Course Project
(Introduction to Reflection Groups)

Weyl Groupoids

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Abstract

This is an introduction to the theory of finite Weyl groupoids, crystallographic arrangements, and their classification.

1 Introduction

Reflections appear in many areas of mathematics. For instance, certain groups generated by involutions may be investigated by representing them as reflection groups. In particular, the Weyl groups belong to this class. They and are fundamental for the classification semisimple Lie groups and semisimple algebraic groups. The Weyl groups are in fact subgroups of $GL(\mathbb{Z}^r)$ for some r. This integrality is a very strong and important restriction; reflection groups with this property are also called crystallographic.

Closely related to a Lie group or an algebraic group is another important structure, the Lie algebra. Lie algebras arise in nature as vector spaces of linear transformations, for example differential operators. It turns out that finite dimensional semisimple complex Lie algebras decompose into a direct sum labeled by roots and a Cartan subalgebra. These roots are (up to signs) the normal vectors defining the reflection hyperplanes of a Weyl group. Again, we have an integrality property for the roots: Let $\mathcal{A}$ be the real hyperplane arrangement given by the orthogonal complements of the roots. Then this is a simplicial arrangement and for each chamber K, the roots labeling the walls of K form a simple system Δ, and in particular all other roots are integer linear combinations of the roots in Δ.

So apparently the combinatorics of root systems and Weyl groups play an important role in mathematics and moreover, a certain integrality is an essential feature of these structures. In the last decades, the concept of a Lie algebra has been generalized in many directions.

For example, deformations of Lie algebras called quantum groups have proved to be useful in physics. More generally the theory of Hopf algebras seems to be a further natural direction. Recent results on pointed Hopf algebras have led to yet another symmetry structure, the *Weyl groupoid*. Again one has vectors called roots, but this time the object acting on the roots is in general a groupoid and not a group anymore. A remarkable fact is that even in this much more general setting, the above integrality still plays a crucial role.

The Weyl groupoid historically appeared as an invariant needed to classify Nichols algebras. Recent observations have considerably increased the importance of the Weyl groupoid.

2 Weyl Groupoids

Generalized Cartan matrix:An integer matrix $C_{r\times r}$ is called a Generalized Cartan matrix if :
1 $c_{ii}=2, c_{jk}\leq 0$
2.$c_{jk}=0 \Rightarrow c_{jk}=0$.

Cartan Scheme: Let A be a nonempty finite set.I any finite set. Suppose for all i in I we have maps $\rho_i : A \to A$ and for any a in A we have a Generalized Cartan matrix C^a .A tuple $\mathcal{C}=\mathcal{C}(I, A, (\rho_i)_{i\in I}, (C^a)_{a\in A})$ is called a Cartan Scheme if :
1. $\rho_i^2 = id$ for all i in I
2. $c_{ij}^a = c_{ij}^{\rho_i(a)}$ for all i,j in I,for all a in A.
Let $\alpha_i (i \in I)$ be the standard basis for $\mathbb{Z}^I$.Define $\sigma_i^a \in Aut(\mathbb{Z}^I)$ by $\sigma_i^a(\alpha_j)=\alpha_j - c_{ij}^a\alpha_i$ (extend by linearity). Then σ_i^a is a reflection.

Weyl Groupoid: The Weyl groupoid of a Cartan Scheme $\mathcal{C}$ is a category $\mathcal{W}(\mathcal{C})$ such that Ob($\mathcal{W}(\mathcal{C})$)=A . Morphisms are compositions of maps $\sigma_i^a \in Hom(a, \rho_i(a))$ (compositions are induced by group structure of $Aut(\mathbb{Z}^I)$). $\mathcal{W}(\mathcal{C})$ is a groupoid since it is a small category and every morphism is an isomorphism.
Example *Weyl groups*:Let (W,S) be a Coxeter system with Cartan matrix C. The Cartan scheme $\mathcal{C}(\{1,...,|S|\},\{a\},(id)_{i\in I},C)$ has Weyl groupoid $\mathcal{W}(\mathcal{C})$=Hom(a,a)=$<\sigma_i|i\in S>$=W. Thus Weyl groups are Weyl groupoids.

