ON THE NOTION OF TOPOLOGICAL COMPLEXITY

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by
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Abstract

Topological complexity of a configuration space X is a homotopy invariant in robotics that measures the minimum number of discontinuities of a continuous motion planning problem. In this thesis, we will define it formally, discuss its properties such as homotopy invariance, and compute it for various spaces. We also discuss notions such as sectional category, Lusternik-Schnirelman category, and other variants of topological complexity that are relevant to our topic at hand.

The notion of sectional category of a fibration, defined by A.S Schwarz in [12], is a very useful concept that generalises the notion of topological complexity. It can be shown that the topological complexity is the sectional category of the fibration $\pi : X^I \longrightarrow X \times X$; $\alpha \mapsto (\alpha(0), \alpha(1))$, where X^I is equipped with compact open topology. We also use the sectional category to define other homotopy invariants, such as the Lusternik-Schnirelman category. The notion of Lusternik-Schnirelman category is very well known and well-investigated on its own right due to the connections it has to the number of critical points of a smooth function from a closed smooth manifold. It is useful for giving bounds for $\mathrm{TC}(X)$ and thus useful in computations.

After computing $\mathrm{TC}(X)$ for some easy examples, we proceed to discuss a problem that is of great practical interest: finding the complexity of collision-free motion planning of n point-like objects in $\mathbb{R}^k$. We consider two cases: the case where there are no obstacles and the case when there are m point-like obstacles, that are possibly moving.

In the last two chapters, we briefly give a survey of some similar notions such as higher topological complexity, and other variants of topological complexity, such as monoidal topological complexity and symmetric topological complexity.

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Contents

Introduction

Consider a mechanical system with a configuration space X. The set X has the structure of a topological space, that formalises how similar two configurations are. As an example, we consider the system of a rigid rod of length r, fixed to a point O in $\mathbb{R}^{k+1}$, free to rotate around the point. See Fig. 1.1. Since, the position of the other end of the rod completely determines the configuration, and this can take any point in the sphere of radius r centred at O, it makes sense to consider the configuration space as the k-sphere, S^k. Given a pair of initial and final configurations, A and B of the system, we can now ask how to take the system from A to B, in a continuous way. This is equivalent to finding a continuous path in X from A to B.

Motion planning problem deals with finding an algorithm that takes in two points A and B and gives us a path joining A to B in such a way, that the algorithm depends as continuously as possible, on the pair (A, B). Topological complexity of X, $\mathrm{TC}(X)$, is a measure of the minimal number of discontinuities such an algorithm should have. It turns out that $\mathrm{TC}(X)$ is a homotopy invariant of X.

The notion of topological complexity was defined by Farber in his paper [2]. Computation of $\mathrm{TC}(X)$ for various spaces have been an active area of research in the following years. Notions similar to $\mathrm{TC}(X)$ have also been investigated extensively.

In this thesis, we introduce topological complexity, discuss its basic properties and its relevance to the motion planning problem. We also give an alternate definition for it, which agrees with the original version on smooth manifolds.

Schwarz genus, also known as the sectional category of fibration, was introduced by A S Schwarz in 1961 in the paper [12]. This turns out to be very useful in the theory of $\mathrm{TC}(X)$, as $\mathrm{TC}(X)$ itself is the sectional category of a specific fibration, known as the path fibration. This, and similar notions such as Lusternik-Schnirelman category of a space are discussed in Chapter 2.

We also discuss the problem of computing the complexity of collision-free motion planning of n point-like particles avoiding collisions with themselves and m obstacles, and its connection to $\mathrm{TC}(X)$. The $m = 0$ case for this problem was investigated by Farber, Grant and Yuzvinsky, in a 2006 paper [6] and was completed by Farber and Grant in [4]. The general m case was completed in [5]. We discuss the collision-free motion problem in detail in Chapter 4.

Other notions similar to $\mathrm{TC}(X)$ are discussed. These include higher topological complexity, which generalises $\mathrm{TC}(X)$ and talks about the complexity of finding a motion planning algorithm with the system travelling through specific configurations in between the motion of the system; symmetric topological complexity, which attempts to treat the starting and the ending points of the motion of the system on an equal footing; and other notions such as monoidal topological complexity. We talk about higher topological complexity in Chapter 5, and briefly give some results for monoidal topological complexity and symmetric topological complexity in Chapter 6.

Chapterwise organisation

Chapter 1 The motion planning algorithm and some examples of configuration spaces are given in Section 1.1. We proceed on to define the topological complexity in Section 1.2, and give an alternate version of topological complexity, using Euclidean Neighborhood retracts, in Section 1.3.

Chapter 2 In this chapter, we discuss the notion of sectional category. After giving the necessary language for it, we define the sectional category in Section 2.1. The properties of sectional category like the product inequality, and the invariance under fiber homotopy equivalence, are discussed in Section 2.2. We also prove these for the particular case of topological complexity as the sectional category of the path fibration. Finally, in Section 2.3, we define the Lusternik-Schnirelman category and give some of its properties and bounds.

Chapter 3 In this chapter, we focus on the computations of topological complexity. First, in Section 3.1, we give some bounds for topological complexity, that helps us in our computations. These include the well-known dimension bound, dimension-connectivity bounds and the cohomological lower bound. After this, we compute the topological complexity of some well-known spaces in Section 3.2.

Chapter 4 Here, we discuss the problem of collision-free motion planning of n point-like objects in $\mathbb{R}^k$. For this, in Section 4.1, we discuss the configuration space of the system, and compute the cohomology groups of the related spaces in Section 4.2. We deal with the problem for the case with no obstacles in Section 4.3, and the case with m possibly-moving obstacles in Section 4.4.

Chapter 5 We give a brief survey of results about higher topological complexity, $\mathrm{TC}_n(X)$, here. In Section 5.1, we define the higher topological complexity, and give its practical interpretation for robotics. The Section 5.2 states some of the properties and bounds of higher topological complexity. We also mention some results in computing $\mathrm{TC}_n(X)$ in Section 5.3.

Chapter 6 We discuss briefly some of the other variants of $\mathrm{TC}(X)$ that are of interest, in the motion planning problem. Specifically, we discuss the monoidal topological complexity in Section 6.1, and the symmetric topological complexity in Section 6.2.

Chapter 1

Preliminaries

In this chapter, we begin with considering the motion planning problem of a robotic system, and giving a mathematical model for it. After giving some examples of interesting mechanical systems and their configuration spaces, we proceed to understand the importance of a notion of complexity for motion planning, and define the topological complexity, $\mathrm{TC}(X)$, formally. We also give another similar but distinct definition for the notion of topological complexity, and give a result about the equivalence of the two definitions on smooth manifolds.

1.1 The motion planning problem

Consider a mechanical system with all the possible configurations of the system already known. We assume that the possible configurations of the system can be modelled as a topological space, called the configuration space, X. In fact, throughout this thesis, we assume that X is a CW complex, unless otherwise mentioned.

Example 1.1. Consider a rigid rod, say, of length r with its one end fixed, to a point in $\mathbb{R}^{k+1}$ such that it is free to rotate around the point, in the space $\mathbb{R}^{k+1}$. We note that the configurations of this space can be given by the coordinate of the second end, and this forms a k-sphere of radius r, around the fixed end. Thus, here, the configuration space can be modelled as S^k. See *Fig*. 1.1.

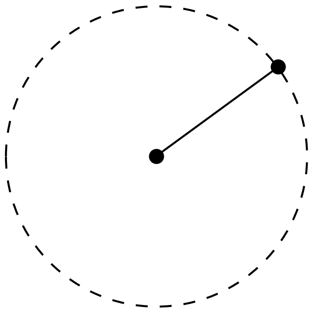

Figure 1.1: Rigid rod in $\mathbb{R}^2$ with one end fixed

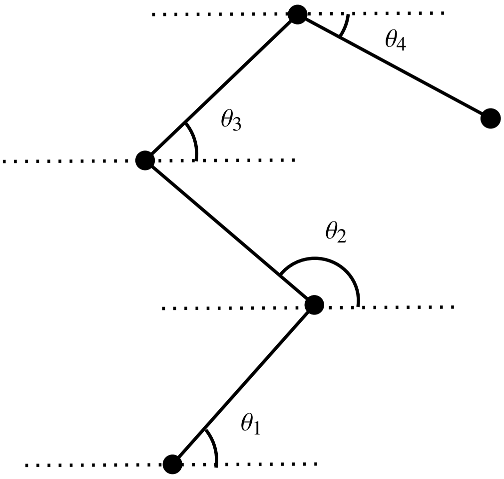

Figure 1.2: Mechanical linkage in $\mathbb{R}^2$ with 4 rods.

Example 1.2. Let us generalise *Example* 1.1, by considering a mechanical linkage with n rods attached one-by-one to the former one's end, whereas the first one is fixed to a fixed point in Euclidean space $\mathbb{R}^{k+1}$. Each rod is allowed to rotate it in space, around its initial point. We simplify our model, by allowing self-intersections of the rods with each other. See Figure Fig. 1.2.

This gives us a system with a configuration space $S^k \times \cdots \times S^k$ (n times).

Example 1.3. Consider a system with n point-like objects in $\mathbb{R}^k$, that tries to avoid collision with a single, point-like obstacle O. Note that the obejcts are allowed to collide with each other (i.e., occupy the same point in space) and move through each other. This is a system where each of the n objects can take any value in $\mathbb{R}^k \backslash \{O\}$. See Fig. 1.3. The configuration space is $(\mathbb{R}^k \backslash \{O\}) \times \cdots \times (\mathbb{R}^k \backslash \{O\})$.

Example 1.4. Consider a rigid rod in $\mathbb{R}^{k+1}$, with its center fixed at a specific point $O \in \mathbb{R}^{k+1}$. It is free to rotate in $\mathbb{R}^{k+1}$, and two configurations are considered equivalent, as long as the rod is pointing in the same direction, irrespective of its orientation. (By this we mean that if the tip of the rod is pointing in the direction of vector $v \in S^k$, or in the direction of $-v$, both are actually considered to be the same configuration.) See Fig. 1.4.

The configuration space of this system is $\mathbb{R}\mathrm{P}^k$, the real projective space of dimension k.

Finding a movement of the system from configuration A to B is modelled as finding a path from point A to point B, in our configuration space. Trying to find such a path from point A to B in X, in a way that the path depends continuously on the pair (A, B), is what the motion planning problem attempts to do.

We now give the relevant mathematical terminology associated to this.

Definition 1.5 (Path fibration)**.** For a topological space X, the path fibration is the map $\pi : X^I \longrightarrow X \times X$; $p \mapsto (p(0), p(1))$, where $I = [0, 1]$ and X^I is equipped with the compact-open topology.

Definition 1.6. A continuous section of the path fibration π, is called a continuous motion planning algorithm on X.

We now proceed on to see how the notion of topological complexity comes into the picture, in the next section.

