho(T_1)$. Hence the first and last numbers are circled.

It can be checked that this is indeed a bijection between permutations of $\mathrm{Av}_n[1324]$ and circled compositions of n. □

Example 7.18. Just as before, the bijection in the above theorem might be more clear when the permutation is represented pictorially. For example, the circled composition

$$\textcircled{1}^2\ \ 2\ \ 1\ \ \textcircled{1}^2\ \ 3\ \ \textcircled{1}$$

corresponds to the circular permutation $[8\ 9\ 10\ 5\ 3\ 4\ 1\ 2\ 6\ 7\ 11] \in \mathrm{Av}_{11}[1342]$ (see Figure 7.3).

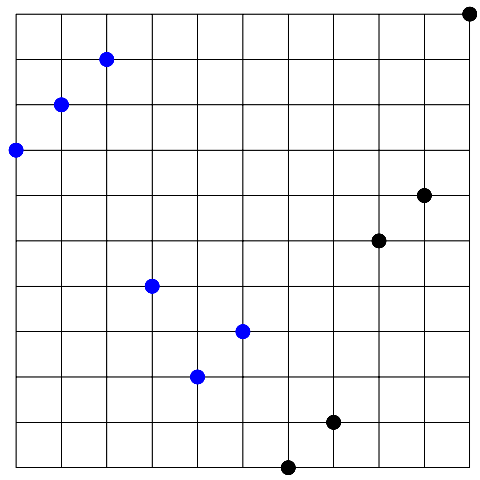


FIGURE 7.3: Permutation in $\mathrm{Av}_{11}[1324]$ corresponding to the circled composition in Example 7.18. The numbers corresponding to circled parts of the composition are colored black.

We know reprove results about Av[1324] using circled compositions. We define for any $n \geqslant 0$, the n^{th} Fibonacci number, F_n, as the number of compositions of n into 1s and 2s, where we set $F_0 = 1$. Hence we have $F_0 = F_1 = 1$ and for $n \geqslant 2$,

$$F_n = F_{n-1} + F_{n-2}.$$

Note that the index of the Fibonacci numbers is different from the usual convention.

Example 7.19. Theorem 7.17 also reflects [12, Theorem 1], which states that for any $n \geqslant 2$,

$$\#\mathrm{Av}_n[1324] = F_{2n-4}.$$

This is because the number of circled compositions of n, say $u(n)$, satisfies the recurrence

$$u(n) = u(n-1) + \sum_{i=1}^{n-2} u(n-i)$$

with initial conditions $u(2) = 1$. This is obtained by deleting the term before the last ①. A combinatorial proof that this implies $u(n) = F_{2n-4}$ is given in the proof of [12, Theorem 1].

Example 7.20. There are

$$\binom{n+i-2}{2i} = \binom{n+i-2}{n-i-2}$$

circled compositions of n with i uncircled parts (see the proof of Proposition 8.33). It can also be checked that the permutation corresponding to a circled compositions with i uncircled numbers has $i+1$ cyclic descents. Hence, we get

$$D_n([1324]; q) = \sum_{i=1}^{n-1} \binom{n+i-3}{n-i-1} q^i,$$

recovering [21, Theorem 3.2].

7.3 Avoiding [1432]: Grassmannian permutations

Just as in Section 7.1, we use binary words to represent the permutations in Av[1432]. The binary words we use in this section actually represent certain special permutations called *Grassmannian permutations* (and their inverses). This representation is the same as the one presented in [58], where subsets of $[n]$ are used instead of binary words of length n.

Definition 7.21. A *Grassmannian permutation* is a permutation which has at most one descent. An *inverse Grassmannian permutation* is a permutation whose inverse is Grassmannian.

Combining [58, Corollary 2.10] and [58, Proposition 3.6], we get the following result.

Theorem 7.22 ([58]). *Let $[\sigma] \in [\mathfrak{S}_n]$ be a permutation written so that σ ends with n. Then $[\sigma]$ avoids $[1432]$ if and only if σ is either Grassmannian or inverse Grassmannian.*

Note that the identity permutation is the only Grassmannian permutation with no descents. The non-identity permutations $\sigma \in \mathfrak{S}_n$ that are Grassmannian and end with n are in bijection with binary words of length n that start with 0 and have at least 3 runs (see Definition 7.1). From such a permutation σ, we obtain the corresponding binary word $w_1 w_2 \cdots w_n$ by setting for each $i \in [n]$, $w_i = 0$ if and only if $n-i+1$ is after the descent of σ.

Example 7.23. The binary word corresponding to the Grassmannian permutation $12356478 \in \mathfrak{S}_8$ is $0^2 1^2 0 1^3$. This permutation is shown pictorially in Figure 7.4, where the dashed line represents the descent. To obtain the binary word associated to such a permutation:

1. read the blue dots in the picture from top to bottom,
2. write 0 if the dot is to the right of the dashed line, and
3. write 1 if the dot is to the left of the dashed line.

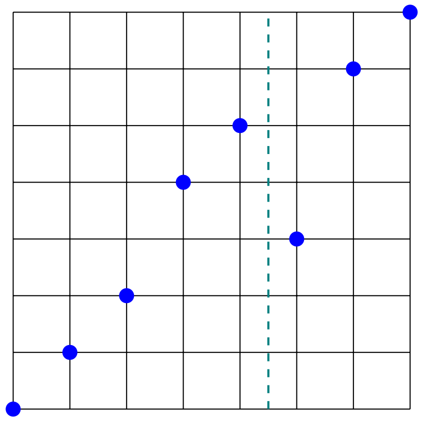


FIGURE 7.4: Grassmannian permutation associated to the binary word $0^2 1^2 0 1^3$.

If w is a binary word starting with 0 and having at least 3 runs, we denote the corresponding Grassmannian permutation by $G(w)$.

Similarly, the non-identity permutations $\sigma \in \mathfrak{S}_n$ that are inverse Grassmannian and end with n are in bijection with binary words of length n that start with 0 and have at least 3 runs. The bijection we will use associates the inverse of $G(w)$, which we denote by $IG(w)$, to such a binary word w.

Example 7.24. For $w = 0^2 1 0^2 1^3$, the permutation $IG(w)$ is the inverse of the permutation $12364578 \in \mathfrak{S}_8$. Hence, $IG(w) = 12356478$. This permutation is shown pictorially in Figure 7.5, where the dashed line represents the descent in the inverse of the permutation. To obtain the binary word associated to such a permutation:

1. read the blue dots in the picture from right to left,
2. write 0 if the dot is above the dashed line, and
3. write 1 if the dot is below the dashed line.

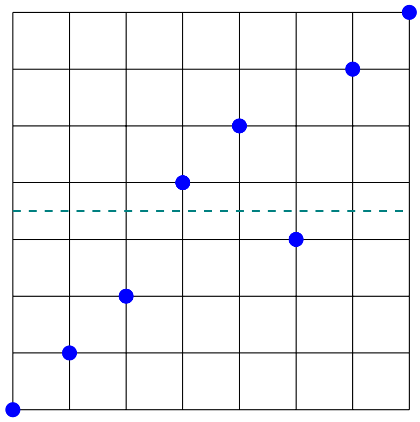


FIGURE 7.5: Inverse Grassmannian permutation associated to the binary word $0^2 1 0^2 1^3$.

Remark 7.25. From the way the bijections are defined it is natural to set $G(w) = IG(w) = \iota_n$ for any binary word w of length n starting with 0 and having at most two runs.

Evidently, there are non-identity permutations that are Grassmannian as well as inverse Grassmannian (see the examples above). We now characterize this overlap.

It can be checked that if the binary word w has either $(2k-1)$ or $2k$ runs for some $k \geqslant 2$, then the permutation $IG(w)$ has exactly $(k-1)$ descents. Hence $IG(w)$ is a Grassmannian permutation if and only if w has either 3 or 4 runs. Similarly, it can be checked that $G(w)$ is an inverse Grassmannian permutation if and only if w has either 3 or 4 runs.

Using the above observations and examining the bijections, we get the following result.

Lemma 7.26. *For any binary words w and w' starting with 0 and having at least 3 runs, $G(w) = IG(w')$ if and only if*

1. $w = 0^a 1^b 0^c$ *and* $w' = 0^a 1^c 0^b$ *for some* $a, b, c \geqslant 1$, *or*
2. $w = 0^a 1^b 0^c 1^d$ *and* $w' = 0^a 1^c 0^b 1^d$ *for some* $a, b, c, d \geqslant 1$.

Combining the above observations, we get that the permutations in Av[1432] are those of the following mutually disjoint types:

1. Type I: The identity permutations.

2. Type E: Permutations of the form $[G(w)]$ (or $[IG(w)]$) where w is a binary word starting with 0 and having either 3 or 4 runs.

3. Type G: Permutations of the form $[G(w)]$ where w is a binary word starting with 0 and having at least 5 runs.

4. Type IG: Permutations of the form $[IG(w)]$ where w is a binary word starting with 0 and having at least 5 runs.

Example 7.27. The above description reflects [12, Theorem 3] which states that for any $n \geqslant 1$,

$$\#\mathrm{Av}_n[1432] = 2^n + 1 - 2n - \binom{n}{3}.$$

This follows since the number of permutations of $\mathrm{Av}_n[1432]$ of different types are

1. Type I: 1.

2. Type E: $\binom{n-1}{2} + \binom{n-1}{3} = \binom{n}{3}$.

3. Type G: $2^{n-1} - (1 + \binom{n-1}{1} + \binom{n-1}{2} + \binom{n-1}{3}) = 2^{n-1} - (n + \binom{n}{3})$.

4. Type IG: $2^{n-1} - (1 + \binom{n-1}{1} + \binom{n-1}{2} + \binom{n-1}{3}) = 2^{n-1} - (n + \binom{n}{3})$.

Example 7.28. From [21, Theorem 3.6], we know that

$$D_n([1432]; q) = q + (2^{n-1} - n)q^2 + \sum_{j \geqslant 3} \binom{n}{2j-1} q^j. \tag{7.2}$$

We now prove this using the above characterization of $\mathrm{Av}[1432]$. It is straightforward to verify that

$$\mathrm{cdes}[\iota_n] = 1 \text{ and } \mathrm{cdes}[G(w)] = 2$$

for any binary word w of length n starting with 0 and having at least 3 runs. Since there are $(2^{n-1} - n)$ such binary words, we have obtained the first two terms on the right-hand side of (7.2). This covers the Type I, E and G. We now compute the cyclic descents in a permutation of Type IG.

Examining the bijection $w \leftrightarrow IG(w)$, it can be checked that the cyclic descents in $[IG(w)]$ correspond to the first 0 in any run of w consisting of 0s. This means that for any binary word w having $(2j-1)$ or $2j$ runs for some $j \geqslant 3$, we have $\mathrm{cdes}[IG(w)] = j$. Since there are

$$\binom{n-1}{2j-2} + \binom{n-1}{2j-1} = \binom{n}{2j-1}$$

binary words having $(2j-1)$ or $2j$ runs for any $j \geqslant 3$, we get the required result.

Chapter 8

Avoiding $[4, k]$-pairs in circular permutations

In this chapter, we define notions of patterns in the objects defined in the previous sections. This is done in such a way that it corresponds to pattern avoidance in circular permutations. This allows us to study avoidance of $[4, k]$-pairs in circular permutations.

In particular, we obtain upper bounds for the number of Wilf equivalence classes of $[4, k]$-pairs. Moreover, we prove that the obtained bound is tight when the pattern of size 4 in consideration is $[1342]$. Using ideas from our general results, we also obtain a complete characterization of the avoidance classes for $[4, 5]$-pairs.

The results in this chapter are from [42], which is joint work with Anurag Singh.

8.1 Subsequences in binary words: Avoiding $[1342, k]$-pairs

In this section, we study the avoidance of pairs of the form $[1342, \sigma]$ for some $\sigma \in \text{Av}[1342]$. In Section 7.1, we saw that circular permutations in $\text{Av}[1342]$ can be represented as binary words. We now see how patterns in permutations of $\text{Av}[1342]$ translate to subsequences in binary words.

Definition 8.1. Let w, w' be binary words. We say that w *contains* w' if $[\sigma(w)]$ contains $[\sigma(w')]$ and that w *avoids* w' if $[\sigma(w)]$ avoids $[\sigma(w')]$.

We now characterize this pattern containment relation. The following theorem is the cyclic analogue of [46, Lemma 6].