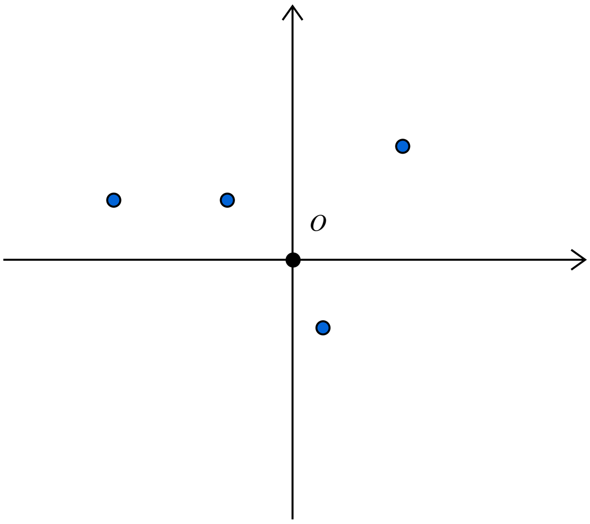

Figure 1.3: The system of 4 point-like objects (blue) in $\mathbb{R}^2$, that tries to avoid collision with the point-like obstacle O (black).

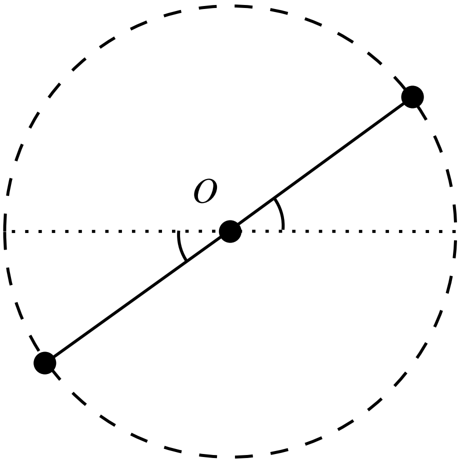

Figure 1.4: Rod in $\mathbb{R}^2$, free to rotate around its centre, without caring about its orientation.

1.2 The definition of TC(X)

Here, we will see that a continuous motion planning algorithm on X is possible only for very limited cases of X, namely the case when X is contractlible. This forces us to consider sections of π that are not globally continuous, however with minimal possible number of discontinuities. This minimal number, defined formally, is called the topological complexity.

We first begin with showing the following theorem:

Theorem 1.7. *A globally continuous section of π is possible if and only if X is contractible.*

Proof. ($\Rightarrow$): Suppose a globally continuous section $s : X \times X \longrightarrow X^I$ of π exists. Fix $c \in X$.

Consider the map $S_c : X \times I \longrightarrow X; (\alpha, s) \mapsto s(\alpha, c)(s)$. S_c turns out to be a continuous map (From properties of compact-open topology [7]), and gives us a homotopy in X between the maps id_X and the constant function c, as

$$S_c(x, 0) = x; \quad S_c(x, 1) = c \quad \forall x \in X$$

proving that X is contractible.

($\Leftarrow$): Suppose S is a homotopy in X between id_X and c, a constant map such that

$$S(x, 0) = x; \quad S(x, 1) = c \quad \forall x \in X.$$

For each $\alpha \in X$, let h_α be the path obtained from S, joining α to S; i.e.,

$$h_\alpha(t) := S(\alpha, t).$$

Now, given $(\alpha, \beta) \in X \times X$, we can consider the path $h_\beta * \overline{h_\alpha}$, the path obtained by concatenating h_β with $\overline{h_\alpha}$, the homotopy inverse path of h_α. It can be seen that the map

$$s : X \times X \longrightarrow X^I; \quad (\alpha, \beta) \mapsto h_\beta * \overline{h_\alpha}$$

is continuous, and thus gives us a continuous section of π. □

Thus, we can see that except in very limited cases, it is impossible to obtain a continuous motion planning algorithm on X. We then attempt to do the next best thing: we divide $X \times X$ to open patches each of which contains a local section of π. However, we would like to do this in a way such that only the minimum number of local patches are being used, so that the number of discontinuities are as less as possible. Topological complexity represents this minimal number of local patches required.

Definition 1.8 (Topological complexity)**.** Consider the space X and the path fibration $\pi : X^I \longrightarrow X \times X$; $p \mapsto (p(0), p(1))$. The minimal number k such that there are open subsets $U_0, U_1, \cdots U_k$ covering $X \times X$, with continuous sections $s_i : U_i \longrightarrow X^I$ of π on each U_i, is called the topological complexity of X. It is denoted by $\mathrm{TC}(X)$.

Corollary 1.9. *We have,* $\mathrm{TC}(X) = 0$ *if and only if X is contractible.*

Proof. Immediate from $Theorem$ 1.7 and the definiton of $\mathrm{TC}(X)$. □

Remark 1.10. We note that $\mathrm{TC}(X)$ we defined here is one less than the value of $\mathrm{TC}(X)$ defined in Farber's paper [2]. This is an intentional choice and we follow the notation from [11], as this makes our equations look simpler. This version is also sometimes called the 'reduced' version of topological complexity, to distinguish it from the [2] version.

There is another way to define the notion of complexity of motion planning, and it turns out that this agrees with the value of $\mathrm{TC}(X)$, atleast when X is a smooth manifold. We discuss this in the next section.

1.3 The ENR Version

In defining $\mathrm{TC}(X)$, we had tried to cover $X \times X$, by open sets with sections on them. However, in doing so, there was some ambiguity in defining a global (possibly discontinuous) section, as two open patches U_i and U_j can intersect and both the local sections s_i and s_j exist on the intersection. To avoid this ambiguity, we try to partition $X \times X$ into disjoint collection of nice enough subsets, each of which has a continuous section of π on it. This gives rise to an alternate version of topological complexity.

We start by precisely defining what we mean by the 'nice enough' subsets:

Definition 1.11 (Euclidean Neighborhood retracts)**.** A Euclidean Neighborhood retract (ENR) is a topological space X, homeomorphic to a subset Y of a Euclidean space $\mathbb{R}^n$, such that there exists an open neighborhood U in $\mathbb{R}^n$ containing Y where Y is a retract of U.

Now, we proceed on to define this alternate notion of topological complexity.

Definition 1.12. $\mathrm{TC}_{ENR}(X)$ is defined as the minimum k such that there are disjoint ENRs $E_0, E_1, \cdots E_k$ covering $X \times X$, with continuous sections $s_i : E_i \longrightarrow X^I$ of π on each E_i.

We state the equivalence of the two notions on smooth manifolds. We do not give a proof here. Interested reader can refer [3] for the proof.

Theorem 1.13 (Theorem 6.1(1), [3])**.** *Suppose X is a connected C^∞ smooth manifold. Then*

$$\mathrm{TC}_{ENR}(X) = \mathrm{TC}(X).$$

Since $E_0, E_1, \cdots E_k$ are disjoint subsets that cover $X \times X$, using ENRs instead of open subsets $U_0, U_1, \cdots U_k$ as in the definition of $\mathrm{TC}(X)$, removes the ambiguity we had about having a global (possibly discontinuous) section.

Chapter 2

The Sectional category

In this chapter, we introduce the sectional category, or Schwarz genus, of a fibration, after setting up the necessary language for it. Introduced by A S Schwarz in his paper [12], this is a very useful notion that can be considered as a generalisation of $\mathrm{TC}(X)$.

After defining it, and showing its basic properties such as invariance under homotopy equivalence and the product inequality, we proceed to define a similar notion, Lusternik-Schnirelman category. We also discuss the relations of these concepts with $\mathrm{TC}(X)$.

2.1 The definition of the sectional category

We first define the notion of a fibration.

Let $p : E \longrightarrow B$ be a continuous map, and Y a topological space. Also, let $i_0 : Y \longrightarrow Y \times I$ be the map $y \mapsto (y, 0)$.

Definition 2.1 (Homotopy Lifting Property)**.** The pair (Y, p) is said to satisfy homotopy lifting property (HLP) if for any homotopy $h : Y \times I \longrightarrow B$ and map $f : Y \longrightarrow E$ such that $p \circ f = h \circ i_0$, it there exists a map $\tilde{h} : Y \times I \longrightarrow E$ such that the following diagram commutes:

$$\begin{tikzcd}
Y \arrow[r, "f"] \arrow[d, "i_0"'] & E \arrow[d, "p"] \\
Y \times I \arrow[r, "h"'] \arrow[ur, dashed, "\tilde{h}"] & B
\end{tikzcd}$$

Definition 2.2 (Fibration)**.** A continuous map $p : E \longrightarrow B$ is called a fibration if for any topological space Y, the pair (Y, p) satisfies HLP.

Remark 2.3. This version of fibration is called a Hurewicz fibration. There is a more general version of fibration where HLP needs to be satisfied only for $Y = I^n$, i.e., n dimensional cubes. This is called a Serre fibration. However, when we talk about fibrations, unless otherwise mentioned, we mean Hurewicz fibration.

For a fibration $p : E \longrightarrow B$ and a continuous function $g : X \longrightarrow B$, let

$$X \times_g E := \left\{ (x, e) \in X \times E \,\middle|\, g(x) = p(e) \right\}$$

with the subspace topology from $X \times E$. We also sometimes denote this space by g^*E.

Definition 2.4 (Pullback fibration, [9])**.** If $p : E \longrightarrow B$ is a fibration and $g : X \longrightarrow B$ is any continuous map, then the projection map $pr_1 : X \times_g E \longrightarrow A$ is a fibration. This is called the Pullback fibration.

Theorem 2.5. *If $p : E \longrightarrow B$, $p' : E' \longrightarrow X$ are fibrations and $g : X \longrightarrow B$ and $f : E \longrightarrow E'$ are continuous maps such that $p \circ f = g \circ p'$, then there exists a continuous map $h : E' \longrightarrow X \times_g E$ such that the following diagram commutes:*

$$\begin{tikzcd}
E' \arrow[rr, "f"] \arrow[dr, dashed, "h"] \arrow[dd, "p'"'] & & E \arrow[dd, "p"] \\
& X \times_g E \arrow[ur, "pr_2"'] \arrow[dl, "pr_1"] & \\
X \arrow[rr, "g"] & & B
\end{tikzcd}$$

For more details on fibrations we can refer to [9].

We now define the concept of fiber-homotopy equivalence. This will turn out to be a very useful concept in understanding when two fibrations have the same sectional category.

Definition 2.6 (Fiber-homotopy equivalence)**.** Suppose $q : D \longrightarrow B$ and $p : E \longrightarrow B$ are continuous maps. A fiber homotopy equivalence is a map $f : D \longrightarrow E$, that has a homotopy inverse $g : E \longrightarrow D$ along a homotopy that satisfies that it is a map over B at each time t.

That is, if $H : E \times I \longrightarrow E$ is the homotopy that takes $f \circ g$ to id_E, and $h_t : E \longrightarrow E$ represents the homotopy H at time t, then the following diagram commutes for each $t \in [0, 1]$.

$$\begin{tikzcd}
E \arrow[rr, "h_t"] \arrow[dr, "p"'] & & E \arrow[dl, "p"] \\
& B &
\end{tikzcd}$$

Similarly for the homotopy $G : D \times I \longrightarrow D$ from $g \circ f$ to id_D also.