Theorem 8.2. *Let w, w' be binary words.*

1. *If w' has more than two runs, then w contains w' if and only if w contains w' as a subsequence.*
2. *If $w' = 0^{a+1}1^b$ for some $a, b \geqslant 0$, then w contains w' if and only if w contains either $0^{a+1}1^b$ or $1^{b+1}0^a$ as a subsequence.*
3. *If $w' = 1^{b+1}0^a$ for some $a, b \geqslant 0$, then w contains w' if and only if w contains either $0^{a+1}1^b$ or $1^{b+1}0^a$ as a subsequence.*

Proof. To prove this result, we examine what binary word corresponds to a pattern in a permutation of $\mathrm{Av}_n[1342]$. Let $w = w_1w_2\cdots w_{n-1}$ and $\sigma(w) = \sigma_1\sigma_2\cdots\sigma_n$. Suppose $A \subseteq [n-1]$ is the set of indices i for which $w_i = 0$ and $B = [n-1] \setminus A$. Then, we have

1. $a \in A,\ b \in B \Rightarrow \sigma_a < \sigma_n < \sigma_b$,
2. $a, a' \in A$ and $a < a' \Rightarrow \sigma_a < \sigma_{a'}$, and
3. $b, b' \in B$ and $b < b' \Rightarrow \sigma_b > \sigma_{b'}$.

This tells us that if $1 \leqslant i_1 < \cdots < i_{k-1} < i_k \leqslant n$, then the pattern $\sigma_{i_1}\cdots\sigma_{i_{k-1}}\sigma_{i_k}$ satisfies the same order relation as the permutation corresponding to the binary word $w_{i_1}\cdots w_{i_{k-1}}$. Hence, the binary word corresponding to $\sigma_{i_1}\cdots\sigma_{i_{k-1}}\sigma_{i_k}$ is $w_{i_1}\cdots w_{i_{k-1}}$.

This means that the circular patterns contained in $[\sigma(w)]$ are those of the form $[\sigma(w')]$ where w' is a subsequence of w. Combining this with Theorem 7.6 gives us the required result. □

The upshot of the above results is the following corollary.

Corollary 8.3. *The permutations in $\mathrm{Av}_n[1342]$ are in bijection with binary words of length $(n-1)$ of the following kinds:*

- *Words with at most two runs and starting with 0, i.e., 0^{n-1}, $0^{n-2}1$, $\ldots$, 01^{n-2}, which we call* exceptional.

- *Words with more than two runs, which we call* non-exceptional.

For any binary word w, we have for any $n \geqslant 2$,

$$\#\text{Av}_n[1342, \sigma(w)] = \#E_{n-1}(w) + \#NE_{n-1}(w) = \#B_{n-1}(w) - \#E_{n-1}(w) \qquad (8.1)$$

where

- $E_{n-1}(w)$ *is the set of exceptional binary words of length* $(n-1)$ *that avoid* w*,*
- $NE_{n-1}(w)$ *is the set of non-exceptional binary words of length* $(n-1)$ *that avoid* w*, and*
- $B_{n-1}(w)$ *is the set of binary words of length* $(n-1)$ *that avoid* w*.*

Proof. Most of the proof follows directly from Theorem 8.2. We only note that the second equality in (8.1) follows since the number of binary words starting with 0

1. having length $n-1$,
2. having at most two runs, and
3. containing a binary word w

is the same as the number of those starting with 1 satisfying the same conditions. □

The results of [21] can be proved using the correspondence between Av[1342] and binary words.

Example 8.4. From [21, Theorem 2.2], we know that for any $n \geqslant 3$,

$$\#\text{Av}_n[1342, 1234] = 2(n-2).$$

We prove this using Corollary 8.3.

Since $[1234] = [\sigma(0^3)] = [\sigma(10^2)]$, we take $w = 0^3$. Any binary word avoiding w can have at most two 0s and if it has two 0s, then the first term should be a 0. Counting based on the number of 0s, for any $n \geqslant 3$, there are

$$1 + (n-1) + (n-2)$$

binary words of length $n-1$ avoiding w. The exceptional such binary words are

$$01^{n-2} \text{ and } 0^2 1^{n-3}.$$

Hence, using Corollary 8.3, we get, as required,

$$\#\mathrm{Av}_n[1342, 1234] = 1 + (n-1) + (n-2) - 2 = 2(n-2).$$

We now study Wilf equivalences among pairs of patterns the form $[1342, \sigma]$ where $[\sigma]$ avoids the pattern $[1342]$.

Theorem 8.5. *For any $k \geqslant 1$, all pairs of patterns of the form $[1342, \sigma(w)]$, where w is a non-exceptional binary word of length k, are Wilf equivalent.*

Proof. The idea of this proof is similar to that of [32, Theorem 2.7]. Let $w = w_1 w_2 \cdots w_k$ be a non-exceptional binary word. We prove this result by describing the cyclic permutations that avoid $[1342]$ but contain $[\sigma(w)]$. By Theorem 8.2, we have to describe the binary words that contain w as a subsequence. Any such binary word will also be non-exceptional and hence we do not have to worry about over-counting exceptional words.

Using the left most occurrence of w, we can see that any binary word v containing w is of the form

$$v = v^{(1)}\ w_1\ v^{(2)}\ w_2\ \cdots\ v^{(k)}\ w_k\ v^{(k+1)}$$

where $v^{(i)}$ is a word whose letters are in $\{0,1\} \setminus \{w_i\}$ for $i \in [k]$ and $v^{(k+1)}$ is a word with letters from $\{0,1\}$. Note that the $v^{(i)}$'s could be empty words as well. This shows that the number of cyclic permutations in $\mathrm{Av}_n[1342]$ that contain $[\sigma]$ is

$$\sum_{(n_1, n_2, \ldots, n_{k+1})} 2^{n_{k+1}}$$

where the sum is over tuples $(n_1, n_2, \ldots, n_{k+1})$ such that $n_i \geqslant 0$ for all $i \in [k+1]$ and $k + n_1 + n_2 \cdots + n_{k+1} = n-1$. Since this only depends on the size of w, we get our result. □

The above proof also gives us the following generating function.

Corollary 8.6. *For any $k \geqslant 1$ and non-exceptional binary word w of length k, we have*

$$\sum_{n=1}^{\infty} \#\mathrm{Av}_n[1342, \sigma(w)]t^n = \sum_{n=1}^{\infty} \#\mathrm{Av}_n[1342]t^n - \left(\frac{t}{1-t}\right)^k \left(\frac{t}{1-2t}\right).$$

Using the fact that $\#\mathrm{Av}_n[1342] = 2^{n-1} - (n-1)$ for $n \geqslant 1$, we get

$$\sum_{n=1}^{\infty} \#\mathrm{Av}_n[1342, \sigma(w)]t^n = \frac{t}{1-2t} - \frac{t^2}{(1-t)^2} - \left(\frac{t}{1-t}\right)^k \left(\frac{t}{1-2t}\right).$$

Lemma 8.7. *For any $a, b \geqslant 0$, the pair $[1342, \sigma(0^{a+1}1^b)]$ is Wilf equivalent to $[1342, \sigma(0^{b+1}1^a)]$.*

Proof. This is a trivial Wilf equivalence and follows since $[1342^c] = [1342]$ and for any $a, b \geqslant 0$, $[\sigma(0^{a+1}1^b)^c] = [\sigma(0^{b+1}1^a)]$. □

We now show that there are no other Wilf equivalences among exceptional patterns.

Theorem 8.8. *Let $0^{a+1}1^b$ and $0^{c+1}1^d$ be such that $a, b, c, d \geqslant 0$ and $\{a, b\} \neq \{c, d\}$. Then the pairs $[1342, \sigma(0^{a+1}1^b)]$ and $[1342, \sigma(0^{c+1}1^d)]$ are not Wilf equivalent.*

Proof. If $a + b \neq c + d$, then the result follows since taking $n = \min\{a+b+2, c+d+2\}$ gives different values for $\#\mathrm{Av}_n[1342, \sigma(0^{a+1}1^b)]$ and $\#\mathrm{Av}_n[1342, \sigma(0^{c+1}1^d)]$.

Suppose $a + b = c + d$. Without loss of generality, we can translate the condition $\{a, b\} \neq \{c, d\}$ to $c < a$ and $c \neq b$. We will show that $\#\mathrm{Av}_n[1342, \sigma(0^{a+1}1^b)]$ and $\#\mathrm{Av}_n[1342, \sigma(0^{c+1}1^d)]$ differ for some $n \geqslant 1$.

Let $k = a + b + 1 = c + d + 1$, the length of the words $0^{a+1}1^b$ and $0^{c+1}1^d$. The exceptional binary words of length $n \geqslant k$ that contain neither $0^{a+1}1^b$ nor $1^{b+1}0^a$ as a subsequence are

$$01^{n-1}, 0^2 1^{n-2}, \ldots, 0^a 1^{n-a}, 0^{n-b+1}1^{b-1}, 0^{n-b+2}1^{b-2}, \ldots, 0^{n-1}1, 0^n.$$

Since $n \geqslant k$, these are all distinct and there are $(a + b)$ of them. This means that, by Corollary 8.3, the number of permutations in $\mathrm{Av}_n[1342, \sigma(0^{a+1}1^b)]$ is $(a + b)$ less than the number of binary words of length n that contain neither $0^{a+1}1^b$ nor $1^{b+1}0^a$ as a subsequence. We get a similar result for the size of $\mathrm{Av}_n[1342, \sigma(0^{c+1}1^d)]$. Since $a + b = c + d$, to prove our result, it is enough to show that for some $n \geqslant k$, the number of binary words of length n that contain neither $0^{a+1}1^b$ nor $1^{b+1}0^a$ as a subsequence is different from the number of those that contain neither $0^{c+1}1^d$ nor $1^{d+1}0^c$ as a subsequence.

We have $c \neq b$, so we consider two cases: $b < c$ and $c < b$. First let us consider $b < c$ and $n = k + b + 1$. Consider the binary words of length n that contain either $0^{a+1}1^b$ or $1^{b+1}0^a$ as a subsequence. The proof of Theorem 8.5 shows that the number that contain $0^{a+1}1^b$ as a subsequence is

$$\sum_{\substack{(n_1, n_2, \ldots, n_{k+1}) \\ k+n_1+n_2\cdots+n_{k+1}=n}} 2^{n_{k+1}}.$$

This same number counts the binary words of length n that contain $1^{b+1}0^a$ as a subsequence. However, the binary word $1^{b+1}0^{a+1}1^b$ is of length n and contains both $0^{a+1}1^b$ and $1^{b+1}0^a$ as subsequences. Hence, the number of binary word of length n that contain either $0^{a+1}1^b$ or $1^{b+1}0^a$ as a subsequence is strictly less than

$$2 \quad \times \sum_{\substack{(n_1, n_2, \ldots, n_{k+1}) \\ k+n_1+n_2\cdots+n_{k+1}=n}} 2^{n_{k+1}}. \tag{8.2}$$

We now show that no binary word of length n can contain both $0^{c+1}1^d$ and $1^{d+1}0^c$ as subsequences. This will then show that the number of binary words containing either $0^{c+1}1^d$ or $1^{d+1}0^c$ as a subsequence is given by (8.2) and hence that $[1342, \sigma(0^{a+1}1^b)]$ and $[1342, \sigma(0^{c+1}1^d)]$ are not Wilf equivalent.

Let w be a binary word of length n containing both $0^{c+1}1^d$ and $1^{d+1}0^c$ as subsequences. Since w contains the subsequence $0^{c+1}1^d$, it must have at least d 1s after the $(c+1)^{\text{th}}$ 0. Similarly, since w contains the subsequence $1^{d+1}0^c$, it must have at least c 0s after the $(d+1)^{\text{th}}$ 1. This means that if the $(d+1)^{\text{th}}$ 1 is before the $(c+1)^{\text{th}}$ 0, then w has at least $(d+1)+(c+1)+d$ letters. On the other hand, if the $(d+1)^{\text{th}}$ 1 is after the $(c+1)^{\text{th}}$ 0, then w has at least $(d+1)+(c+1)+c$ letters. But we have

$$n = c+d+b+2 < \begin{cases} d+2c+2, & \text{since } b < c \\ c+2d+2, & \text{since } b < d. \end{cases}$$

This is a contradiction to the length of w being n. Hence, no binary word of length n can contain both $0^{c+1}1^d$ and $1^{d+1}0^c$.

Next, we have to consider the case when $c < b$. But this follows just as before by taking $n = k + c + 1$. □

Corollary 8.9. *If w and w' are binary words where w is exceptional and w' is not, then $[1342, \sigma(w)]$ and $[1342, \sigma(w')]$ are not Wilf equivalent.*

Proof. Just as in the previous theorem, we can assume that w and w' have the same length, say $k \geqslant 1$. Suppose $w = 0^{a+1}1^b$ for some $a, b \geqslant 0$. Hence, $k = a + b + 1$ and by the proof of Theorem 8.8, the number of permutations in $\mathrm{Av}_{k+2}[1342]$ that contain $[\sigma(w)]$ if $\{a, b\} \neq \{0, k-1\}$ is

$$2 \times \sum_{\substack{(n_1, n_2, \dots, n_{k+1}) \\ k+n_1+n_2 \cdots + n_{k+1} = k+1}} 2^{n_{k+1}} - (k + 1 - (a + b)) = 2k + 2.$$

This is because, if $\{a, b\} \neq \{0, k-1\}$, there are no binary words of length $(k+1)$ that contain both $0^{a+1}1^b$ and $1^{b+1}0^a$ as subsequences. Otherwise, there are exactly two binary words of length $k+1$ that contain both $0^{a+1}1^b$ and $1^{b+1}0^a$ as subsequences. If $a = k-1$ and $b = 0$, they are 10^k and 010^{k-1} and if $a = 0$ and $b = k-1$, they are 01^k and 101^{k-1}. Hence, if $\{a, b\} = \{0, k-1\}$, the number of permutations in $\mathrm{Av}_{k+2}[1342]$ that contain $[\sigma(w)]$ is

$$2 \times \sum_{\substack{(n_1, n_2, \dots, n_{k+1}) \\ k+n_1+n_2 \cdots + n_{k+1} = k+1}} 2^{n_{k+1}} - (k + 1 - (a + b)) - 2 = 2k.$$

Also, by the proof of Theorem 8.5, the number of permutations in $\mathrm{Av}_{k+2}[1342]$ that contain $[\sigma(w')]$ is

$$\sum_{\substack{(n_1, n_2, \dots, n_{k+1}) \\ k+n_1+n_2 \cdots + n_{k+1} = k+1}} 2^{n_{k+1}} = k + 2.$$

Since there exist non-exceptional words only for $k \geqslant 3$, we get that $\#\mathrm{Av}_{k+2}[1342, \sigma(w)] < \#\mathrm{Av}_{k+2}[1342, \sigma(w')]$. Hence $[1342, \sigma(w)]$ and $[1342, \sigma(w')]$ are not Wilf equivalent. □

We call a pair of patterns $[1342, \sigma]$ a $[1342, k]$-pair if $[\sigma] \in \mathrm{Av}_k[1342]$.

Theorem 8.10. *For $k \geqslant 4$, the number of Wilf equivalence classes of $[1342, k]$-pairs is $\lceil \frac{k}{2} \rceil$.*

Proof. Combining Theorem 8.5, Lemma 8.7, Theorem 8.8, and Corollary 8.9, we get that for $k \geqslant 4$, the number of Wilf equivalence classes of $[1342, k]$-pairs is

$$1 \quad + \quad \#\{(a,b) : a \geqslant b \geqslant 0, \ a+b = k-1\}.$$

The first term counts the equivalence class consisting of non-exceptional patterns and the second counts the exceptional classes. □

We now compute the sequence $(\#\mathrm{Av}_n[1342,\sigma])_{n\geqslant 1}$ for various $[\sigma] \in \mathrm{Av}[1342]$.

Proposition 8.11. *For any* $k \geqslant 1$, *we have* $[1342, \iota_{k+1}] \equiv [1342, \delta_{k+1}]$ *and for any* $n \geqslant k$,

$$\#\mathrm{Av}_{n+1}[1342, \iota_{k+1}] = \binom{n-1}{k-2} - (k-1) + \sum_{i=0}^{k-2} \binom{n}{i}.$$

Proof. Since $\sigma(0^k) = \iota_{k+1}$, we have to count the binary words of length n that contain neither 0^k nor 10^{k-1} as subsequences. Such words either have strictly less than $(k-1)$ 0s, or have exactly $(k-1)$ 0s and start with 0. Clearly, there are

$$\binom{n-1}{k-2} + \sum_{i=0}^{k-2} \binom{n}{i}$$

such binary words of length n. Since there are $(k-1)$ exceptional words of length n that avoid 0^k and 10^{k-1}, we get the required result. □

Proposition 8.12. *Let w be a non-exceptional binary word of length* $k \geqslant 1$. *For any* $n \geqslant k$, *we have*

$$\#\mathrm{Av}_{n+1}[1342, \sigma(w)] = 1 + \sum_{i=2}^{k-1} \binom{n}{i}. \tag{8.3}$$

Proof. This follows from the fact that there are

$$\sum_{i=0}^{k-1} \binom{n}{i}$$

binary words of length n that do not contain 0^k as a subsequence. By the proof of Theorem 8.5, this is the same as the number of those that do not contain w as a subsequence. The equality in (8.3) then follows from Corollary 8.3 since all n exceptional binary words of length n avoid w. □

Remark 8.13. The following, admittedly complicated, generating function can be obtained for the size of avoidance classes of $[1342, \sigma(w)]$-pairs where w is exceptional. If $w = 0^{a+1}1^b$, then

$$\sum_{n \geqslant 1} \#\mathrm{Av}_n[1342, \sigma(w)]t^n = t[G_1(t) + G_2(t) + G_3(t)] - E(t)$$

where the terms are defined as follows.

1. $G_1(t)$ accounts for those binary words with at most b 1s and is given by

$$\left(\frac{1-t^{a+1}}{1-t}\left(\frac{1}{1-t}\right)^b + \sum_{k=1}^{b}\left(\frac{1}{1-t}\right)^k\right).$$

2. $G_2(t)$ accounts for those binary words with $(b+k)$ 1s where $k \in [b]$ and is given by

$$\sum_{k=1}^{b}\left[\left(\sum_{i=0}^{a}\binom{i+k}{k}t^i\right) \times \left(\frac{1}{1-t}\right)^{b-k} \times \left(\sum_{i=0}^{a-1}\binom{i+k-1}{i}t^i\right)\right].$$

3. $G_3(t)$ accounts for those binary words with $(b+k)$ 1s where $k > b$ and is given by

$$\sum_{k>b}\sum_{j=0}^{a-1}\left[\binom{b-k-1+j}{j}t^j \times \left(\sum_{i=0}^{a-j}\binom{i+b}{b}t^i\right) \times \left(\sum_{i=0}^{a-j-1}\binom{i+b-1}{i}t^i\right)\right].$$

4. $E(t)$ accounts for the exceptional over-counting and is given by

$$\sum_{n=0}^{a+b+1}(n-1)t^n + \sum_{n>a+b+1}(a+b)t^n.$$

Using the results of this section we also get the following result about linear pattern avoidance.

Corollary 8.14. *Let* $k \geqslant 1$*. All sets of linear patterns of the form* $\{213, 231, \sigma(w)\}$*, where* w *is a binary word of length* k*, are Wilf equivalent. For any* $n \geqslant k$*,*

$$\#\mathrm{Av}_{n+1}(213, 231, \sigma(w)) = \sum_{i=0}^{k-1}\binom{n}{i}.$$

We also have

$$\sum_{n=1}^{\infty} \#\mathrm{Av}_n(213, 231, \sigma(w))t^n = \frac{t}{1-2t} - \left(\frac{t}{1-t}\right)^k \left(\frac{t}{1-2t}\right).$$

Proof. This follows from:

1. The proof of [46, Lemma 6] (linear version of Theorem 8.2), which shows that (linear) pattern avoidance in $(213, 231)$-avoiding permutations corresponds to linear pattern avoidance in the corresponding binary words.

2. The proof of Theorem 8.5, which shows that the number of binary words of size n linearly containing a pattern only depends on the size of the pattern.

3. The fact that there are
$$\sum_{i=0}^{k-1} \binom{n}{i}$$
binary words of length n that do not contain 0^k as a subsequence.

4. The fact that for any $n \geqslant 1$, $\#\mathrm{Av}_n(213, 231) = 2^{n-1}$.

□

8.1.1 Avoiding [1342] and a pattern of size 5

We now use our results to study avoidance of pairs $[1342, \sigma]$ where $[\sigma] \in \mathrm{Av}_5[1342]$.

The first two results are special cases of Propositions 8.11 and 8.12 respectively.

Corollary 8.15. *We have* $[1342, 12345] \equiv [1342, 15432]$ *and for any* $n \geqslant 5$,

$$\#\mathrm{Av}_n[1342, 12345] = (n-3) + \binom{n-1}{2} + \binom{n-2}{2}.$$

Corollary 8.16. *For any* $\sigma \in \{12435, 12534, 13254, 14235, 14325, 15234, 15243, 15423\}$ *and* $n \geqslant 5$, *we have*

$$\#\mathrm{Av}_n[1342, \sigma] = 1 + \binom{n-1}{2} + \binom{n-1}{3}.$$

Proposition 8.17. *We have* $[1342, 12354] \equiv [1342, 12543]$ *and for* $n \geqslant 6$,

$$\#\mathrm{Av}_n[1342, 12354] = 3n - 1.$$

Proof. Since $[12354] = [\sigma(0^3 1)] = [\sigma(1^2 0^2)]$ we have to count binary words of length $(n-1)$ that contain neither $0^3 1$ nor $1^2 0^2$ as a subsequence. The result follows from the following facts which can be verified.

1. There are 4 such binary words with at most one 1.

2. There are $(3n-2)$ such binary words with at least two 1s.

3. There are 3 such binary words that are exceptional.

□

From these computations, we get the following result.

Result 8.1. *There are* 3 *Wilf equivalence classes among* $[1342, \sigma]$*-pairs where* $[\sigma] \in \mathrm{Av}_5[1342]$.

Note that this is just a special case of Theorem 8.10.

8.2 Domination in circled compositions: Avoiding $[1324, k]$-pairs

In Section 7.2, we saw that the permutations in $\mathrm{Av}[1324]$ correspond to circled compositions. We not define a notion called *domination* in circled compositions and show that this corresponds to patterns in permutations of $\mathrm{Av}[1324]$. We use this to study avoidance of $[4, k]$-pairs where the pattern of size 4 is $[1324]$.

Definition 8.18. A circled composition X is said to *dominate* a circled composition Y if Y can be obtained from X via the following procedure:

1. Select a subsequence of X.

2. Replace any uncircled number k in this sequence by some number in $[k]$.

3. If either the first or last term is an uncircled number k, then replace it with k ①s.

Example 8.19. The circled composition X given by

$$\textcircled{1}^2 \quad 5 \quad \textcircled{1} \quad 1 \quad \textcircled{1}^2 \quad 3 \quad 1 \quad \textcircled{1}^3 \quad 2 \quad 1 \quad \textcircled{1}^3$$

dominates the circled composition Y given by

$$\textcircled{1}^8 \quad 2 \quad \textcircled{1}^2 \quad 1 \quad \textcircled{1}.$$

One possible procedure corresponding to the steps in Definition 8.18 that illustrates this is as follows:

1. Select the highlighted subsequence of X in

$$\textcircled{1}^2 \quad \textcolor{red}{5} \quad \textcolor{red}{\textcircled{1}} \quad 1 \quad \textcolor{red}{\textcircled{1}^2} \quad \textcolor{red}{3} \quad 1 \quad \textcolor{red}{\textcircled{1}^2} \quad \textcircled{1} \quad 2 \quad \textcolor{red}{1} \quad \textcolor{red}{\textcircled{1}} \quad \textcircled{1}^2.$$

2. Replacing the uncircled 3 by 2, we get

$$5 \quad \textcircled{1}^3 \quad 2 \quad \textcircled{1}^2 \quad 1 \quad \textcircled{1}.$$

3. After replacing the the first term, the uncircled 5, with 5 $\textcircled{1}$s, we get Y.

Theorem 8.20. *Given a circled composition* X *of* n, *let* $[\sigma(X)]$ *be the associated permutation in* $\text{Av}_n[1324]$. *For any two circled compositions* X *and* Y, *we have that* $[\sigma(X)]$ *contains* $[\sigma(Y)]$ *if and only if* X *dominates* Y.

Proof. We have to show that pattern containment in permutations of Av[1324] corresponds to domination of circled compositions. We do this by showing that finding a pattern in a permutation of Av[1324] corresponds to the steps in Definition 8.18 in the associated circled composition.