Now, we are in a position to define the sectional category and discuss its properties:

Definition 2.7 (Sectional category)**.** Consider a fibration $p : E \longrightarrow B$. The sectional category or Schwarz genus of p is defined as the minimum k such that there are open subsets $U_0, U_1, \cdots U_k$ covering B, with continuous sections $s_i : U_i \longrightarrow E$ of p on each U_i. It is denoted by $\mathrm{secat}(p)$.

Remark 2.8. Recall that the path fibration $\pi : X^I \longrightarrow X \times X$ is the map $p \mapsto (p(0), p(1))$. We note that $\mathrm{TC}(X)$, defined as the minimal number of open sets $U_0, U_1, \cdots U_k$ covering $X \times X$, with continuous sections $s_i : U_i \longrightarrow X^I$ of π on each U_i, is equal to $\mathrm{secat}(\pi)$.

2.2 Properties of the sectional category

Here, we continue with our discussion of sectional category and its properties. We first give a product inequality for the sectional category, and then use this to prove the product inequality for $\mathrm{TC}(X)$.

Definition 2.9. If $p : E \longrightarrow B$ and $p' : E' \longrightarrow B'$ are two fibrations , then their product is the map $p \times p' : E \times E' \longrightarrow B \times B'; (e, e') \mapsto (p(e), p'(e'))$.

Theorem 2.10. *Given two fibrations p and p',*

$$\mathrm{secat}(p \times p') \leq \mathrm{secat}(p) + \mathrm{secat}(p')$$

Proof. See [12]. □

Corollary 2.11 (Product inequality)**.** *We have,*

$$\mathrm{TC}(X \times Y) \leq \mathrm{TC}(X) + \mathrm{TC}(Y) \tag{2.1}$$

Proof. We recall the path fibration $\pi_X : X^I \longrightarrow X \times X; p \mapsto (p(0), p(1))$. We note that $\pi_{X\times Y} = \pi_X \times \pi_Y$, and hence we have the result. □

We would like to discuss how the sectional category behaves under homotopy invariance. As a first step to this, we compare the sectional categories of a fibration and its pullback fibration.

Theorem 2.12. *Suppose $\xi = \{p : E \longrightarrow B\}$ be a fibration, and $f : X \longrightarrow B$ be a continuous map. We can consider the pullback f^*E and obtain a fibration $pr_1 : f^*E \longrightarrow B$ such that the following diagram commutes :*

$$\begin{tikzcd}
f^*E \arrow[r, "pr_2"] \arrow[d, "pr_1"'] & E \arrow[d, "p"] \\
X \arrow[r, "f"'] & B
\end{tikzcd}$$

We have,

$$\mathrm{secat}(f^*\xi) \leq \mathrm{secat}(\xi).$$

Proof. Suppose $U_0, U_1, \cdots, U_n$ (where $n := \mathrm{secat}(p)$) is a collection of open sets on B, with a local section $s_i : U_i \longrightarrow E$, of p, on each U_i. We try to construct local sections of pr_1 on $f^{-1}(U_i)$s, so that we can conclude $\mathrm{secat}(pr_1) \leq n = \mathrm{secat}(p)$.

For this, for i fixed, consider the map

$$t_i := (s_i \circ f, id_X) : f^{-1}(U_i) \longrightarrow E \times X$$

Since $f^*E := \left\{(x, e) \in E \times X \,\middle|\, p(e) = f(x)\right\}$ and $p \circ s_i \circ f = f \circ id_X$, we have that image of t_i is in f^*E. Since it is clear that $pr_1 \circ t_i = id_{f^{-1}(U_i)}$, we have t_i as a local section of pr_1 on $f^{-1}(U_i)$. □

Theorem 2.13. *Suppose there are two fibrations* $p : E \longrightarrow B$ *and* $p' : E' \longrightarrow B$ *such that the following diagram commutes:*

$$\begin{tikzcd} E \arrow[r, "f"] \arrow[d, "p"] & E' \arrow[d, "p'"] \\ B \arrow[r, equal] & B \end{tikzcd}$$

Then, we have $\operatorname{secat}(p') \leq \operatorname{secat}(p)$. *Also if* f *is a fiber homotopy equivalence, then* $\operatorname{secat}(p) = \operatorname{secat}(p')$.

Proof. Suppose $U_0, U_1, \cdots, U_n$ (where $n := \operatorname{secat}(p)$) is a collection of open sets on B, with a local section $s_i : U_i \longrightarrow E$, of p, on each U_i. We try to construct local sections of p' on U_is, so that we can conclude $\operatorname{secat}(p') \leq n = \operatorname{secat}(p)$.

For this consider the maps $t_i := f \circ s_i : B \longrightarrow E'$. We know, from the diagram *Theorem* 2.13, that :

$$\begin{aligned} p' \circ t_i &= p' \circ f \circ s_i \\ &= p \circ s_i \\ &= id_{U_i} \end{aligned}$$

showing that t_is form local sections of p' on U_i s.

If f is a fiber homotopy equivalence, we just consider a fiber homotopy inverse of f, say $h : E' \longrightarrow E$, in place of f and use a similar argument to get $\operatorname{secat}(p) \leq \operatorname{secat}(p')$. □

Theorem 2.14 (Homotopy invariance)**.** *If two fibrations* $p : E \longrightarrow B$ *and* $p' : E' \longrightarrow B'$ *are such that the following diagram commutes:*

$$\begin{tikzcd} E \arrow[r, "f"] \arrow[d, "p"'] & E' \arrow[d, "p'"] \\ B \arrow[r, "g"'] & B' \end{tikzcd}$$

Suppose f *is a fiber homotopy equivalence and* g *is a homotopy equivalence, then* $\operatorname{secat}(p) = \operatorname{secat}(p')$.

Proof. Using *Theorem* 2.5, the above diagram induces a new map from E to g^*E' such that the following diagram commutes:

$$\begin{tikzcd}
E \arrow[rd, dashed, "q"] \arrow[rrd, "f"] \arrow[rdd, "p"'] & & \\
& E'\times_{B'} B \arrow[r, "pr_2"] \arrow[d, "pr_1"] & E' \arrow[d, "p'"] \\
& B \arrow[r, "g"'] & B'
\end{tikzcd}$$

Now the commuting square

$$\begin{tikzcd}
g^*E' \arrow[r, "pr_2"] \arrow[d, "pr_1"] & E' \arrow[d, "p'"] \\
B \arrow[r, "g"'] & B'
\end{tikzcd}$$

gives us that $\mathrm{secat}(pr_1) \leq \mathrm{secat}(p')$, using *Theorem* 2.12; whereas the commuting square:

$$\begin{tikzcd}
E \arrow[r, "q"] \arrow[d, "p"] & g^*E' \arrow[d, "pr_1"] \\
B \arrow[r, equal] & B
\end{tikzcd}$$

gives us that $\mathrm{secat}(p) = \mathrm{secat}(pr_1)$, using *Theorem* 2.13. Thus, $\mathrm{secat}(p) \leq \mathrm{secat}(p')$. Proceeding similarly with their respective homotopy inverses, we also get that $\mathrm{secat}(p') \leq \mathrm{secat}(p)$, proving that

$$\mathrm{secat}(p) = \mathrm{secat}(p').$$

□

Corollary 2.15. *We have that* $\mathrm{TC}(X)$ *is invariant under homotopy equivalence.*

Proof. Suppose X, Y are homotopy equivalent spaces with $f : X \longrightarrow Y$ and $g : Y \longrightarrow X$ such that $g \circ f \simeq id_X$ and $f \circ g \simeq id_Y$. Consider the diagram

$$\begin{tikzcd}
X^I \arrow[r, "f_*"] \arrow[d, "\pi_X"'] & Y^I \arrow[d, "\pi_Y"] \\
X \times X \arrow[r, "f\times f"] & Y \times Y
\end{tikzcd}$$

where f_* is the map $\alpha \mapsto f \circ \alpha$, $f \times f$ is the map $(\alpha,\beta) \mapsto (f(\alpha), f(\beta))$ and π_X and π_Y are the corresponding path fibrations for X and Y respectively. We can see that both of them are homotopy equivalences. (Compositions with g_* and $g \times g$ gives maps homotopic to corresponding identity maps.)

It can also be proved that f_* is a fiber-homotopy equivalence [9]. Thus, using *Theorem* 2.14, we get that $\mathrm{secat}(\pi_X) = \mathrm{secat}(\pi_Y)$ and thus we get,

$$\mathrm{TC}(X) = \mathrm{TC}(Y).$$

□

Now that we have discussed the sectional category in some detail, and have mentioned how $\mathrm{TC}(X)$ is a specific example of a sectional category, we proceed to give another relevant example in the next section.

2.3 The Lusternik-Schnirelman category

In this section, we briefly discuss a similar notion, Lusternik-Schnirelman category or LS category. This is a similar concept to that of $\mathrm{TC}(X)$ that is useful in its own right, mainly due to the connections it has with the number of critical points of a smooth function $f : M \longrightarrow \mathbb{R}$ on a closed smooth manifold M.

However, here we use this concept to give some bounds and inequalities for $\mathrm{TC}(X)$, which will turn out to be useful for us for $\mathrm{TC}(X)$ computations.

Fix $x_0 \in X$. Consider the the space

$$P(X, x_0) = \Big\{ \alpha \in X^I \,|\, p(\alpha) = \alpha(0) \Big\}.$$

The map $p : PX \longrightarrow X$; $\alpha \mapsto \alpha(0)$ is a fibration.

Definition 2.16 (Lusternik-Schnirelman category)**.** Lusternik-Schnirelman category of X is the sectional category of the fibration p defined above. We denote it by $\mathrm{cat}(X)$.

Remark 2.17. An equivalent way to define Lusternik-Schnirelman category is, it is the minimal integer k such that there exist open sets $U_0, \cdots, U_k$ that cover X, such that the inclusion maps $U_i \longrightarrow X$ are each null-homotopic.

Theorem 2.18. *(Homotopy invariance) The Lusternik-Schnirelman category,* $\mathrm{cat}(X)$*, is a homotopy invariant of X.*

Proof. Similar to the proof of *Corollary* 2.15. □

Here, we briefly give some inequalities that involve the notions of $\mathrm{TC}(X)$, $\mathrm{cat}(X)$ and the sectional category of a fibration p.

Theorem 2.19 (Theorem 7.1, [11])**.** *For any fibration $p : E \longrightarrow B$, we have,*

$$\mathrm{secat}(p) \leq \mathrm{cat}(B).$$

Theorem 2.20 (Theorem 5, [2])**.** *If X is a path-connected CW complex, then*

$$\mathrm{cat}(X) \leq \mathrm{TC}(X) \leq 2\mathrm{cat}(X).$$

Definition 2.21. A closed subset X of Y is called a neighborhood retract of Y, if X is a retract of some open neighborhood that contains X.