Recall from the proof of Theorem 7.17 that the terms in a circled composition of n correspond to the numbers $1, \ldots, n$ in order. That is, the first part, which is always 1, corresponds to the number 1 in the permutation. If the second part is b, it corresponds to the numbers $2, \ldots, b+1$ in the permutation, and so on. Also, the circled parts correspond to the numbers in the final run when the permutation is written with the largest number at the end.

Let X be a circled composition and suppose we have chosen an occurrence of a pattern in $[\sigma(X)]$. Example 8.21 below illustrates the steps that follow and hence might make them easier to understand.

1. Since an occurrence of a pattern in $[\sigma(X)]$ corresponds to choosing some elements of the permutation, let $A \subseteq [n]$ be the elements chosen. Highlight the subsequence of X that consists of parts whose corresponding permutation elements have elements of A.

2. We rewrite the subsequence by replacing each uncircled number with the number of corresponding permutation elements that are in A. Call this sequence Y'. The pattern obtained by selecting the numbers in A can be extracted from Y'. This is done in a similar fashion to how a permutation in $\text{Av}[1324]$ is obtained from a circled composition (drawing the permutation with contiguous runs corresponding to the uncircled parts and adding terms at the end of the permutation at the appropriate places using the $\textcircled{1}$s).

3. To get the circled composition corresponding to this pattern, we have to cyclically shift it so that it ends with the largest number. To do so, it can be checked that

   (a) we do not have to cyclically shift the pattern if the last part of Y' is circled, and

   (b) if the last part of Y' is a and is uncircled, we have to cyclically shift the pattern a steps to the left.

   Since at most one interval of numbers is cyclically shifted to the end, the only numbers in the final run are those corresponding to the first and last term of Y' and those corresponding to circled parts of Y'. Also, the circled composition Y corresponding to this pattern is the one obtained by replacing the first and last part of Y' with the appropriate number of $\textcircled{1}$s.

□

Example 8.21. Let X be the circled composition given by

$$\textcircled{1}^2 \quad 1 \quad 2 \quad 1 \quad \textcircled{1}^3 \quad 1 \quad 3 \quad 1 \quad \textcircled{1}.$$

Consider the pattern induced by the numbers $A = \{4, 5, 6, 7, 9, 12, 13\}$. This permutation is shown in Figure 8.1 with the numbers in A highlighted using red boxes.

1. Looking at which parts have corresponding numbers that intersect A, we highlight the sequence shown below:

$$\textcircled{1}^2 \quad 1 \quad 2 \quad 1 \quad \textcircled{1} \quad \textcircled{1} \quad \textcircled{1} \quad 1 \quad 3 \quad 1 \quad \textcircled{1}$$

2. Since all numbers from those corresponding to the uncircled 2 and the uncircled 1 are chosen, they are left unchanged. Since only 2 numbers are chosen from those corresponding to the uncircled 3, it is replaced by an uncircled 2. Hence Y' is the sequence

$$2 \quad 1 \quad \textcircled{1}^2 \quad 2.$$

The pattern corresponding to Y' is shown on the left in Figure 8.2.

3. After writing the pattern in the required form, we see that its corresponding circled composition is

$$\textcircled{1}^2 \quad 1 \quad \textcircled{1}^4.$$

This is shown on the right in Figure 8.2. Note that this circled composition is the same one that is obtained from the sequence Y' in the previous point after replacing the first and last part with the appropriate number of $\textcircled{1}$s.

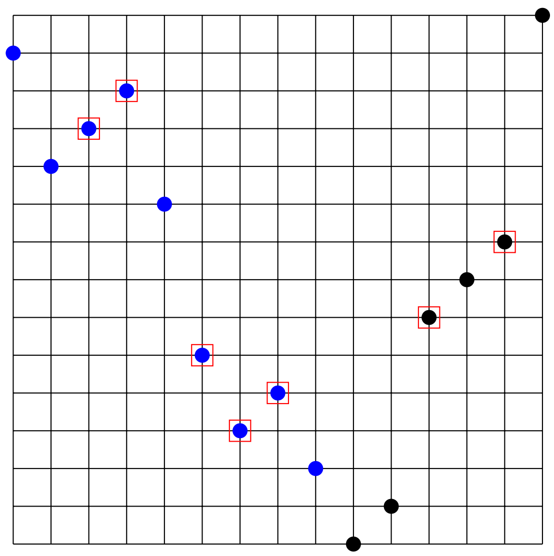


FIGURE 8.1: Step 1 in Example 8.21.

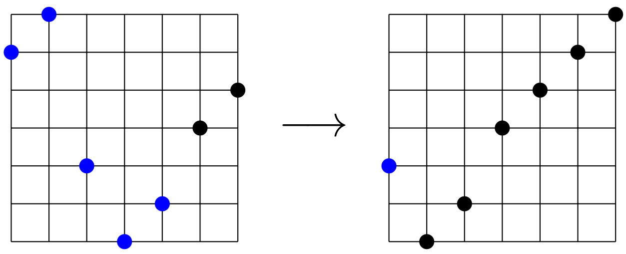


FIGURE 8.2: Step 2 and 3 in Example 8.21.

Example 8.22. From [21, Theorem 2.3], we know that for any $n \geqslant 3$,

$$\#\mathrm{Av}_n[1324, 1234] = 2(n-2).$$

We now prove this using circled compositions. Since the circled composition corresponding to [1234] is $\textcircled{1}^4$, for any $n \geqslant 3$, we have to count the circled compositions of n that do not dominate $\textcircled{1}^4$. Such a circled composition has either two or three $\textcircled{1}$s.

If the circled composition has three $\textcircled{1}$s, it is of the form

$$\textcircled{1} \quad A \quad \textcircled{1} \quad B \quad \textcircled{1}$$

where A and B consist of only uncircled numbers. If A has some number $k \geqslant 2$, then we can use it along with the two $\textcircled{1}$s after it to obtain the circled composition $\textcircled{1}^4$. Hence A consits of just 1s. Similarly B also consists of just 1s. We can check that such circled compositions do not domiate $\textcircled{1}^4$. It is easy to check that there are $(n-2)$ such circled compositions.

Similarly, the circled compositions of n that have two $\textcircled{1}$s and do not dominate $\textcircled{1}^4$ are of the form

$$\textcircled{1} \quad A \quad \textcircled{1}$$

where A consits of all 1s or has exactly one 2 and all other terms 1. Again, the number of such compositions is $(n-2)$. Combining these counts, we get the required result.

By the above results, studying pattern avoidance among permutations in Av[1324] is the same as studying the domination poset of circled compositions. Hence, we now focus on domination in circled compositions.

Definition 8.23. Two circled compositions X and Y are said to be *Wilf equivalent*, written as $X \equiv Y$, if the number of circled compositions of n that dominate X is equal to the number that dominate Y for any $n \geqslant 1$.

Lemma 8.24. *If X and Y are circled compositions such that $X \equiv Y$, then they are compositions of the same number.*

Proof. This follows from the fact that if $m < n$, any circled composition of m avoids all circled compositions of n. □

Lemma 8.25. *Any circled composition is Wilf equivalent to its reverse.*

Proof. This follows from the fact that a circled composition X dominates a circled composition Y if and only if the reverse of X dominates the reverse of Y. □

Remark 8.26. In terms of permutations, the above lemma translates to the trivial Wilf equivalence $[1324, \sigma] \equiv [1324, \sigma^{rc}]$.

Suppose X is a circled composition with at least one uncircled number. This means that X is of the form

$$\textcircled{1}^{r} \quad A \quad \textcircled{1}^{s}$$

where $r, s \geqslant 1$ and A is a non-empty sequence of $\textcircled{1}$s and uncircled numbers that starts and ends with an uncircled number. Suppose A has n parts such that the parts indexed by $C \subseteq [n]$ are circled. Using the leftmost occurrence of

$$\textcircled{1}^{r} \quad A,$$

we can see that a circled compositions that dominates X can be written uniquely as

$$D^{r} \quad D_1 \quad D_2 \quad \cdots \quad D_n \quad D^{s} \tag{8.4}$$

where the following conditions hold:

1. D^r is a sequence of $\textcircled{1}$s and uncircled numbers that starts with a $\textcircled{1}$ such that the following hold:
   (a) If the number of $\textcircled{1}$s in D^r is m, we have $m \leqslant r$. Also, if $m = r$, then D^r ends with a $\textcircled{1}$ and if D^r ends with an uncircled number, then $m < r$.

(b) For any non-final uncircled number k in D^r, k more than the number of ①s after it is at most r.

(c) If D^r ends with a ① and $m < r$, the value r is attained at least once in the procedure given in (b).

(d) If D^r ends with an uncircled number, this number is at least r and the value r is never attained in the procedure given in (b).

2. If $i \in C$, then D_i is a sequence that ends with a ① and all other terms are uncircled.

3. If $i \in [n] \setminus C$ and the corresponding uncircled number in A is k, then D_i is a sequence that ends with an uncircled number whose value is at least k and all other terms are either ①s or uncircled numbers less than k.

4. D^s is a sequence of ①s and uncircled numbers that ends with a ① such that at least one of the following hold:

(a) The number of ①s in D^s is at least s.

(b) There is some uncircled number k in D^s such that k more than the number of ①s before it is at least s.

Remark 8.27. Before using this description to prove results about Wilf equivalence, we note that it gives us a generating function for the circled compositions that dominate X. Denoting this generating function by F_X, we have

$$F_X = F^r F_1 F_2 \cdots F_n F^s$$

where the terms are defined as follows.

1. F^r corresponds to D^r and $F^r(t)$ is given by

$$\sum_{m=1}^{r} \left[t \prod_{i=1}^{m-1} \frac{1-t}{1-t^{r-m+i}} \right] - \sum_{m=1}^{r-1} \left[t \left(\frac{1-2t^{r-1}}{1-t^{r-1}} \right) \prod_{i=0}^{m-2} \frac{1-t}{1-t^{r-m+i}} \right].$$

2. For $i \in [n]$, F_i corresponds to D_i. If $i \in C$, $F_i(t)$ is given by

$$\frac{t(1-t)}{1-2t}$$

and if $i \in [n] \setminus C$, and the corresponding uncircled number in A is k, then $F_i(t)$ is given by

$$\frac{t^k}{(1-t)^2}.$$

3. F^s corresponds to D^s and $F^s(t)$ is given by

$$\frac{t(1-2t)}{1-3t+t^2} - \sum_{m=1}^{s-1} \left[t \prod_{i=0}^{m-2} \frac{1-t}{1-t^{s-m+i}} \right].$$

This description also tells us that the circled composition X is Wilf equivalent to any circled composition obtained by permuting A such that it is still starts and ends with an uncircled number. If $\sigma \in \mathfrak{S}_n$ is such a permutation, then the circled composition dominating X written in the form (8.4) is mapped to

$$D^r \quad D_{\sigma(1)} \quad D_{\sigma(2)} \quad \cdots \quad D_{\sigma(n)} \quad D^s.$$

This means that we can combine all except the first and last strings of circled numbers and rearrange the uncircled numbers. We adopt the convention of reordering the uncircled numbers in decreasing order and placing the combined circled numbers (if any exist) before the last uncircled number. Also, by Lemma 8.25, we can make sure that the initial string of $①$s is longer than the final string.

Example 8.28. The circled composition

$$①^2 \quad 5 \quad ① \quad 1 \quad ①^2 \quad 3 \quad 1 \quad ①^6 \quad 2 \quad 1 \quad ①^3$$

is Wilf equivalent to the circled composition

$$①^3 \quad 5 \quad 3 \quad 2 \quad 1 \quad 1 \quad ①^9 \quad 1 \quad ①^2.$$

The above discussions gives us the following lemma.

Lemma 8.29. *Any circled composition with at least one uncircled number is Wilf equivalent to a circled composition of one of the following forms:*

1. *A circled composition*

$$①^{k_0} \quad a_1 \quad a_2 \quad \cdots \quad a_k \quad ①^{k_1} \quad a_{k+1} \quad ①^{k_2}$$

where $k_0 \geqslant k_2$ *and* $a_1 \geqslant a_2 \geqslant \cdots \geqslant a_k \geqslant a_{k+1}$.

2. *A circled composition*

$$\textcircled{1}^{k_0} \quad a_1 \quad a_2 \quad \cdots \quad a_k \quad \textcircled{1}^{k_2}$$

where $k_0 \geqslant k_2$ *and* $a_1 \geqslant a_2 \geqslant \cdots \geqslant a_k$.