Definition 2.22. A metrizable space X is called an absolute neighbourhood retract if for any metrizable space Y such that X is a closed subset of Y, we have that X is a neighborhood retract of Y.

Theorem 2.23 (Theorem 7.7, [11])**.** *If X, Y are two absolute neighbourhood retracts, then we have,*

$$\max\left\{\mathrm{TC}(X), \mathrm{TC}(Y), \mathrm{cat}(X \times Y)\right\} \leq \mathrm{TC}(X \vee Y) \leq \mathrm{TC}(X) + \mathrm{TC}(Y) + 3.$$

Chapter 3

Some computations

In this chapter, we try to see how to compute the $\mathrm{TC}(X)$ for some interesting spaces X. For this, in the first section, we give some of the known upper bounds and lower bounds for $\mathrm{TC}(X)$. Second section involves some explicit computations of $\mathrm{TC}(X)$ using these bounds.

3.1 Bounds for topological complexity

Since $\mathrm{TC}(X)$ is a homotopy invariant of X that takes values in non-negative integers, getting sharp enough upper and lower bounds is a way to compute it. Here, we give some upper bounds of $\mathrm{TC}(X)$ in terms of dimension and connectivity (*Theorem* 3.1 and *Theorem* 3.2). We restrict our attention to spaces homotopy equivalent to a path-connected CW complex.

Cohomology ring of a space is a very useful homotopy invariant of a topological space, that helps us give a lot of information about its topology. It turns out that this is very useful in giving a good lower bound for $\mathrm{TC}(X)$ (*Theorem* 3.7).

3.1.1 The dimension bound

Theorem 3.1. *For a path-connected CW complex X of dimension* $\dim X$*, we have,*

$$\mathrm{TC}(X) \leq 2\dim X.$$

Proof. It is known that $\mathrm{cat}(X) \leq \dim(X)$. (See [8].) This, along with the right bound in *Theorem* 2.20 gives us the result. □

3.1.2 The dimension-connectivity bound

Theorem 3.2 (Theorem 5.2, [3])**.** *If X is an r-connected CW polyhedron of dimension* $\dim X$*, then*

$$\mathrm{TC}(X) < \frac{2\dim X + 1}{r+1}.$$

Corollary 3.3. *If X is a simply-connected CW polyhedron, then*

$$\mathrm{TC}(X) \leq \dim X.$$

Proof. Since X is simply connected, it is 1-connected. From *Corollary* 3.3, we have

$$\begin{aligned}\mathrm{TC}(X) &< \frac{2\dim X+1}{1+1}\\ &= \dim X + 1/2\end{aligned}$$

giving us that $\mathrm{TC}(X) \leq \dim X$. □

3.1.3 The cohomological lower bound

Given a space X, and a field k, we can consider the map induced by the cup product, in $H^*(X;k)$:

$$H^*(X;k)\otimes H^*(X;k) \xrightarrow{\smile} H^*(X;k).$$

We note that $H^*(X;k)\otimes H^*(X;k)$ is also a graded k algebra with the multiplication

$$(u_1\otimes v_1)\cdot(u_2\otimes v_2) = (-1)^{|v_1||u_2|}(u_1u_2\otimes v_1v_2)$$

where $|v_1|$ and $|u_2|$ are the degrees of v_1 and u_2 respectively; and that $\smile$ forms a graded k-algebra homomorphism.

Definition 3.4. The kernel of the map $\smile$ is called the ideal of zero-divisors of $H^*(X;k)$. We denote it by $I_k(X)$.

Definition 3.5. The length of the longest non-trivial product in the ideal of zero-divisors of $H^*(X;k)$ is called the zero divisor cup length of $H^*(X,k)$. We denote it by $zcl_k(X)$.

Let $p: E \longrightarrow B$ be a fibration.

Theorem 3.6 (Theorem 4, [12]). *If $\xi_1 \in H^*(B,A_1)$, $\xi_2 \in H^*(B,A_2)$,... ,$\xi_n \in H^*(B,A_n)$ are such that $p^*\xi_i = 0$ for each i, and $\xi_1 \smile \cdots \smile \xi_n \in H^*(B, \bigotimes_{i=1}^n A_i)$ is non zero, then* $\mathrm{secat}(p) \geq n$.

Theorem 3.7 (Cohomological lower bound). *We have a lower bound for* $\mathrm{TC}(X)$ *in terms of the zero divisor cup length of $H^*(X,k)$:*

$$\mathrm{TC}(X) \geq zcl_k(X).$$

Proof. Consider the following commutative diagram, where $c: X \longrightarrow X^I$ is the map $x \mapsto c_x$, c_x being the constant path x from I to X; and $\Delta: X \longrightarrow X\times X$ is the diagonal map $x \mapsto (x,x)$.

$$\begin{tikzcd}
X \arrow[r, "c"] \arrow[dr, "\Delta"'] & X^I \arrow[d, "\pi"] \\
 & X\times X
\end{tikzcd}$$

We note that c is a homotopy equivalence. Thus, applying the cohomology functor $H^*(\_;k)$ where k is a field, and composing $\Delta^* := H^*(\Delta;k)$ with the Kunneth isomorphism $\varphi: H^*(X;k)\otimes H^*(X;k) \longrightarrow$

$H^*(X \times X; k)$, we get a map $\psi : H^*(X;k) \otimes H^*(X;k) \longrightarrow H^*(X;k)$ such that the following diagram commutes:

$$\begin{tikzcd}
H^*(X,k) & & H^*(X^I,k) \arrow[ll, "c*"'] \\
H^*(X,k) \otimes H^*(X,k) \arrow[u, dashed, "\psi"] \arrow[rr, "\varphi"'] & & H^*(X \times X,k) \arrow[u, "\pi^*"'] \arrow[ull, "\Delta^*" description]
\end{tikzcd}$$

We note that the induced map ψ is the same as the one we obtained from cup product $\smile$. Now since φ and c^* are k-algebra isomorphisms (c was a homotopy equivalence, as we noted earlier), we have that the *Theorem* 3.6 applied to the path fibration π can be translated easily to a result regarding the cup-product :

$$\mathrm{TC}(X) \geq zcl_k(X).$$

□

Remark 3.8. Here, we have assumed that the coefficients of the cohomology ring is in some field k. However, this can be generalised to other rings also. Specifically, the inequality holds if the coefficients are in $\mathbb{Z}$.

3.2 Topological complexity computations

In this section, we compute the topological complexity for some interesting spaces, using the tools we have already mentioned.

3.2.1 Spheres

$$\mathrm{TC}(S^n) = \begin{cases} 1, & \text{for } n \text{ odd,} \\ 2, & \text{for } n \text{ even.} \end{cases}$$

Proof. We know,

$$H^*(S^n; \mathbb{Q}) = \mathbb{Q}[x]/(x^2) \quad \deg x = n$$

$a := x \otimes x$, $b := 1 \otimes x - x \otimes 1$ are zero-divisors of $H^*(X;\mathbb{Q})$.

$$a^2 = ab = 0 \quad \text{as} \quad x^2 = 0.$$

Also,

$$\begin{aligned} b^2 &= (1 \otimes x - x \otimes 1)^2 \\ &= \Big((-1)^{n^2} - 1\Big)(x \otimes x) \\ &= \Big((-1)^n - 1\Big) x \otimes x. \end{aligned}$$

Thus, $b^2 = 0$ if n is even, and $b^2 \neq 0$ if n is odd. From cohomological bound *Theorem* 3.7, $\mathrm{TC}(S^n) \geq 2$ for n odd. For n even, $\mathrm{TC}(S^n) \geq 1$ as S^n is not contractible.

Now we show that these inequalities are, in fact equalities. We prove this by giving explicit motion planning algorithms. Since S^k is a C^∞ smooth manifold, using *Theorem* 1.13, we can use $\mathrm{TC}_{ENR}(X)$ in the place of $\mathrm{TC}(X)$.

- Case: k is odd.

  Consider $E_0 := \{(A,B) \,|\, A \neq B\}$ and for a pair $(A,B) \in E_0$, let (A,B) maps to the shortest geodesic path between A and B under $s_0 : E_0 \longrightarrow X^I$, for $X = S^k$.

  Also, define $E_1 := \{(-A,A) \,|\, A \in X\}$. To define the section s_1, we consider a continuous non-vanishing vector field v, on S^k. Existence of such a field for S^k, as k is odd, is a well known result. Now we define $s_i : E_1 \longrightarrow X^I$ as

  $$s_1(-A,A)(t) := -\cos(\pi t)\cdot A + \sin(\pi t)\cdot v(A)$$

  Note that E_0 and E_1 are ENRs with continuous sections of the path fibration π, namely s_0 and s_1 respectively, on each of them. This gives us the inequality

  $$TC(S^k) \leq 1 \text{ for } n \text{ odd.}$$

  by using the fact that $\mathrm{TC}_{ENR}(X) = \mathrm{TC}(X)$ for smooth manifolds X.

- Case: k is even.

  We define E_0 and s_0 similarly to the odd case: as $E_0 := \{(A,B) \,|\, A \neq B\}$ and $s_0 : E_0 \longrightarrow X^I$, for $X = S^k$ as the map taking (A,B) to the shortest geodesic path between A and B.

  For defining E_1 and s_1, unlike the case k is odd, there might not exist a continuous non-vanishing vector field on S^k for k even. However, we can get a vector field v that vanishes at only one point, say A_0. We define $E_1 := \{(-A,A) \,|\, A \in X, A \neq A_0\}$ and $s_1 : E_1 \longrightarrow X^I$ as above.

  Now $(-A_0,A_0)$ is the only point remaining that is not in either of the E_is. Define $E_2 := \{(-A_0,A_0)\}$ and $s_2 : E_2 \longrightarrow X^I$ as a map taking $(-A_0,A_0)$ to some fixed, continuous map from $-A_0$ to A_0.

  We note that E_0, E_1 and E_3 are disjoint ENRs with continuous sections of the path fibration π, on each of them. This gives us the inequality

  $$TC(S^k) \leq 2 \text{ for } n \text{ even.}$$

□

3.2.2 Tori and product of m-dimensional spheres

We know, the n-torus is the topological space $S^1 \times \cdots \times S^1$ (n times). In more generality, we can consider the space $X = S^m \times \cdots \times S^m$ and try to compute the $\mathrm{TC}(X)$ for it. We have,

$$\mathrm{TC}(X) = \begin{cases} \text{n}, & \text{if } m \text{ is odd,} \\ \text{2n}, & \text{if } m \text{ is even.} \end{cases}$$

Proof. Using product inequality Eq. (2.1) repeatedly, $\mathrm{TC}(X) \leq n \cdot \mathrm{TC}(S^m)$. Thus,

$$\mathrm{TC}(X) \leq \begin{cases} \text{n}, & \text{if } m \text{ is odd,} \\ \text{2n}, & \text{if } m \text{ is even.} \end{cases}$$

Also, for a lower bound, we use the cohomological lower bound. Note that,

$$H^*(S^m;\mathbb{Q}) = \frac{\mathbb{Q}[x]}{(x^2)}$$

with $\deg x = m$. If $p_i : X \longrightarrow S^m$ is the ith projection, let $a_i := {p_i}^*(x)$. We see that $1 \otimes a_i - a_i \otimes 1 \in I_{\mathbb{Q}}(X)$. Since in $H^*(X \times X;\mathbb{Q})$

$$\prod_{i=1}^{n}(1 \otimes a_i - a_i \otimes 1) \neq 0$$

for S^m, and

$$\prod_{i=1}^{n}(1 \otimes a_i - a_i \otimes 1)^2 \neq 0$$

for m even, we obtain that

$$\mathrm{TC}(X) \geq \begin{cases} \text{n}, & \text{if } m \text{ is odd,} \\ \text{2n}, & \text{if } m \text{ is even,} \end{cases}$$

using the cohomological lower bound (*Theorem* 3.7). This, thus, completes our proof.