Lemma 8.30. *Suppose* X *is a circled composition of the form*

$$A \quad 2 \quad B$$

where A *and* B *are nonempty sequences of* $\textcircled{1}$*s and uncircled numbers such that* A *starts and* B *ends with a* $\textcircled{1}$*. If* B *has an uncircled number, then* $X \equiv Y$ *where* Y *is the circled composition*

$$A \quad 1 \quad \textcircled{1} \quad B.$$

Proof. Using the same ideas as the discussion above, we can prove this result by showing that there are a same number of the following objects for each $n \geqslant 1$:

1. Sequences D of circled and uncircled numbers such that
   (a) D ends with an uncircled number whose value is at least 2 and all other terms are either $\textcircled{1}$s or uncircled 1s, and
   (b) the sum of the parts in D is n.
2. Pairs of the form (D_1, D^1) where
   (a) D_1 is a sequence that ends with an uncircled number with all terms before it being $\textcircled{1}$s, and
   (b) D^1 is a sequence that ends with a $\textcircled{1}$ and all terms before it are uncircled

   such that the sum of parts in D_1 added to the sum of parts in D^1 is n.

It can be checked that both these objects are counted by $2^{n-1} - 1$. For objects of type (1), we can use the recurrence $a(n) = a(n-1) + 2^{n-2}$ obtained based on whether the last number is greater than 2. For the objects of type (2) concatenating D_1 and D^1 and deleting the $\textcircled{1}$s gives a bijection with compositions of numbers less than n. □

Lemma 8.31. *Suppose* X *is a circled composition of the form*

$$\textcircled{1}^2 \quad c \quad B$$

where c *is an uncircled number and* B *is a nonempty sequence of* $\textcircled{1}$*s and uncircled numbers that ends with a* $\textcircled{1}$*. Then* $X \equiv Y$ *where* Y *is the circled composition*

$$\textcircled{1} \quad 1 \quad c \quad B.$$

Proof. Let Z be a circled composition of n that dominates X. Using the left most occurrence of $\textcircled{1}^2$, we can write Z as

$$Z_1 \quad Z_2$$

where Z_1 is a sequence of the form

$$\textcircled{1} \quad 1 \quad 1 \quad \cdots \quad 1 \quad \textcircled{1} \quad \text{or} \quad \textcircled{1} \quad 1 \quad 1 \quad \cdots \quad 1 \quad a \tag{8.5}$$

where $a \geqslant 2$.

Let Z' be the circled composition

$$Z_1' \quad Z_2$$

where Z_1' is obtained by replacing all terms between the first and last term of Z_1 with $\textcircled{1}$ and the last term with 1 if the last term of Z_1 is $\textcircled{1}$. This means that if Z_1 is as in (8.5), then Z_1' is

$$\textcircled{1} \quad \textcircled{1} \quad \textcircled{1} \quad \cdots \quad \textcircled{1} \quad 1 \quad \text{or} \quad \textcircled{1} \quad \textcircled{1} \quad \textcircled{1} \quad \cdots \quad \textcircled{1} \quad a$$

respectively. It can be checked that this is a bijection between circled compositions of n that dominate X and those that dominate Y. □

Combining Lemma 8.29, Lemma 8.30, and Lemma 8.31, we get the following theorem.

Theorem 8.32. *Any circled composition of* n *is Wilf equivalent to a circled composition of* n *of one of the following forms:*

1. *The circled composition* $\textcircled{1}^n$.

2. *A circled composition*

$$\textcircled{1}^{k_0} \quad a_1 \quad a_2 \quad \cdots \quad a_k \quad \textcircled{1}^{k_1} \quad a_{k+1} \quad \textcircled{1}^{k_2}$$

where $k_0 \geqslant k_2$, $a_1 \geqslant a_2 \geqslant \cdots \geqslant a_k \geqslant a_{k+1}$, *and* $a_1, \ldots, a_{k+1}, k_0, k_2 \neq 2$.

3. *A circled composition*

$$\textcircled{1}^{k_0} \quad a_1 \quad a_2 \quad \cdots \quad a_k \quad \textcircled{1}^{k_2}$$

where $k_0 \geqslant k_2$, $k_0, k_2 \neq 2$, $a_1 \geqslant a_2 \geqslant \cdots \geqslant a_k$, *and if* $k \geqslant 2$, *then* $a_1, \ldots, a_k \neq 2$.

We now compute the sequence $(\#\mathrm{Av}_n[1324, \sigma])_{n \geqslant 1}$ for various $[\sigma] \in \mathrm{Av}[1324]$.

Proposition 8.33. *We have for* $n \geqslant 2$ *and* $k \geqslant 1$,

$$\#\mathrm{Av}_n[1324, \delta_{k+2}] = \sum_{i=0}^{k-1} \binom{n+i-2}{2i}.$$

Proof. Note that for any $k \geqslant 1$, $[\delta_{k+2}]$ is the permutation corresponding to the circled composition

$$\textcircled{1} \quad 1 \quad 1 \quad \cdots \quad 1 \quad \textcircled{1}$$

where there are k uncircled 1s. Hence, $\#\mathrm{Av}_n[1324, \delta_{k+2}]$ is the number of circled compositions of n that have less than k uncircled numbers. We count these circled compositions based on the number of uncircled parts.

The following procedure can be used to specify a circled composition of n with i uncircled parts.

1. Consider a sequence of length $n+i$ consisting of 1s, with the first and last 1 circled.

2. Select $2i$ out of the remaining $(n+i-2)$ uncircled 1s. Suppose these are the 1s with indices $a_1 < a_2 < \cdots < a_{2i}$.

3. Replace the string of 1s having indices in $[a_{2j-1}, a_{2j}]$ by $a_{2j} - a_{2j-1} + 1$ for all $j \in [i]$ and circle the remaining 1s.

4. Reduce the value of any uncircled number by 1.

This shows that there are

$$\binom{n+i-2}{2i}$$

circled compositions of n with i uncircled parts and hence proves the result. □

Remark 8.34. Setting $a(n,k) = \#\mathrm{Av}_n[1324, \delta_{k+2}]$ for all $k \geqslant 0$ and $n \geqslant 2$, we have for any $n \geqslant 3$ and $k \geqslant 1$,

$$a(n,k) = a(n-1,k) + \sum_{i=1}^{n-2} a(n-i,k-1)$$

with $a(2,k) = 1$ for $k \geqslant 1$ and $a(n,0) = 0$ for $n \geqslant 2$. This recurrence can be obtained by deleting the last term in A of a circled composition

$$\textcircled{1} \quad A \quad \textcircled{1}$$

that has less than k uncircled parts. The first term on the right-hand side corresponds to the last term being a $\textcircled{1}$ and the second corresponds to it being an uncircled number.

We also note that $a(n,k) = T(n+k-3, 2n-4)$ where T is the triangle of numbers listed in the OEIS [53] as A027926.

Proposition 8.35. *Let X be the circled composition given by*

$$\textcircled{1} \quad k \quad \textcircled{1}$$

for some $k \geqslant 1$. Setting $a(n) = \#\mathrm{Av}_n[1324, \sigma(X)]$ for all $n \geqslant 2$, we have for any $n \geqslant k+1$,

$$a(n) = a(n-1) + \sum_{i=1}^{k-1} a(n-i)$$

with $a(n) = F_{2n-4}$ for $n \in [2,k]$.

Proof. It is clear that $a(n)$ is the number of circled compositions of n of the form

$$\textcircled{1} \quad A \quad \textcircled{1}$$

where any term in A is either $\textcircled{1}$ or some uncircled number in $[k-1]$. Deleting the last term of A gives the required recurrence relation. The initial conditions follow from the fact that $\#\mathrm{Av}_n[1324] = F_{2n-4}$ for $n \geqslant 2$ (see Example 7.19). □

8.2.1 Avoiding [1324] and a pattern of size 5

We now use our results to study avoidance of pairs $[1324, \sigma]$ where $[\sigma] \in \mathrm{Av}_5[1324]$.

The first two results are special cases of Propositions 8.33 and 8.35 respectively.

Corollary 8.36. *We have* $[1324, \sigma]$ *for* $\sigma \in \{12453, 12543, 14532, 15432\}$ *are all Wilf equivalent and for* $n \geqslant 4$,

$$\#\mathrm{Av}_n[1324, 15432] = 1 + \binom{n-1}{2} + \binom{n}{4}.$$

Corollary 8.37. *Setting* $a(n) = \#\mathrm{Av}_n[1324, 15234]$, *we have for* $n \geqslant 4$,

$$a(n) = 2a(n-1) + a(n-2)$$

with $a(2) = 1$ *and* $a(3) = 2$.

Proposition 8.38. *We have for* $n \geqslant 4$,

$$\#\mathrm{Av}_n[1324, 12345] = F_{n+1} - 4 + \sum_{i=0}^{n-4} (n-3-i)F_i.$$

Proof. Note that [12345] is the permutation corresponding to the circled composition given by

$$\textcircled{1}^5.$$

Hence, we have to count the circled compositions of n that avoid $\textcircled{1}^5$. We count them based on the number of $\textcircled{1}$s. Such a circled composition can have at most four $\textcircled{1}$s.

If the circled composition has four $\textcircled{1}$s, it is of the form

$$\textcircled{1} \quad A \quad \textcircled{1} \quad B \quad \textcircled{1} \quad C \quad \textcircled{1}$$

where A and C consist of just 1s and B consists of 1s and 2s. The number of such circled compositions of n is

$$\sum_{i=0}^{n-4} (n-3-i)F_i.$$

This is because the size of B can take values in $[0, n-4]$ and there are $(n-3-i)$ ways to choose the sizes of A and C so that B can have size $i \in [0, n-4]$.

If the circled composition has three ①s, it is of the form

$$① \quad A \quad ① \quad B \quad ①$$

where A and B consist of 1s and 2s and at most one of them can have a 2. The number of such circled compositions of n is

$$(n-2) + 2 \times \sum_{i=2}^{n-3} (F_i - 1).$$

This is because there are $(n-2)$ such circled compositions without a 2 and if A has a 2 and is of size $i \in [2, n-3]$, there are $F_i - 1$ possibilities for A and B should contain just 1s. A similar argument holds if B has a 2.

If the circled composition has only two ①s, it is of the form

$$① \quad A \quad ①$$

where A consists of 1s and 2s or has exactly one 3 and all other terms 1. The number of such circled compositions is

$$(n-4) + F_{n-2}.$$

Combining all these counts, we get the required result, using the fact that for any $k \geqslant 0$,

$$\sum_{i=0}^{k} F_i = F_{k+2} - 1. \tag{8.6}$$

□

Corollary 8.39. *Setting* $a(n) = \#\mathrm{Av}_n[1324, 12345]$, *we have for* $n \geqslant 6$,

$$a(n) = a(n-1) + a(n-2) + (n+1)$$

with $a(4) = 5$ *and* $a(5) = 12$.

Proposition 8.40. *We have* $[1324, 12354] \equiv [1324, 13452]$. *Setting* $a(n) = \#\mathrm{Av}_n[1324, 12354]$, *we have for* $n \geqslant 3$,

$$a(n) = a(n-1) + F_{n+1} - (n+1)$$

with $a(2) = 1$.

Proof. Note that [12354] is the permutation corresponding to the circled composition X given by

$$\textcircled{1}^3 \quad 1 \quad \textcircled{1}.$$

The circled compositions of n that do not dominate X are those which have no uncircled number or are of the form

$$\textcircled{1} \quad A \quad c \quad \textcircled{1}^k \tag{8.7}$$

where $c \geqslant 1$ is the last uncircled number, and A is a sequence such that

1. all uncircled numbers are either 1 or 2,
2. there is at most one $\textcircled{1}$, and
3. there are no $\textcircled{1}$s after an uncircled 2.

Note that $k \geqslant 1$ is fixed once A is fixed.

It is clear that $a(2) = 1$ and that for $n \geqslant 3$, those circled compositions counted by $a(n)$ which have no uncircled numbers or whose last uncircled number c is at least 2 is given by $a(n-1)$ (replace c by $c-1$). We now have to count the circled compositions of n of the form described by (8.7) where $c = 1$.

Suppose A has a $\textcircled{1}$. Then A is of the form

$$B \quad \textcircled{1} \quad C$$

where B consists of just 1s and C consists of 1s and 2s. Hence, the number of circled compositions of n of the form (8.7) where $c = 1$ and A contains a $\textcircled{1}$ is

$$\sum_{i=0}^{n-3} \sum_{j=0}^{i-1} F_j.$$

Here $i \in [0, n-3]$ represents the size of A and $j \in [0, i-1]$ represents the size of C.