□

3.2.3 Orientable surfaces

If Σ_g is the orientable surface of genus g, then :

$$\mathrm{TC}(\Sigma_g) = \begin{cases} 2, & g \leq 1, \\ 4, & g > 1. \end{cases}$$

Proof. The case $g = 0$ was already considered in the spheres case, as genus g surface with $g = 0$ is just S^2. The case $g = 1$ is a torus, and is also already considered seperately.

For $g \geq 2$, using Theorem 3.1, we get that $\mathrm{TC}(\Sigma_g) \leq 4$. Also,

$$H^*(\Sigma_g;\mathbb{Q}) = \frac{\mathbb{Q}\langle u_1, v_1, \cdots, u_g, v_g, A\rangle}{R}$$

where $R := \langle u_i^2,\ v_i^2,\ u_iu_j,\ u_iv_j,\ v_iv_j,\ u_iv_i - A,\ u_iv_i + v_iu_i \,|\, i \neq j\rangle$ and $\deg u_i = \deg v_i = 1$ for each i, and $\deg A = 2$. Here $\mathbb{Q}\langle S\rangle$ represents the non-commutative $\mathbb{Q}$- algebra generated by the elements of S. Here $1 \otimes u_i - u_i \otimes 1, 1 \otimes v_i - v_i \otimes 1 \in I_{\mathbb{Q}}(\Sigma_g)$, and

$$\prod_{i=1}^{2}(1 \otimes u_i - u_i \otimes 1)(1 \otimes v_i - v_i \otimes 1) = 2A \otimes A \neq 0$$

and thus, using the cohomological lower bound, we get $\mathrm{TC}(\Sigma_g) \geq 4$. Thus,

$$\mathrm{TC}(\Sigma_g) = 4.$$

□

3.2.4 Complex projective spaces

$$\mathrm{TC}(\mathbb{CP}^n) = 2n.$$

Proof. Since $\mathbb{CP}^n$ is simply connected, from the dimension-connectivity bound *Theorem* 3.2, we get that

$$\begin{aligned}\mathrm{TC}(\mathbb{CP}^n) &< \frac{2 \cdot 2n + 1}{1 + 1} \\ &= 2n + 1/2.\end{aligned}$$

Also,

$$H^*(\mathbb{C}P^n; \mathbb{Q}) = \frac{\mathbb{Q}[x]}{(x^{n+1})} \quad \deg x = 2.$$

Now, $1 \otimes x - x \otimes 1 \in I_{\mathbb{Q}}(\mathbb{C}P^n)$ and

$$(1 \otimes x - x \otimes 1)^{2n} = (-1)^n \binom{2n}{n} x^n \otimes x^n \neq 0.$$

By cohomological lower bound, we get,

$$\mathrm{TC}(\mathbb{C}P^n) \geq 2n.$$

Thus, we have,

$$\mathrm{TC}(\mathbb{CP}^n) = 2n.$$

□

Chapter 4

The collision-free motion planning

In this chapter, we try to discuss the complexity of motion planning problem for n point-like objects in a Euclidean space $\mathbb{R}^k$, avoiding collisions. We consider two cases: the case when there are no obstacles to avoid, and then the case when there are m point-like obstacles that are moving in time, to avoid.

For doing this, first we learn about the configuration space $F(\mathbb{R}^k, n)$, for n point-like objects in $\mathbb{R}^k$ avoiding collisions with themselves.

4.1 The configuration space

Consider n distinct objects in $\mathbb{R}^k$, each represented by a point in our space. Two objects are said to be colliding when they occupy the same point in $\mathbb{R}^k$.

Since each one of the objects can take up any point of $\mathbb{R}^k$, except the ones already occupied by another object, it is natural to define the configuration space of the problem as the space

$$F(\mathbb{R}^k, n) := \left\{(x_1, \cdots, x_n) \in (\mathbb{R}^k)^n \middle| x_i \neq x_j \,\forall\, i \neq j\right\}$$

considered as a subspace of the Euclidean space $(\mathbb{R}^k)^n = \mathbb{R}^{kn}$.

Note that $F(\mathbb{R}^k, n)$ can be considered as the complement in $\mathbb{R}^{kn}$ of the hyperplane arrangement $\mathcal{A} := \{H_{ij}\}_{i \neq j}$ where H_{ij} is the hyperplane $\left\{(x_1, \cdots, x_n) \in (\mathbb{R}^k)^n \middle| x_i = x_j\right\}$.

Remark 4.1. We note that this system has a configuration that is very different from that in *Example* 1.3, simply because we now do not allow objects to collide with, or pass through each other.

In the case $k = 2$, we have that $\mathbb{R}^2$ can be identified with $\mathbb{C}$. Thus our configuration space is the space:

$$F(\mathbb{C}, n) = \left\{ (z_1, \cdots, z_n) \middle| z_i \in \mathbb{C};\ z_i \neq z_j \text{ if } i \neq j\right\}$$

i.e., it is the complement of the central hyperplane arrangement $\{H_{ij}\}_{i \neq j}$ where $H_{ij} = \left\{(z_1, \cdots, z_n) \in \mathbb{C}^n \middle| z_i = z_j\right\}$.

Now since our configuration space is a complement of a complex hyperplane arrangement, we define some terms and notations regarding complex hyperplane arrangements, that are useful in our computations.

Suppose $\mathcal{A} = \{V_i \,|\, i = 1, \cdots, m\}$ is a hyperplane arrangement in $\mathbb{C}^n$. Let $\alpha_i \in (\mathbb{C}^n)^*$ be such that $V_i = \ker \alpha_i$.

Definition 4.2. The rank of $\mathcal{A}$ is the cardinality of the maximal independent subset in $\{\alpha_i \,|\, i = 1, \cdots, m\}$. In other words, it is the dimension of the subspace generated by α_i s in $(\mathbb{C}^n)^*$.

4.2 Cohomology of the configuration spaces

The cohomology group of the configuration space, specifically the zero-divisors of the cohomology, is very useful in computing lower bounds for the topological complexity; as we saw in *Theorem* 3.7. Hence we try to investigate the cohomology of $F(\mathbb{R}^k, n)$ and some related configuration spaces.This will turn out to be useful for our problem of collision-free motion planning.

Consider the space $F(\mathbb{R}^k, n) \subset \mathbb{R}^{kn}$. We would like to consider the contiuous map

$$\varphi_{ij} : F(\mathbb{R}^k, n) \longrightarrow S^{k-1}; (x_1, \cdots x_n) \mapsto \frac{x_i - x_j}{|x_i - x_j|}.$$

Applying the cohomology functor, $H^*(\_, R)$ to φ_{ij}, we get the map

$$\varphi_{ij}^* : H^*(S^{k-1}, R) \longrightarrow H^*(F(\mathbb{R}^k, n), R)$$

We recall that by taking R $= \mathbb{Z}$, we have that $H^*(S^{k-1}, \mathbb{Z}) = \mathbb{Z}[x]/(x^2)$. We define e_{ij} s in the cohomology ring of $F(\mathbb{R}^k, n)$ as: the image of x under φ_{ij}, i.e., $e_{ij} := \varphi_{ij}^*(x)$, for each $1 \leq i < j \leq n$. We can give the structure of $H^*(F(\mathbb{R}^k, n), R)$ in terms of these e_{ij}s.

Theorem 4.3 (Theorem 4.1, [5]). *The cohomology ring $H^*(F(\mathbb{R}^k, n))$ is the free associative graded-commutative algebra generated by e_{ij} s where $1 \leq i < j \leq n$, subject to the relations :*

$$\begin{aligned} &i) \;\; {e_{ij}}^2 = 0, \\ &ii) \;\; e_{ij}e_{i\ell} - e_{ij}e_{j\ell} + e_{i\ell}e_{j\ell} = 0, \qquad \text{for any triple } i < j < \ell. \end{aligned}$$

The space $X := F(\mathbb{R}^k - S_m, n)$, where S_m is a collection of m distinct points in $\mathbb{R}^k$, turns out to be very useful in the moving obstacle case *Section* 4.4. Hence we give results about the cohomology $H^*(F(\mathbb{R}^k - S_m, n))$ also.

Let $S_m = \{q_1, \cdots, q_m\}$. Note that we can think of $F(\mathbb{R}^k - S_m, n)$ as the subspace of $F(\mathbb{R}^k, n + m)$, obtained by taking the n-dimensional cross section of $F(\mathbb{R}^k, n + m)$ got by fixing the last m coordinates as $q_1, \cdots, q_m$.

Theorem 4.4 (Theorem 4.4, [5]). *The homomorphism i^*, induced by the inclusion $i : F(\mathbb{R}^k - S_m, n) \longrightarrow F(\mathbb{R}^k, n + m)$,*

$$i^* : H^*(F(\mathbb{R}^k, n + m); \mathbb{Z}) \longrightarrow H^*(F(\mathbb{R}^k - S_m, n); \mathbb{Z})$$

is an epimorphism, with $\ker i^* = \langle\, e_{ij} \,|\, i, j > n \,\rangle$.

Corollary 4.5. *In $H^*(F(\mathbb{R}^k - S_m, n); \mathbb{Z})$, the following relations are also satisfied :*

$$e_{ij}\, e_{i\ell} = 0$$

for i, j, ℓ such that $i \leq n$ and $j, \ell > n$.

Proof. This follows from the fact that $e_{ij}e_{i\ell} - e_{ij}e_{j\ell} + e_{i\ell}e_{j\ell} = 0$ for all $i < j < \ell$ (See *Theorem* 4.3) and that $e_{j\ell} = 0$ if both j and ℓ are greater than n. (They are in the kernel of i^* using *Theorem* 4.4.) □

We state a result for a basis for the cohomology ring of $F(\mathbb{R}^k - S_m, n)$. We do not give the proof here. Interested readers can check [1] for the details.