Suppose A has no $\textcircled{1}$s. Then, A just consists of 1s and 2s. Hence, the number of circled compositions of n of the form (8.7) where $c = 1$ and A does not contain a $\textcircled{1}$

is

$$\sum_{i=0}^{n-3} F_i.$$

Combining the above counts, we get that the number of circled compositions of n of the form (8.7) where $c = 1$ is

$$\sum_{i=0}^{n-3}\sum_{j=0}^{i-1} F_j + \sum_{i=0}^{n-3} F_i = F_{n+1} - (n+1)$$

where the equality is obtained by repeatedly using (8.6). This gives us the required result. □

Corollary 8.41. *We have for $n \geqslant 2$,*

$$\#\mathrm{Av}_n[1324, 12354] = F_{n+3} - 1 - \binom{n+2}{2}.$$

Proposition 8.42. *We have* $[1324, \sigma]$ *for* $\sigma \in \{12534, 13542, 14523, 15342, 15423\}$ *are all Wilf equivalent and for* $n \geqslant 1$,

$$\#\mathrm{Av}_n[1324, 13542] = 2^{n-1} - (n-1).$$

Proof. Note that $[13542]$ is the permutation corresponding to the circled composition X given by

$$\textcircled{1}\quad 1\quad \textcircled{1}\quad 1\quad \textcircled{1}.$$

The circled compositions of n that do not dominate X are those where all uncircled numbers are consecutive. Such circled compositions can be specified as follows.

1. Consider a sequence of length n consisting of 1s.
2. Select either none or at least two spaces from the $(n-1)$ spaces between the 1s.
3. Circle all 1s before the first selected space and after the last. If no spaces are selected, circle all 1s.
4. Combine all 1s between any two consecutive selected spaces to form an uncircled number.

This shows that there are $2^{n-1} - (n-1)$ circled compositions of n where all uncircled numbers are consecutive. □

From these computations, we get the following result.

Result 8.2. *There are* 5 *Wilf equivalence classes among* $[1324, \sigma]$*-pairs where* $[\sigma] \in \mathrm{Av}_5[1324]$.

This shows that when $[\sigma] \in \mathrm{Av}_5[1324]$, there are no Wilf equivalences other than those given in Theorem 8.32.

8.3 Patterns in Grassmannian permutations: Avoiding $[1432, k]$-pairs

In Section 7.3, we saw that permutation in Av[1432] can be represented using binary words via Grassmannian permutations and their inverses. We now translate pattern avoidance among [1432]-avoiding permutations to the corresponding binary words and use this to study avoidance of $[4, k]$-pairs where the pattern of size 4 is [1432].

From [21, Theorem 1.3], we know that for any $k \geqslant 1$, $\#\mathrm{Av}_n[1432, \iota_k] = 0$ for all $n \geqslant 2k - 2$. However, if $[\sigma] \in \mathrm{Av}_k[1432]$ is not of Type I, then $\#\mathrm{Av}_n[1432, \sigma] \geqslant 1$ for all $n \geqslant 1$ (in particular, $[\iota_n] \in \mathrm{Av}_n[1432, \sigma]$). Hence, we only focus on pairs of the form $[1432, \sigma]$ where $[\sigma] \in \mathrm{Av}[1432]$ is of Type E, G or IG.

Definition 8.43. For a binary word w, the *complement* of w, denoted by w^c, is the binary word obtained by changing all 0s to 1s and vice versa.

Theorem 8.44. *Let* w_1 *be a binary word starting with* 0 *and having at least* 3 *runs. The permutations contained in* $[G(w_1)]$ *are those of the form* $[G(w_2)]$ *where either* w_2 *or* w_2^c *is a subsequence of* w_1.

Proof. Let the length of w_1 be n. Suppose σ is the pattern in $G(w_1)$ formed using the numbers in the set $A \subseteq [n]$.

If the largest number of A is after the descent of $G(w_1)$, then σ is already in the required form (largest number at the end). Also, the numbers before the descent of σ are precisely those elements of A that are before the descent of $G(w_1)$. Hence the binary word corresponding to σ is the subsequence of w_1 corresponding to the numbers in A.

If the largest number of A is before the descent of $G(w_1)$, then the permutation σ has to be rotated to end with the largest number. Call this permutation σ'. It can be checked that this process results in the numbers before the descent of σ' being precisely those elements of A that are after the descent of $G(w_1)$. Hence the binary word corresponding to σ' is the complement of the subsequence of w_1 corresponding to the numbers in A.

Note that even if σ (or σ') were the identity permutation, using the convention of Remark 7.25, the statement of the lemma would still hold. □

Example 8.45. The pattern in $[G(01^3010^21)] = [146782359]$ obtained by choosing the set $A \subseteq [9]$ when

1. $A = \{2, 3, 4, 7, 9\}$ is $[G(01^20^2)]$, which is shown in Figure 8.3, and when
2. $A = \{1, 2, 5, 6, 8\}$ is $[G(0^21^20)]$, which is shown in Figure 8.4.

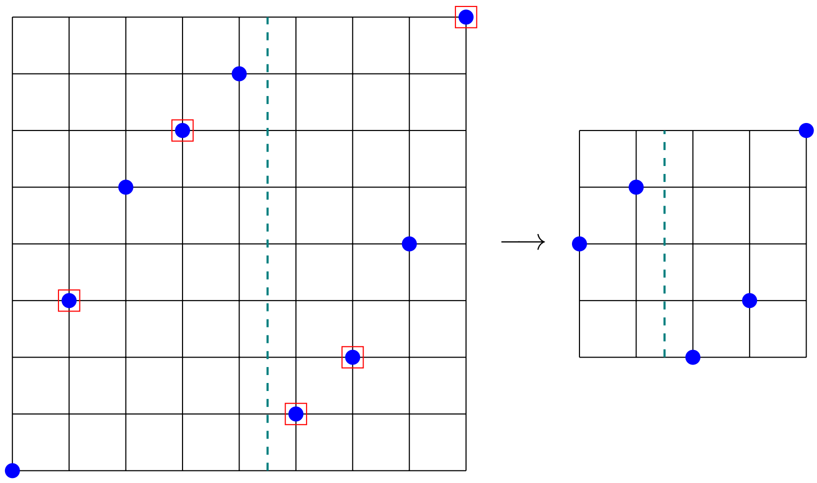


Figure 8.3: The pattern $[G(01^20^2)]$ in $[G(01^3010^21)]$.

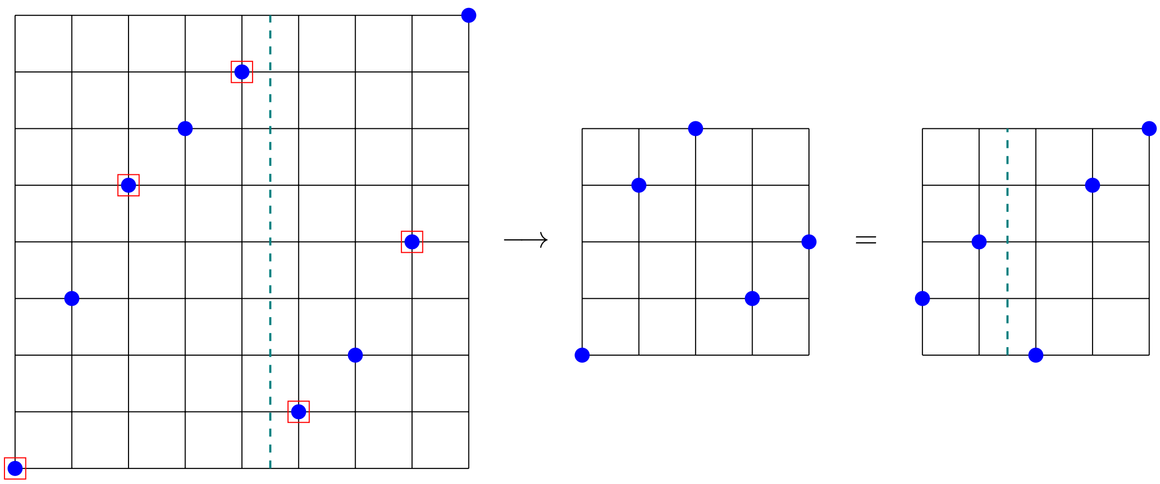


Figure 8.4: The pattern $[G(0^21^20)]$ in $[G(01^3010^21)]$.

Theorem 8.46. *Let w_1 be a binary word starting with 0 and having at least 3 runs. The permutations contained in $[\mathrm{IG}(w_1)]$ are those of the form $[\mathrm{IG}(w_2)]$ where $w_2 = 0^{n_1}1^{n_2}\dots 1^{n_k}0^m$ where all $n_i \geqslant 1$, $m \geqslant 0$, and*

$$1^i 0^{n_1} 1^{n_2} \dots 1^{n_k} 0^{m-i}$$

is a subsequence of w_1 for some $i \in [0, m]$.

Proof. Just as in the proof of Theorem 8.44, we consider the rightmost number (in $\mathrm{IG}(w_1)$) in the set A used to form a pattern in $\mathrm{IG}(w_1)$. If this number is after the descent of the inverse permutation (above the dashed line), then the pattern is already in the required form. As before, it can be checked that the corresponding binary word is the subsequence of w_1 corresponding to numbers used to form the pattern.

If the rightmost number is before the descent of the inverse permutation (below the dashed line), then the pattern has to be cyclically shifted to the required form. Again, it can be checked that the corresponding binary word is the one obtained from the subsequence of w_1 corresponding to A by cyclically shifting the first string of 1s to the end and changing them to 0s. □

Example 8.47. The pattern in $[\mathrm{IG}(01^3010^21)] = [167283459]$ obtained by choosing the set $A \subseteq [9]$ when

1. $A = \{2, 3, 4, 7, 9\}$ is $[\mathrm{IG}(01^30)]$, which is shown in Figure 8.5, and when
2. $A = \{1, 3, 5, 6, 8\}$ is $[\mathrm{IG}(0^210^2)]$, which is shown in Figure 8.6.

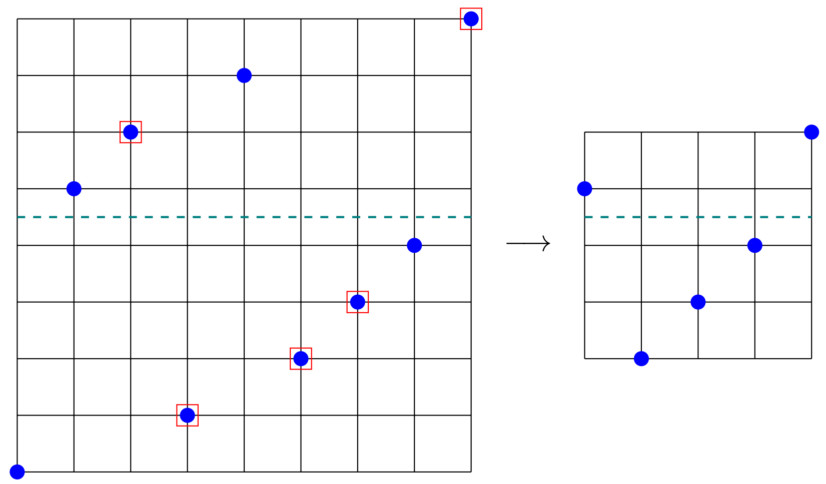


FIGURE 8.5: The pattern $[\mathrm{IG}(01^30)]$ in $[\mathrm{IG}(01^3010^21)]$.