Theorem 4.6 (Theorem 4.4, [5]). *A basis for* $H^*(F(\mathbb{R}^k - S_m, n); \mathbb{Z})$ *is given by the monomials*

$$e_{i_1 j_1} e_{i_2 j_2} \cdots e_{i_r j_r}$$

where $1 \leq i_1 < i_2 < \cdots < i_r \leq n$*, and* $i_\ell < j_\ell \leq n + m$ *for each* $1 \leq \ell \leq r$*.*

Example 4.7. For the cohomology computations that we are about to do, it is very important to understand the basis elements given above. In this spirit, we give some explicit examples and non examples for the basis elements mentioned in *Theorem* 4.6, for the case of $H^*(F(\mathbb{R}^2 - S_m, 5))$.

Examples:

- e_{12}, e_{23}, e_{25}, etc.
- $e_{12}\, e_{25}\, e_{34}$.
- $e_{15}\, e_{25}\, e_{35}\, e_{45}$, one of the largest length basis elements if $m = 0$.
- $e_{18}\, e_{28}\, e_{38}\, e_{48}\, e_{58}$, one of the largest length basis elements if $m = 3$.

Non-examples:

- e_{16}, if $m = 0$. Here, $6 > n + m = 5 + 0 = 5$.
- $e_{12}\, e_{13}\, e_{34}$. This is because i_ℓ s needs to be strictly increasing, whereas here i_ℓ is same for the first two elements.
- $e_{15}\, e_{25}\, e_{35}\, e_{45}\, e_{58}$ if $m = 1$. Here, e_{58} is not in one of the generators we mentioned, as j_ℓ needs to be less than $n + m = 5 + 1 = 6$ for each ℓ.

We note that the length of a basis element in *Theorem* 4.6 is n if $m \geq 1$ and $n - 1$ if $m = 0$. This is because, in the $m \geq 1$ case, we can take $i_\ell = l$ for each $1 \leq \ell \leq n$ and j_ℓ to be, say, $n + 1$; whereas in the $m = 0$ case, even if we take $i_\ell = l$ for each ℓ, we will have to leave out atleast the last place, for the last j_ℓ.

Each e_{ij} has a grade $k - 1$ due to the fact that $e_{ij} = \varphi_{ij}(x)$ where x is of grade $k - 1$. This gives us that the maximal non-zero graded component in $H^*(F(\mathbb{R}^k - S_m, n); \mathbb{Z})$ is $(k - 1)(n - 1)$ if $m = 0$ and $(k - 1)n$ if $m \geq 1$.

We note that $H^*(F(\mathbb{R}^k - S_m, n); \mathbb{Z})$ have no torsion elements,from *Theorem* 4.6. Now, if this space is simply-connected, then we will have the following result, using a theorem in [7], that says that such a space is homotopy equivalent to a CW complex that will have a k cell corresponding to each k-graded element in the basis.

Theorem 4.8. *If $F(\mathbb{R}^k - S_m, n)$ is simply-connected, then it is homotopy equivalent to a CW complex X of dimension:*

$$\dim X = \begin{cases} (k-1)(n-1), & \text{if } m = 0, \\ (k-1)\,n, & \text{if } m \geq 1. \end{cases}$$

Now we proceed on to our actual problem, in the following section.

4.3 Collision free motion planning of n objects: without obstacles

Without any obstacles, our problem is the same as finding the topological complexity of the configuration space $F(\mathbb{R}^k, n)$.There are 3 cases, each of which we deal with seperately :

1. $k = 2$,
2. k is odd,
3. k is even with $k > 2$.

For the first case, we recall that $\mathbb{R}^2$ can be considered the same as the complex plane $\mathbb{C}$, and hence $F(\mathbb{R}^2, n)$ can be considered as the complement of a complex hyperplane arrangement in $\mathbb{C}^n$. In fact, the arrangement given by hyperplanes $z_i = z_j$ for each distinct pairs i and j, is a very known one and is known as the braid arrangement.

To compute $\mathrm{TC}(F(\mathbb{R}^2, n))$, we first note the following result:

Theorem 4.9 (Theorem 6, [6])**.** *If M is the complement of central complex hyperplane arrangement of rank r, in $\mathbb{C}^n$, then*

$$\mathrm{TC}(M) \leq 2r - 1.$$

We use this theorem to obtain an upper bound for our case. For this, we note:

Theorem 4.10. *Rank of the braid arrangement in $\mathbb{C}^n$, is $n-1$.*

Proof. The map $\alpha_{ij} := e_i^* - e_j^*$, where e_i^* is the dual for the i th standard coordinate vector in $\mathbb{C}^n$, satisfies the condition that the hyperplane $z_i - z_j = 0$ is $\ker \alpha_{ij}$. Note that α_{1j}s for $j = 2, ..., n$ forms a linearly independent subset, since e_j^* are themselves linearly independent. Noting that $\alpha_{ij} = \alpha_{1j} - \alpha_{1i}$ gives us that it also forms a basis of the span of α_{ij}s.

Thus we have that the rank is $n-1$. □

Suppose H_{ij} is the hyperplane in $\mathbb{C}^n$ with $z_i = z_j$. We, once again, note that $\alpha_{ij} = e_i^* - e_j^* \in \mathbb{C}^{n*}$ satisfies $H_{ij} = \ker \alpha_{ij}$ for each pair (i, j). The following theorem is stated in [6], in the general matroid language. Here, we translate it to our specific case:

Theorem 4.11. *If S is a union of two disjoint sets T_1 and T_2 of α_{ij} s, that satisfies the conditions that T_1 is linearly independent, and $T_2 \cup \{\alpha_{ij}\}$ is linearly independent for any $\alpha_{ij} \in S$, then, the product*

$$\mu = \prod_{(i,j)\in S} \overline{e_{ij}}$$

is non-zero in $H^*(X) \otimes H^*(X)$ where $X = \mathbb{C}^n \setminus \bigcup_{i<j} H_{ij}$ and $\overline{e_{ij}} = 1 \otimes e_{ij} - e_{ij} \otimes 1$ where e_{ij} are the generators of the cohomology ring of X, as in Theorem 4.6.

Theorem 4.12. *The topological complexity of the collision free motion planning of n point-like objects in* $\mathbb{R}^k, k > 1$ *is given by :*

$$\mathrm{TC}(F(\mathbb{R}^k, n)) = \begin{cases} 2n-2, & \text{for } k \text{ odd} \\ 2n-3, & \text{for } k \text{ even} \end{cases}$$

Proof. We give the proofs only for the two most practically interesting k; i.e., $k = 2$ and $k = 3$. The case $k = 3$ is just a specific case of the general odd k case. For obtaining the proof for the general odd k case, we just replace $k = 3$ with a general odd integer $k > 2$, and the proof will go through.

However, the general even case for $k > 2$, is a bit more complicated. We do not prove it here. Interested reader can refer to [4] for the proof.

- Case: $k = 2$.

  Using *Theorem* 4.9, and the fact that the braid arrangement has rank $n - 1$, we have an upper bound

  $$\mathrm{TC}(F(\mathbb{R}^2, n)) \le 2(n-1) - 1 = 2n - 3.$$

  For the reverse inequality, we consider the the lower bound *Theorem* 3.7. Using the same notation as *Theorem* 4.11, we get that $T_1 := \{\alpha_{in} \,|\, i = 1, \cdots n-1\}$ and $T_2 := \{\alpha_{in-1} \,|\, i = 1, \cdots n-2\}$ satisfies the conditions of the theorem. Thus, if $S := T_1 \cup T_2$, then, the product

  $$\mu = \prod_{\alpha_{ij} \in S} \overline{e_{ij}}$$

  is non-zero. By *Theorem* 3.7, this gives us that:

  $$\mathrm{TC}(F(\mathbb{R}^2, n)) \ge |S| = 2n - 3$$

  and this proves our theorem for $k = 2$ case.

- Case: $k = 3$.

  Note that since each of the $H_{ij} = \left\{(x_1, \cdots, x_n) \in (\mathbb{R}^3)^n \middle| x_i = x_j\right\}$, is of codimension 3 in $\mathbb{R}^{3n}$, our configuration space $F(\mathbb{R}^3, n) = \mathbb{R}^{3n} \setminus \bigcup_{i \ne j} H_{ij}$ is simply connected. Hence, from *Theorem* 4.8, $F(\mathbb{R}^3, n)$ is homotopy equivalent to a CW complex of dimension $2(n-1)$.

  Now, using *Theorem* 3.2, we get

  $$\begin{aligned} \mathrm{TC}(F(\mathbb{R}^k, n)) &< \frac{2 \cdot 2(n-1) + 1}{1 + 1} \\ &= 2(n-1) + \frac{1}{2}. \end{aligned}$$

  Thus, $\mathrm{TC}(F(\mathbb{R}^3, n)) \le 2n - 2$.

  For the lower bound, once again, we consider the one given by zero-divisor-cup length.

  For e_{ij}s as defined in *Theorem* 4.6, we define $\overline{e_{ij}} := 1 \otimes e_{ij} - e_{ij} \otimes 1$. We note that $\overline{e_{ij}}$ is a zero-divisor, for each e_{ij}.

Now, if we compute $\overline{e_{ij}}^2$, for any i and j, we can see that

$$\overline{e_{ij}}^2 = -2e_{ij} \otimes e_{ij}$$

Thus,

$$\prod_{i=1}^{n-1} \overline{e_{in}}^2 = (-2)^{n-1} \mu \otimes \mu,$$

where $\mu := \prod_{i=1}^{n-1} e_{in}$. Since μ is a basis vector of $H^*(F(\mathbb{R}^3, n))$ by *Theorem* 4.6, we have that $\mu \otimes \mu$ is a basis vector of $H^*(F(\mathbb{R}^3, n)) \otimes H^*(F(\mathbb{R}^3, n))$, and thus non-zero.

This, thus gives us

$$\mathrm{TC}(F(\mathbb{R}^3, n)) \geq 2(n-1)$$

completing our proof.

□

4.4 Collision free motion planning of n objects avoiding obstacles

We consider the case of n point-like obstacles in $\mathbb{R}^k$ that tries to avoids collisions with m moving obstacles, and with themselves.

Note that since the obstacles are also moving in time, the possible configurations (ie, the possible positions the objects can occupy) and thus, the configuration space, is also changing in time. Thus, it is no longer just a problem of just computing $\mathrm{TC}(X)$. Instead, we will define formally what the appropriate analogue of $\mathrm{TC}(X)$, the complexity of the motion planning of the system, means in this case, and then compute it. The problem is discussed in [5].