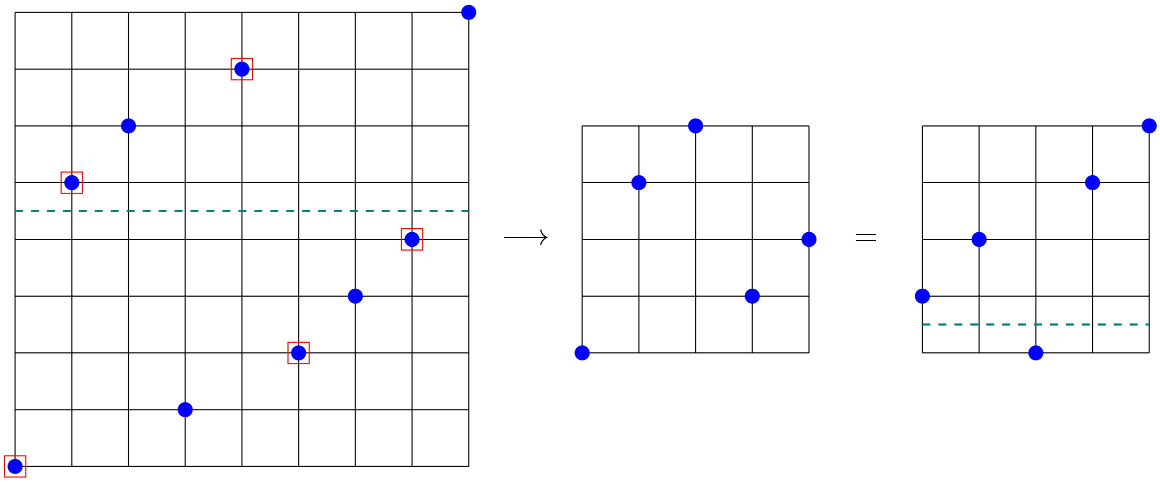


FIGURE 8.6: The pattern $[\mathrm{IG}(0^2 10^2)]$ in $[\mathrm{IG}(01^3 010^2 1)]$.

Using Theorem 8.44 and Theorem 8.46, we get the following result about containment among different types of permutations in Av[1432]. Recall that patterns in identity permutations are again identity permutations.

Corollary 8.48. *We have*

1. *Type I permutations can be found as patterns in Type I, E, G and IG permutations.*
2. *Type E permutations can be found as patterns only in Type E, G and IG permutations.*
3. *Type G permutations can be found as patterns only in Type G permutations.*
4. *Type IG permutations can be found as patterns only in Type IG permutations.*

Example 8.49. From [21, Theorem 2.4], we know that for any $n \geqslant 1$,

$$\#\mathrm{Av}_n[1432, 1324] = 1 + \binom{n-1}{2}.$$

We now prove this using the above description of Av[1432]. Note that [1324] is a Type E permutations and $[\mathrm{G}(0101)] = [\mathrm{IG}(0101)] = [1324]$. It is clear that if w is a binary word starting with 0 and having at least 4 runs, then $[\mathrm{G}(w)]$ and $[\mathrm{IG}(w)]$ will contain [1324]. Using this, we get that the only permutations in Av[1432, 1324] are those of Type I and those of Type E of the form $[\mathrm{G}(w)]$ where w is a binary word starting with 0 having exactly 3 runs. This gives us the required result.

Theorem 8.44 also gives the following relation with [1342]-avoiding permutations.

Corollary 8.50. *If* $[G(v)]$ *and* $[G(w)]$ *are Type G permutations, then* $[1432, G(v)] \equiv [1432, G(w)]$ *if and only if* $[1342, \sigma(v), \sigma(v^c)] \equiv [1342, \sigma(w), \sigma(w^c)]$.

Proof. The result is a consequence of the following facts.

1. The binary words corresponding to Type G permutations are non-exceptional (in the sense of Corollary 8.3).

2. A binary word w_1 contains w_2 or w_2^c as a subsequence if and only if w_1^c contains w_2 or w_2^c as a subsequence.

□

We now turn to Wilf equivalences among $[1432, k]$-pairs. The following lemmas are trivial Wilf equivalences which follow since $[1432^{rc}] = [1432]$. They could also be derived using Theorem 8.44 and Theorem 8.46.

Lemma 8.51. *Let* w *be a binary word starting with* 0 *and having* $k \geqslant 3$ *runs. If run sizes of* w *are* $n_1, n_2, \ldots, n_k$*, then we have* $[1432, G(w)] \equiv [1432, G(w')]$ *where* w' *is the binary word starting with* 0 *having run sizes* $n_k, \ldots, n_2, n_1$.

Lemma 8.52. *For any binary word* w *starting with* 0 *and having at least* 3 *runs, we have* $[1432, IG(w)] \equiv [1432, IG(w')]$ *where*

$$w = 0^{n_1} 1^{n_2} \cdots 1^{n_k} 0^m \text{ and } w' = 0^{n_k} \cdots 0^{n_2} 1^{n_1} 0^m.$$

That is w' *is obtained from* w *by reversing and complementing the portion of the binary word up to the last run consisting of* 1*s.*

We now obtain some non-trivial Wilf equivalences. We use the notation $(01)^k$ for the alternating binary word of length $2k$ starting with 0.

Lemma 8.53. *Let* w *and* w' *be binary words of the same length starting with* 0 *and having at least* 5 *runs. Suppose* w *ends with a* 1*. Then if* w' *either ends with a* 1 *or* $w' = (01)^k 0$ *for some* $k \geqslant 2$ *then* $[1432, IG(w)] \equiv [1432, IG(w')]$.

Proof. Let $w = w_1 w_2 \cdots w_m$ where $w_1 = 0$ and $w_m = 1$. The permutations in $\text{Av}_n[1432]$ that contain $[IG(w)]$ are in bijection with binary words of length n starting with 0 that contain w as a subsequence. Such words are clearly in bijection with binary words of length $(n-1)$ that contain $w_2 \cdots w_m$ as a subsequence. Using the idea in the proof of Theorem 8.5, we know that the number of such binary words only depends on m. Hence, if w' also ends with a 1, we get $[1432, IG(w)] \equiv [1432, IG(w')]$.

The permutations in $Av_n[1432]$ that contain $[IG((01)^k 0)]$ are in bijection with binary words of length n that contain either $(01)^k 0$ or $1(01)^k$ as a subsequence. But any binary word starting with 0 that contains $1(01)^k$ will automatically contain $(01)^k 0$ as well. Hence, the permutations in $Av_n[1432]$ that contain $[IG((01)^k 0)]$ are in bijection with binary words of length n that contain $(01)^k 0$. The result now follows just as before. □

As suggested by the statement of Lemma 8.51, it will be convenient to represent binary words as compositions, at least for Type G. For a composition $(n_1, n_2, n_3, \ldots)$, the corresponding binary word is $B(n_1, n_2, n_3, \ldots) = 0^{n_1} 1^{n_2} 0^{n_3} \cdots$.

Lemma 8.54. *Let* $w = B(1, n_2, \ldots, n_k)$ *be a binary word starting with* 0 *and having* $k \geqslant 4$ *runs with first run of size* 1*. Then we have* $[1432, G(w)] \equiv [1432, G(w')]$ *where* $w' = B(n_2, \ldots, n_k, 1)$.

Proof. When $k \geqslant 5$, the result is a consequence of Theorem 8.44 and the fact that the following statements are equivalent for a given composition $(m_1, m_2, m_3 \ldots, m_p)$.

1. The binary word $B(m_1, m_2, m_3 \ldots, m_p)$ contains $B(1, n_2, \ldots, n_k)$ or $B(1, n_2, \ldots, n_k)^c$ as a subsequence.
2. The binary word $B(m_2, m_3 \ldots, m_p)$ contains $B(n_2, \ldots, n_k)$ or $B(n_2, \ldots, n_k)^c$ as a subsequence.
3. The binary word $B(m_2, m_3 \ldots, m_p, m_1)$ contains $B(n_2, \ldots, n_k, 1)$ or $B(n_2, \ldots, n_k, 1)^c$ as a subsequence.

When $k = 4$, the fact that the number of Type E and Type G permutations in $Av_n[1432, G(w)]$ is the same as the number in $Av_n[1432, G(w)]$ follows just as before. For Type IG, we use the logic of Lemma 8.53 to show that the number of permutations containing $[G(w)]$ is the same as the number of those containing $[G(w')]$. We can do so because

1. any binary word with 4 runs that starts with a 0 must end with a 1, and

2. the number of binary words with 4 runs containing $01^{n_2}0^{n_3}1^{n_4}$ as a subsequence is the same as the number of those containing $0^{n_2}1^{n_3}0^{n_4}1$.

□

Lemma 8.55. *If $w = 0101\cdots$, an alternating binary word, then we have $[1432, G(w)] \equiv [1432, IG(w)]$.*

Proof. If w has length less than 5, then $G(w) = IG(w)$, and we are done. If w has length $k \geqslant 5$, then it can be checked that the permutations in Av[1432] that contain $[G(w)]$ are in bijection with binary words starting with 0 that contain at least k runs. This can be done using the fact that a binary word starting with 0 containing $w^c = 1010\cdots$ must contain $w = 0101\cdots$. A similar argument shows that the permutations in Av[1432] that contain $[IG(w)]$ are in bijection with binary words starting with 0 that contain at least k runs. □

Combining the results above, we get the following result. Note that $<_{\text{lex}}$ is the usual lexicographic ordering. This can be replaced with any convenient total order.

Theorem 8.56. *Any pair $[1432, \sigma]$ is Wilf equivalent to a pair $[1432, \tau]$ where $[\tau]$ has one of the following forms:*

1. *$[G(w)]$ where $w = 0^a1^b0^c$ where $a \geqslant c$.*
2. *$[G(w)]$ where w is an alternating binary word starting with 0 having at least 4 runs.*
3. *$[G(w)]$ where $w = B(n_1, n_2, \ldots, n_k, 1^r)$ has at least 4 runs, $r \geqslant 0$, $n_1, n_k \neq 1$, and $(n_k, \ldots, n_2, n_1) \leqslant_{\text{lex}} (n_1, n_2, \ldots, n_k)$.*
4. *$[IG(w)]$ where $w = 0^{n_1}1^{n_2}\cdots 1^{n_k}0^m$ is not an alternating binary word, has at least 5 runs, $m \geqslant 1$, and $(n_k, \ldots, n_2, n_1) \leqslant_{\text{lex}} (n_1, n_2, \ldots, n_k)$.*

We now compute the sequence $(\#\text{Av}_n[1432, \sigma])_{n\geqslant 1}$ for various $[\sigma] \in \text{Av}[1432]$.

Proposition 8.57. *Let u be the alternating binary word of length $k \geqslant 5$ starting with 0. For any binary word w of length k, starting with 0, having at least 5 runs, and ending with 1, we have $[1432, G(u)] \equiv [1432, IG(u)] \equiv [1432, IG(w)]$ and for $n \geqslant 5$,*

$$\#\text{Av}_n[1432, G(u)] = 2^{n-1} - (n-1) + \sum_{i=4}^{k-2} \binom{n-1}{i}.$$

Proof. The Wilf equivalences are by Lemma 8.53 and Lemma 8.55. All Type I, E and IG permutations avoid $[G(u)]$. Also, by the proof of Lemma 8.55, we can see that the number of Type G permutations in $Av_n[1432]$ that avoid $[G(u)]$ is the number of binary words of length n, starting with 0, and having at least 5 but less than k runs. The number of such words is

$$\sum_{i=4}^{k-2} \binom{n-1}{i}.$$

Using the counts given in Example 7.27, we get the required result. □

Proposition 8.58. *Let* c *be a composition with* $(k+1) \geqslant 5$ *parts. If* k *of the parts are* 1 *and the other is* $m \geqslant 2$*, then setting* $w = B(c)$*, the generating function* $\sum_{n=1}^{\infty} \#Av_n[1432, G(w)]x^n$ *is given by*

$$\frac{3x^3 - 3x^2 + x}{(1-2x)(1-x)^2} + \left(\frac{x}{1-x}\right)^k \left(\sum_{i=0}^{m-2}\sum_{j=0}^{m-1} \binom{i+j}{i} x^{i+j+1}\right) + \sum_{i=5}^{k} \left(\frac{x}{1-x}\right)^i.$$

Proof. By Lemma 8.54, we can assume $c = (1, 1, \ldots, 1, m)$. We already know that all Type I, E and IG permutations avoid $[G(w)]$. Also, for any binary word v with at most k runs, $[G(v)]$ avoids $[G(w)]$. This gives the first and last term of the proposed generating function.

We have to study those binary words v with at least $(k+1)$ runs such that $[G(v)]$ avoids $[G(w)]$. The result follows since any such v has the form $v_1\, v_2$ where v_1 is a binary word with k runs starting with 0 and v_2 is a non-empty binary word starting with 0 (respectively 1) if k is even (respectively odd) and has at most $(m-1)$ 0s and at most $(m-1)$ 1s. □

Proposition 8.59. *Let* c *be a composition with* 4 *parts. If three of the parts are* 1 *and the other is* $m \geqslant 2$*, then setting* $w = B(c)$*, the generating function* $\sum_{n=1}^{\infty} \#Av_n[1432, G(w)]x^n$ *is given by*

$$\left(\frac{x}{1-x}\right) + \left(\frac{x}{1-x}\right)^3 \left(1 + \sum_{j=0}^{m-2}\left[\left(\frac{x}{1-x}\right)^{j+1} - x^{j+1}\right] + \sum_{i=0}^{m-2}\sum_{j=0}^{m-1}\binom{i+j}{i} x^{i+j+1}\right).$$

Proof. By Lemma 8.54, we can assume $w = 0101^m$. Note that $[G(w)]$ is a Type E permutation and $[G(w)] = [IG(w)]$ (see Lemma 7.26). We now compute the contribution of each type to the generating function.