Let us denote the time by the variable t. The motion planning problem here requires us to take the input :
i) The ordered n-tuples $A = (A_1, A_2, \cdots A_n), B = (B_1, B_2, \cdots, B_n) \in F(\mathbb{R}^k, n)$ where each A_i and B_i represents the initial and final position of the i th object respectively.
ii) The path $C(t) = (C_1(t), C_2(t), \cdots, C_m(t))$ in $F(\mathbb{R}^k, m)$, where each $C_j(t)$ is the position of j th obstacle at time t, for every $t \in [0, 1]$. Since, the objects are at different positions than that of obstacles at time $t = 0$, as well as at $t = 1$, we also require $C(t)$ to have that $A_i \neq C_j(0)$ and $B_i \neq C_j(1)$ for each pair i and j .
Our output should give us a path $\gamma : I \longrightarrow F(\mathbb{R}^k, n)$; $\gamma(t) = (\gamma_1(t), \cdots, \gamma_n(t))$ where each $\gamma_i(t)$ represents the position of the i th object at time t, avoiding collisions with obstacles and other objects. ie, $\gamma(t)$ satisfies $\gamma_i(t) \neq C_j(t)$ for each $1 \leq i \leq n, 1 \leq j \leq m$, and $\gamma_i(t) \neq \gamma_j(t)$ for each distinct pair of i, j in 1 to n and for each $t \in [0, 1]$.

We define:

$$E(C) := \left\{ \gamma : I \longrightarrow F(\mathbb{R}^k, n) \,\middle|\, \gamma_i(t) \neq C_j(t) \text{ for any } i \text{ and } j \text{ and for each } t \in [0,1] \right\}$$

and

$$B(C) := \left\{ (A, B) \in F(\mathbb{R}^k, n) \times F(\mathbb{R}^k, n) \,\middle|\, A_i \neq C_j(0),\ B_i \neq C_j(1) \text{ for each } i \text{ and } j \right\}$$

If the obstacle path $C : I \longrightarrow F(\mathbb{R}^k, m)$ is fixed first, $E(C)$ represents the possible motions the system can take, and $B(C)$ is the collection of possible endpoints (ie, initial and final configurations) for the motions.

Similar to what we did in the case when the configuration space did not change with time, here also we consider the map taking the possible motions to its endpoints :

$$\pi(C) : E(C) \longrightarrow B(C) ; \quad \gamma \mapsto \big(\gamma(0), \gamma(1)\big)$$

It turns out that this map is a fibration. See *Theorem* 4.14.

Finding a continuous motion planning algorithm in our case, is equivalent to finding a continuous section of $\pi(C)$. We define the complexity of this motion planning problem in a similar way as we did for $\mathrm{TC}(X)$:

Definition 4.13. The complexity of motion planning problem for n objects avoiding collisions with themselves and with m obstacles moving along the paths $C = (C_1, \cdots, C_m)$, is defined as the sectional category of $\pi(C)$.

We now complete the proof for the fact that $\pi(C)$ is a fibration and give a result that relates our complexity of motion planning to $\mathrm{TC}(X)$:

Theorem 4.14. *$\pi(C)$ is a fibration and is fiber-homotopy equivalent to the path fibration of the configuration space when the n objects try to avoid collisions with m obstacles that are not moving. Thus, we have that the complexity of motion planning is:*

$$\mathrm{secat}(\pi(C)) = \mathrm{TC}(F(\mathbb{R}^k - S_m, n)).$$

Proof. We try to show that $\pi(C)$ is the same as the path fibration, π, of the configuration space $X := F(\mathbb{R}^k - S_m, n)$, upto homeomorphisms of the total space and the base space that commutes with the maps $\pi(C)$ and π. This will give us that $\pi(C)$ is indeed a fibration, and that it is fiber-homotopy equivalent to π, giving us that $\mathrm{secat}(\pi(C)) = \mathrm{secat}(\pi)$, and thus that $\mathrm{secat}(\pi(C)) = \mathrm{TC}(X)$.

To do this, we consider a continuous family of homeomorphisms $\psi_t : \mathbb{R}^k \longrightarrow \mathbb{R}^k$ that satisfies $\psi_0 = id_{\mathbb{R}^k}$ and $\psi_t(C(t)) = C(0)$ for each $t \in [0, 1]$. Here, ψ_t means ψ_t applied to each of the coordinates, ie, $\psi_t(C(t)) = (\psi_t(C_1(t)), \cdots, \psi_t(C_m(t)))$. Existence of such a family of ψ_t follows from the well-known isotopy extension theorem.

We consider the diagram :

$$\begin{tikzcd}
E(C) \arrow[r, "F"] \arrow[d, "\pi(C)"'] & X^I \arrow[d, "\pi"] \\
B(C) \arrow[r, "G"'] & X \times X
\end{tikzcd}$$

where $F(\gamma)(t) = \psi_t(\gamma(t))$ and $G(A, B) = \big(\psi_0(A), \psi_1(B)\big)$. Since ψ_t are homeomorphisms and $\psi_t(C(t)) = C(0)$ for each $t \in [0, 1]$, it can be noted that F and G are both homeomorphisms themselves.

Also using the equations,

$$\begin{aligned}\pi \circ F(\gamma) &= (\, F(\gamma)(0), F(\gamma)(1)\,) \\ &= (\, \psi_0(\gamma(0)),\, \psi_1(\gamma(1))\,)\end{aligned}$$

and

$$\begin{aligned}G \circ \pi(C)\,(\gamma) &= G(\gamma(0), \gamma(1)) \\ &= (\, \psi_0(\gamma(0)),\, \psi_1(\gamma(1))\,)\end{aligned}$$

we note that $\pi \circ F = G \circ \pi(C)$, and that the diagram above commutes.

Thus, $\pi(C)$ is the same as π, upto homeomorphisms of the total space and the base space that commutes with the maps $\pi(C)$ and π. Thus, $\pi(C)$ is a fibration, and we have that

$$\operatorname{secat}(\pi(C)) = \operatorname{secat}(\pi)$$

completing our proof. □

Thus, computing the complexity of motion planning in this case also, turns out to be computing the $\mathrm{TC}(X)$ for a specific X, namely $F(\mathbb{R}^k - S_m, n)$.

Remark 4.15. Note that this proves that the complexity of motion planning for n objects avoiding collisions with themselves and with m moving obstacles, is independent of the paths C_j s which the obstacles take.

To complete the computation of complexity, we try to compute $\mathrm{TC}(F(\mathbb{R}^k - S_m, n))$. We restrict our attention to the cases $k = 2$ and $k = 3$, as these are the cases where we are most likely to have practical interests in.

Theorem 4.16. *The complexity of motion planning for n point-like objects avoiding m obstacles in* $\mathbb{R}^2$ *is given by*

$$\mathrm{TC}(F(\mathbb{R}^2 - S_m, n)) := \begin{cases} 2n-3, & \text{if } m = 0, \\ 2n-1, & \text{if } m = 1, \\ 2n, & \text{if } m \geq 2. \end{cases}$$

Proof. The first two cases follow from $Theorem$ 4.12, and the fact that $F(\mathbb{R}^2 - S_1, n)$ is homotopy equivalent to $F(\mathbb{R}^2, n+1)$ (See [1]).

For the $m \geq 2$ case, we note that $F(\mathbb{R}^k - S_m, n)$ is homotopy equivalent to a connected CW complex of dimension n, by $Theorem$ 4.8. Thus, by using the dimension bound $Theorem$ 3.1, we get the upper bound

$$\mathrm{TC}(F(\mathbb{R}^2 - S_m, n)) \leq 2n.$$

Now to get a lower bound, we use the cohomological lower bound, $Theorem$ 3.7. Let $K := \{1, \cdots, n\} \times \{n+1, n+2\}$. We consider the product

$$\mu = \prod_{(i,j) \in K} (1 \otimes e_{ij} - e_{ij} \otimes 1),$$

where e_{ij}s are the generators of the cohomology ring, as given in *Theorem* 4.6. If we expand this product out, we can write μ as a linear combination of μ_J s where

$$\mu_J = \Big(\prod_{(i,j)\in J} e_{ij} \Big) \otimes \Big(\prod_{(i,j)\in J^c} e_{ij} \Big),$$

and where J varies over all subsets of K and $J^c = K \backslash J$. Now since, μ_J is in the tensor of the cohomology ring $H^*(F(\mathbb{R}^2, n))$ with itself, and since by *Theorem* 4.6, the maximum non-zero graded part of the ring is of grade $n(2-1) = n$, we have that both J and J^c are of cardinality n, in any non-zero μ_J.

Moreover, if $\mu_J \neq 0$, we should have that

$$a(J) := \prod_{(i,j)\in J} e_{ij}$$

contains all other possible i s from 1 to n. This is because, if not, there will be repetition of some i, and by *Corollary* 4.5, the product, as well as μ_J will be 0. Similarly for $a(J^c)$ as well.

Thus, J s with non-zero μ_Js look like

$$J = \Big\{(1, j_1), (2, j_2), \cdots, (n, j_n)\Big\} \text{ and } J^c = \Big\{(1, \ell_1), (2, \ell_2), \cdots, (n, \ell_n)\Big\}$$

where $\Big\{j_i, \ell_i\Big\} = \Big\{n+1,\ n+2\Big\}$ for each i. Thus, non-zero μ_Js are of the form

$$\Big(\prod_{i=1}^{n} e_{ij_i}\Big) \otimes \Big(\prod_{i=1}^{n} e_{i\ell_i}\Big).$$

Since $(\prod_{i=1}^{n} e_{ir_i})$ s with $r_i = n+1$ or $n+2$ are part of the basis elements mentioned in *Theorem* 4.6, we have that μ_Js of the above form, are distinct basis elements for $H^*\Big(F(\mathbb{R}^k - S_m, n)\Big) \otimes H^*\Big(F(\mathbb{R}^k - S_m, n)\Big)$, and thus cannot cancel each other in the expansion of μ. Thus, we have, $\mu \neq 0$, and gives us the lower bound

$$\mathrm{TC}(F(\mathbb{R}^k - S_m, n)) \geq 2n$$

as we needed. □

Theorem 4.17. *The complexity of motion planning for n point-like objects avoiding m obstacles in* $\mathbb{R}^3$ *is given by*

$$\mathrm{TC}(F(\mathbb{R}^3 - S_m, n)) := \begin{cases} 2n-2, & \text{if } m = 0, \\ 2n, & \text{if } m \geq 1. \end{cases}$$

Proof. Again, the first case is obtained from the without-obstacles case, i.e., *Theorem* 4.12.

We note that since the configuration space $F(\mathbb{R}^3 - S_m, n)$ can be considered as the complement of finitely many codimension 3 hyperplanes in $\mathbb{R}^{3n}$, we have that it is simply-connected. Now using *Theorem* 4.8, we have that $F(\mathbb{R}^3 - S_m, n)$ is homotopy equivalent to a CW complex X of dimension $2n$. Thus, using *Corollary* 3.3, we have that

$$\mathrm{TC}(F(\mathbb{R}^3 - S_m, n)) \leq \dim X = 2n.$$

Now, for using the cohomology lower bound *Theorem* 3.7, we note that $\overline{e_{ij}}^2 = -2e_{ij} \otimes e_{ij}$ for the zero-divisors $\overline{e_{ij}} = 1 \otimes e_{ij} - e_{ij} \otimes 1$ defined for the generators e_{ij}s of the cohomology ring. Thus, we get that

$$\prod_{i=1}^{n} (\overline{e_{in}})^2 = (-2)^n \mu \otimes \mu$$

where $\mu = \prod_{i=1}^{n} e_{in}$. Since μ is a basis vector of $H^*(F(\mathbb{R}^3 - S_m, n))$ from *Theorem* 4.6, $\mu \otimes \mu$ is a basis vector of $H^*(F(\mathbb{R}^3 - S_m, n)) \otimes H^*(F(\mathbb{R}^3 - S_m, n))$, and thus non-zero. We thus have,

$$\mathrm{TC}(F(\mathbb{R}^3 - S_m, n)) \geq 2n,$$

completing our proof. □

Chapter 5

Higher topological complexity

Higher topological complexity ($\mathrm{TC}_n(X)$) is a series of numerical invariants, of X, of which $\mathrm{TC}(X)$ forms a part. We try to define it and mention its connection to the robot motion planning problem. We also give some useful bounds, in the way we already did for $\mathrm{TC}(X)$, and mention the computations for spheres S^n and torus. Results are from [10] and [11].

Throughout this chapter, we will assume that X is a CW space of finite type, unless othewise mentioned.

5.1 The definition of higher topological complexity

Let X be a CW space of finite type, and J_n be wedge of n copies of $I = [0, 1]$, with 0 as the base point.

Consider the continuous map $e_n : X^{\mathrm{J}_n} \longrightarrow X^n; e_n(\alpha) = (\alpha(1_1), ..., \alpha(1_n))$ where 1_i is the 1 in the ith copy of $[0, 1]$. It can be seen that e_n is fibration. Whenever we would like to make the dependence of e_n on X explicit, we will denote it by $e_{n,X}$.

Definition 5.1 (Higher topological complexity)**.** Higher topological complexity of order n, of X ($\mathrm{TC}_n(X)$), is the sectional category of the fibration $e_{n,X}$.

$$\mathrm{TC}_n(X) := \mathrm{secat}(e_{n,X}).$$

Now consider the map $e'_n : X^I \longrightarrow X^n$ given by:

$$e'_n(\alpha) = \left(\alpha(0), \cdots, \alpha\left(\frac{k}{(n-1)}\right), \cdots, \alpha(1)\right).$$

This is also a fibration. We can prove that e'_n is fiber homotopy equivalent to e_n. This gives rise to an equivalent definition for $\mathrm{TC}_n(X)$, due to *Theorem* 2.14 :

Theorem 5.2. *We have,*

$$\mathrm{TC}_n(X) = \mathrm{secat}(e'_n) = \mathrm{secat}(e_n).$$

Remark 5.3. Note that

$$\mathrm{TC}_2(X) = \mathrm{TC}(X).$$

This shows that $\mathrm{TC}_n(X), n \in \mathbb{N}$ is a generalisation of the notion of topological complexity, $\mathrm{TC}(X)$.

Remark 5.4. $\mathrm{TC}_n(X)$ also have a practical interpretation in terms of robot motion planning. It is the minimum number of discontinuities (counting from 0) in creating a motion planning algorithm for the system such that not just the initial and final configuration of the system, but also $n-2$ intermediate steps as given as the input, in trying to construct the algorithm.

5.2 Properties and bounds

In this section, we briefly mention some of the properties and results related to $\mathrm{TC}_n(X)$. For details or proofs, we can see [10].

It turns out $(\mathrm{TC}_n(X))_{n\in\mathbb{N}}$ is an increasing sequence:

Theorem 5.5 (Proposition 3.3, [10]). *We have,*

$$\mathrm{TC}_n(X) \leq \mathrm{TC}_{n+1}(X).$$

Similar to the cohomological lower bound for $\mathrm{TC}(X)$, we have an analogous result for general $\mathrm{TC}_n(X)$.

Theorem 5.6 (Proposition 3.4, [10]). *Let $d_n : X \longrightarrow X^n$ be the diagonal map $x \mapsto (x, ..., x)$. If there are $x_i \in H^*(X; A_i), i = 1, ..., m$ such that $d_n^* x_i = 0$ and*

$$x_1 \smile x_2 \smile ... \smile x_n \neq 0$$

in $H^*\Big(X^n; \bigotimes_{i=1}^m A_i\Big)$, then $\mathrm{TC}_n(X) \geq m$.

We also have,

Theorem 5.7 (Proposition 3.5, [10]). *If X is connected and not contractible, then $\mathrm{TC}_n(X) \geq n-1$.*

5.3 Higher topological complexity of spheres and torus

Here, we just mention the values of $\mathrm{TC}_n(X)$ for the spheres and the torus. The computations are found in [10].

Theorem 5.8. *We have,*

$$\mathrm{TC}_n(S^k) = \begin{cases} n, & \text{for } k \text{ even}, \\ n-1, & \text{for } k \text{ odd}. \end{cases}$$

Theorem 5.9 (Proposition 5.1, [10]). *If T^2 is the 2-torus $S^1 \times S^1$, then we have,*

$$\mathrm{TC}_n(T^2) \geq 2n-1.$$

Now, in the next chapter, we just give a short survey of some other variants of topological complexity.

Chapter 6

Other variants of topological complexity

When we defined $\mathrm{TC}(X)$, our aim was to obtain the number of discontinuities for constructing a motion planning algorithm that takes in endpoints (A, B) as input and gives us a path from A to B. We did not really care about how complicated the paths between the endpoints are, nor did we really place any other restrictions on the path from A to B. However, there are several natural conditions that we can have about a possible path from A to B, and this can give to some other variants of topological complexity. We discuss two such examples in this chapter. These are discussed in [11]. We assume that X is a CW complex, unless otherwise mentioned.

6.1 Monoidal topological complexity

Suppose we consider only the motin planning algorithms with the following property: If the initial and final configurations are the same, $A = B$; then the path connecting them should be the constant path at $A = B$. This gives us the notion of monoidal topological complexity.

Definition 6.1 (Monoidal topological complexity)**.** For a CW complex X, the monoidal topological complexity of X is the minimal number k such that there are open subsets $U_0, U_1, \cdots U_k$ covering $X \times X$, with continuous sections $s_i : U_i \longrightarrow X^I$ of the path fibration π on each U_i, that also satisfies the condition that $s(x, x) = c_x$, the constant path at x, for each $(x, x) \in U_i$. We denote it by $\mathrm{TC}_M(X)$.

Theorem 6.2 (Proposition 16.4, [11])**.** *We have the inequality :*

$$\mathrm{TC}(X) \leq \mathrm{TC}_M(X) \leq \mathrm{TC}(X) + 1.$$

Theorem 6.3 (Proposition 16.5, [11])**.** *The equality* $\mathrm{TC}(X) = \mathrm{TC}_M(X)$ *holds for all k-connected simplicial complexes X with*

$$(k + 1)(\mathrm{TC}(X) + 2) \geq \dim X + 1.$$

We recall the defintion of absolute neighbourhood retract, as a metrizable space X such that if X is a closed subset of a metrizable space Y, then we have that X is a neighborhood retract of Y. We then have the following result:

Theorem 6.4 (Proposition 16.6, [11])**.** *Let X, Y be two absolute neighbourhood retracts. Then,*

$$\max\Big\{\mathrm{TC}(X), \mathrm{TC}(Y), \mathrm{cat}(X \times Y)\Big\} \leq \mathrm{TC}(X \vee Y) \leq \mathrm{TC}_M(X \vee Y)$$
$$\leq \mathrm{TC}_M(X) + \mathrm{TC}_M(Y) + 1 \leq \mathrm{TC}(X) + \mathrm{TC}(Y) + 3.$$

Now, we proceed on to discuss the symmetric topological complexity in the next section.

6.2 Symmetric topological complexity

Consider a motion planning problem where we require our motion planning algorithm to have that the path from A to B is the inverse path of the path from B to A.This is a reasonable expectation to have, as, in nice enough systems, the system should be able to retrace the path from A to B backwards, to get a path from B to A. A variant of topological complexity that corresponds to this, is the notion of symmetric topological complexity.

In fact, it turns out that there are two different ways we could define a notion of symmetric topological complexity. We give both the definitions here, and discuss some of their properties briefly.

We consider the involutions $\tau : X \times X \longrightarrow X \times X; (x, y) \mapsto (y, x)$ and $\bar{\tau} : X^I \longrightarrow X^I; \alpha \mapsto \alpha(1-t)$. Note that $\tau^2 = id_{X \times X}$ and $\bar{\tau}^2 = id_{X^I}$.

Definition 6.5 (Symmetric subsets of $X \times X$)**.** A subset A of $X \times X$ is called symmetric if it satisfies $\tau A = A$.

Definition 6.6 (Equivariant maps)**.** Suppose A and A' are spaces with involutions $\tau' : A' \longrightarrow A'; \tau'^2 = 0$ on them. A map $s : A \longrightarrow A'$ is called equivariant if it satisfies $\tau' \circ s = s \circ \tau$.

Now we proceed on to define the first version of symmetric topological complexity, $\mathrm{TC}_\Sigma(X)$.

Definition 6.7. $\mathrm{TC}_\Sigma(X)$ is the minimal integer k such that there exist symmetric open subsets $U_0 \cdots U_k$ covering $X \times X$, with continuous equivariant sections $s_i : U_i \longrightarrow X^I$ of the path fibration π, on each U_i.

It turns out that $\mathrm{TC}_\Sigma(X)$ is invariant under homotopy equivalence.

Now, for the second version: Suppose $\Delta(X)$ denote the diagonal of X, i.e.,

$$\Delta(X) := \Big\{ (x, x) \,\Big|\, x \in X \Big\}$$

and $C(X)$ be the complement of $\Delta(X)$ in $X \times X$, i.e.,

$$C(X) := X \times X \Big\backslash \Delta(X).$$

Now we consider the path fibration π and restrict it to the inverse of $C(X)$ to get:

$$\pi : \pi^{-1}(C(X)) \longrightarrow C(X).$$

Here, we note that π here satisfies $\tau \circ \pi = \pi \circ \bar{\tau}$. Thus π is an equivariant map with free $\mathbb{Z}_2$-actions on both the domain as well as the range.

This gives rise to a map :

$$\xi := \pi \big/ \mathbb{Z}_2 : \pi^{-1}(C(X)) \big/ \mathbb{Z}_2 \longrightarrow C(X) \big/ \mathbb{Z}_2.$$

It turns out that the above map is a fibration. Using this, we define our second version of symmetric topological complexity.

Definition 6.8. We define $\mathrm{TC}_S(X)$ as the sectional category of ξ plus 1. ie,

$$\mathrm{TC}_S(X) := \mathrm{secat}(\xi) + 1.$$

Remark 6.9. Unike $\mathrm{TC}_\Sigma(X)$, $\mathrm{TC}_S(X)$ is not invariant under homotopy equivalences.

Now, we would like to know the relations between both the notions of symmetric topological complexity. We give one such relation below:

Theorem 6.10 (Proposition 17.3, [11])**.** *If X is an ENR, then we have :*

$$\mathrm{TC}_S(X) - 1 \leq \mathrm{TC}_\Sigma(X) \leq \mathrm{TC}_S(X).$$

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