1. Since all Type I permutations avoid $[G(w)]$, Type I contributes

$$\left(\frac{x}{1-x}\right).$$

2. The permutations of Type E and Type G that avoid $[G(w)]$ are in bijection with binary words starting with 0 that have at least 3 runs and contain neither w nor w^c as subsequences. Such binary words either have 3 runs or are of the form

$$0^a \quad 1^b \quad 0^c \quad v$$

where v is a binary word starting with 1 that has at most $(m-1)$ 1s and at most $(m-1)$ 0s. Hence, the contribution of Type E and Type G to the generating function is

$$\left(\frac{x}{1-x}\right)^3 \left(1 + \sum_{i=0}^{m-2} \sum_{j=0}^{m-1} \binom{i+j}{i} x^{i+j+1}\right).$$

3. The permutations of Type IG that avoid $[IG(w)]$ correspond to binary words that start with 0, have at least 5 runs, and do not contain w as a subsequence. Such binary words are of the form

$$0^a \quad 1^b \quad 0^c \quad 1 \quad v$$

where v is a binary word having at most $(m-2)$ 1s and at least one 0. Hence the contribution of Type IG to the generating function is

$$\left(\frac{x}{1-x}\right)^3 \left(\sum_{j=0}^{m-2} \left[\left(\frac{x}{1-x}\right)^{j+1} - x^{j+1}\right]\right).$$

□

Proposition 8.60. *Let* $w = 010^m$ *for some* $m \geqslant 2$. *Then,*

$$\sum_{n=1}^{\infty} \#\mathrm{Av}_n[1432, G(w)]x^n = \left(\frac{x}{1-x}\right) A(x) + \left(\frac{x}{1-x}\right)^2 B(x), \tag{8.8}$$

where

$$A(x) = 1 + \sum_{i=0}^{m-2} \left[x^{m+i} \left[\left(\frac{1}{1-x}\right)^{m-i-1} - 1\right] + \left(\frac{x}{1-x}\right)^{i+1} - x^{i+1}\right]$$

and

$$B(x) = \frac{x(x^m-1)(1-x^{m-1})}{(1-x)^2} + \sum_{i=0}^{m-2}\sum_{j=0}^{m-1}\binom{i+j}{i}x^{i+j+1}.$$

Proof. The first term in (8.8) is the contribution of Type I, E and IG permutations. The Type I permutations contribute

$$\left(\frac{x}{1-x}\right).$$

By Lemma 7.26, we have $[G(w)] = [IG(01^m0)]$. Hence, the Type E and IG permutations correspond to binary words starting with 0 and having at least 3 runs that do not contain 01^m0 or 101^m as subsequences. We consider two cases. Such binary words that do not contain 01^m as a subsequence are of the form

$$0^a\ 1\ v$$

for some $a \geqslant 1$ and a binary word v containing at least one 0 and at most $(m-2)$ 1s. They contribute

$$\left(\frac{x}{1-x}\right)\sum_{i=0}^{m-2}\left[\left(\frac{x}{1-x}\right)^{i+1} - x^{i+1}\right].$$

Such binary words that contain 01^m as a subsequence are of the form

$$0 \quad 0^{a_1} \quad 1 \quad 0^{a_2} \quad 1 \quad \cdots \quad 0^{a_m} \quad 1 \quad 1^i$$

where $a_1, a_2, \ldots, a_m \geqslant 0$ with $a_k \neq 0$ for some $k \geqslant 2$, $i \in [0, m-2]$, and $a_{k+1} = 0$ for $k \in [i]$. Hence, they contribute

$$\sum_{i=0}^{m-2} x^{m+1+i}\left(\frac{1}{1-x}\right)\left[\left(\frac{1}{1-x}\right)^{m-i-1} - 1\right]$$

The second term in (8.8) is the contribution of Type G permutations. These correspond to binary words starting with 0, having at least 5 runs, and not containing 010^m or 101^m as subsequences. These are binary words of the form

$$0^a\ 1^b\ 0\ v$$

where $a, b \geqslant 1$ and v is a binary word with at most $(m-2)$ 0s and $(m-1)$ 1s. Omitting such binary words that have 3 or 4 runs, gives the required result. □

8.3.1 Avoiding $[1432]$ and a pattern of size 5

The first three results are special cases of [21, Theorem 1.3], Proposition 8.57, and Proposition 8.59 respectively.

Corollary 8.61. *We have for* $n \geqslant 8$,

$$\#\mathrm{Av}_n[1432, 12345] = 0.$$

Corollary 8.62. *We have* $[1432, 13524] \equiv [1432, 14253]$ *and for* $n \geqslant 5$,

$$\#\mathrm{Av}_n[1432, 13542] = 2^{n-1} - (n-1).$$

Corollary 8.63. *We have* $[1432, \sigma]$ *for* $\sigma \in \{12435, 13245, 13425, 14235\}$ *are all Wilf equivalent and for* $n \geqslant 4$,

$$\#\mathrm{Av}_n[1432, 13425] = 1 + \binom{n}{3} + \binom{n-3}{2}.$$

Before proving other results, we note the following interesting corollary to [12, Theorem 2], Proposition 8.42, and Corollary 8.62.

Corollary 8.64. *We have* $[1342] \equiv [1324, \sigma] \equiv [1432, 13524] \equiv [1432, 14253]$ *for all* $\sigma \in \{12534, 13542, 14523, 15342, 15423\}$.

Proposition 8.65. *We have for* $n \geqslant 6$,

$$\#\mathrm{Av}_n[1432, 15234] = 11n - 43.$$

Proof. Note that $[15234] = [G(01^30)] = [IG(010^3)]$. We count the permutations in $\mathrm{Av}_n[1432]$ that avoid $[15234]$ by type.

Type E and G permutations avoiding $[15234]$ correspond to binary words starting with 0, having at least 3 runs, and not containing the subsequences 01^30 and 10^31. It can be checked that such a binary word must have at most 6 runs. The following facts about such binary words are easy to verify.

1. Those with 3 runs are of the form $0^a1^b0^c$ for some $a, b, c \geqslant 1$ where $b \leqslant 2$.
2. Those with 4 runs are of the form $0^a1^b0^c1^d$ for some $a, b, c, d \geqslant 1$ where $b, c \leqslant 2$.
3. Those with 5 runs are of the form $0^a10^b10^c$ for some $a, b, c \geqslant 1$ where $b \leqslant 2$.

4. Those with 6 runs are of the form 0^a10101^b for some $a, b \geqslant 1$.

Type IG permutations avoiding [15234] correspond to binary words starting with 0, having at least 5 runs, and not containing the subsequences 010^3, 1010^2, 1^2010 and 1^301. Again, it can be checked that such binary words have at most 6 runs and that the following facts about such binary words are true.

1. Those with 5 runs are of the form 0^a101^b0 for some $a, b \geqslant 1$.
2. Those with 6 runs are of the form 0^a10101^b for some $a, b \geqslant 1$.

Counting such binary words of length n and including the Type I permutation as well, we get the required result. □

The proofs of the following results are similar.

Proposition 8.66. *We have* $[1432, 12534] \equiv [1432, 14523]$ *and for* $n \geqslant 6$,

$$\#\text{Av}_n[1432, 12534] = 8n - 31 + \binom{n-2}{2}.$$

Proposition 8.67. *We have for* $n \geqslant 6$,

$$\#\text{Av}_n[1432, 12453] = 10n - 39 + \binom{n-3}{2}.$$

Proposition 8.68. *We have* $[1432, 12354] \equiv [1432, 13452]$ *and for* $n \geqslant 6$,

$$\#\text{Av}_n[1432, 12354] = 9n - 34 + \binom{n-4}{2} + \binom{n-3}{2}.$$

Proof. This result could also be proved using Proposition 8.60. □

From the above computations, we get the following result.

Result 8.3. *There are 7 Wilf equivalence classes among* $[1432, \sigma]$*-pairs where* $[\sigma] \in \text{Av}_5[1432]$.

This shows that when $[\sigma] \in \text{Av}_5[1432]$, there are no Wilf equivalences other than those given in Theorem 8.56.

Chapter 9

Future directions

In the proofs of results in Section 8.1, we have shown that for subsequence pattern avoidance in binary words, all patterns of a given size are Wilf equivalent. Also, we have shown that there are only trivial Wilf equivalences among pairs of patterns of the form $\{0^{a+1}1^b, 1^{b+1}0^a\}$.

Question 9.1. *What more can be said about pattern avoidance in binary words and what implications do they have for avoidance of* $[1342, k]$ *and* $[1432, k]$*-pairs?*

For example, what can be said about avoiding pairs of the form $\{w_1, w_2\}$ where w_1 and w_2 are non-exceptional binary words? This corresponds to studying $\mathrm{Av}[1342, \sigma(w_1), \sigma(w_2)]$. In the special case when $w_2 = w_1^c$, this corresponds to studying $\mathrm{Av}[1432, G(w_1)]$. Also, is there a simple formula for $\#\mathrm{Av}_n[1342, \sigma(0^{a+1}1^b)]$ for arbitrary $a, b \geqslant 1$ (see Remark 8.13)? We note that in [43, 44], which are both joint work with Anurag Singh, we study patterns in Grassmannian permutations as well as subsequences in binary words.

In Section 8.2, for a circled composition that has at least one uncircled number, we were able to use its left-most occurrence to give a description of the circled compositions that dominate it. This description could also be used to get a generating function for such circled compositions (see Remark 8.27). This method does not seem to work for circled compositions of the form $\textcircled{1}^n$.

Question 9.2. *Is there a general way to describe or count the circled compositions that dominate* $\textcircled{1}^n$ *for* $n \geqslant 1$*?*

This corresponds to studying the avoidance class $\mathrm{Av}[1324, \iota_n]$. Therefore, Corollary 8.39 might be useful if it can be combinatorially proved and generalized.

We have proved various Wilf equivalences among $[1324, k]$-pairs as well as $[1432, k]$-pairs. However, unlike for $[1342, k]$-pairs, we have not shown that these are the only equivalences.

Question 9.3. *Are there any Wilf equivalences among $[1324, k]$-pairs (respectively $[1432, k]$-pairs) other than those described in Theorem 8.32 (respectively Theorem 8.56)?*

The methods we have used made it natural to study Wilf equivalences among $[4, k]$-pairs for which the pattern of size 4 is the same (or trivially Wilf equivalent). This raises the following problem.

Question 9.4. *What can be said about Wilf equivalences among $[4, k]$-pairs where the patterns of size 4 are different?*

From the computations in Sections 8.1.1, 8.2.1 and 8.3.1, we note that there is only one Wilf equivalence among $[4, 5]$-pairs where the patterns of size 4 are not trivially Wilf equivalent, *i.e.*, $[1324, 12534] \equiv [1432, 13542]$ (see Corollary 8.64). Hence, there are 14 Wilf equivalence classes of $[4, 5]$-pairs.

Most of the sequences that enumerate avoidance classes for $[4, 5]$-pairs are available in the OEIS [53]. We list them in Table 9.1 where we specify a representative from a Wilf equivalence class and its corresponding OEIS sequence number. Studying other descriptions for these sequences mentioned in the OEIS might yield interesting combinatorial questions.

Finally, we note that in Examples 7.9, 7.20 and 7.28, we used the combinatorial descriptions for circular permutations avoiding a pattern of size 4 to study cyclic descent generating functions. Similarly, these descriptions might make it easier to obtain enumerative results for other statistics on circular permutations avoiding patterns of size 4 (see [21, Section 5.2]).

[4, 5]-pair	OEIS sequence number
[1342, 12345]	A028387
[1342, 12435]	A050407
[1342, 12354]	A016789
[1324, 12453]	A027927
[1324, 15234]	A000129
[1324, 12345]	A210673
[1324, 12354]	A116717
[1324, 12534]	A000325
[1432, 12435]	A116721
[1432, 15234]	A017401

Table 9.1: OEIS sequences appearing in the pattern avoidance of [4, 5]-pairs.

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