We say a Cartan Scheme is **connected** if its Weyl groupoid is connected, that is if for all a,b∈A there exists w∈Hom(a,b).
We say a Cartan Scheme is **simply connected** if its Weyl groupoid is simply connected ,that is if Hom(a,a)=$\{id\}$ for all $a\in A$.

$\mathcal{C}(I, A, (\rho_i)_{i\in I}, (C^a)_{a\in A})$ and $\mathcal{C}'(I', A', (\rho_{i'})_{i'\in I'}, (C^{a'})_{a'\in A'})$ are said to be **equivalent** if there exist bijections :$\phi_0 : I \to I'$ and $\phi_1 : A \to A'$ such that $\phi_1(\rho_i(a)) = \rho_{\phi_0(i)}(\phi_1(a))$ and $c_{ij}^a = c_{\phi_0(i)\phi_0(j)}^{\phi_1(a)}$.
Let $\mathcal{C}$ be a Cartan scheme. For all $a\in A$ let

$$(R^{re})^a = \{id^a\sigma_{i_1}...\sigma_{i_i}(\alpha_j)|k\in\mathbb{N}_0, i_1,...,i_k, j\in I\}$$

The elements of this set are called real roots (at a).
If the real roots of a Weyl groupoid form a finite root system (that is if each $(R^{re})^a$ is a finite set), then we will say that the Weyl groupoid is **finite**.

3 Crystallographic arrangements

Let $r \in \mathbb{N}$,V=$\mathbb{R}^r$.For $\alpha \in V^*$ write $\alpha^\perp$=Ker(α)
An arrangement of hyperplanes $\mathcal{A}$ is a finite set of hyperplanes in V. Let $\mathcal{K}(\mathcal{A})$ denote the set of chambers.If every chamber is a open simplicial cone , we say that $\mathcal{A}$ is a **simplicial** arrangement.
Suppose $\mathcal{A} = \{H_1, ..., H_n\}$ be simplicial.For each $1 \leq i \leq n$, chose $x_i \in V^*$ such that $H_i = x_i^\perp$;let $R = \{\pm x_i\}$.For any chamber $K \in \mathcal{K}(\mathcal{A})$ we define $B^K = \{$ normal vectors in R of the walls of K pointing inside $\}$

Crystallographic arrangements Let $\mathcal{A}$ be a simplicial arrangement, $R \subset V^*$ is a finite set such that $\mathcal{A} = \{\alpha^\perp | \alpha \in R\}$ and $\mathbb{R}\alpha \cap R = \{\pm\alpha\}$ for all $\alpha \in R$.We say $(\mathcal{A}, R)$ is crystallographic if for all $K \in \mathcal{K}(\mathcal{A})$ we have $R \subset \sum_{\alpha \in B^K} \mathbb{Z}\alpha$
Two crystallographic arrangements $(\mathcal{A}, R)$ and $(\mathcal{A}', R')$ are said to be **equivalent** if there exist $\psi \in Aut(V^*)$ with $\psi(R) = R'$.

Examples1.Suppose R is set of roots of a crystallographic root system.Then $(\{\alpha^\perp | \alpha \in R\}, R)$ is a crystallographic arrangement.
2.Let R=$\pm\{(1,0),(3,1),(2,1),(5,3),(3,2),(1,1),(0,1)\}$.Then $(\{\alpha^\perp | \alpha \in R\}, R)$ is a crystallographic arrangement.

4 From arrangements to groupoids

Lemma:Let $(\mathcal{A}, R)$ be a crystallographic arrangement.Let K_0, K be adjacent chambers. Let $B^{K_0} = \{\alpha_1, \alpha_2, ..., \alpha_r\}$,then we have $B^K = \{-\alpha_1, \beta_2, ..., \beta_r\}$.Then there exists a permutation $\tau \in S_r$ with $\tau(1) = 1$ and such that $\beta_i = c_{\tau(i)}\alpha_1 + \alpha_{\tau(i)}$, i = 2,...,r for certain $c_2, ..., c_r \in \mathbb{N}_0$.
Proof:Let σ be the linear map:

$$\sigma : V \to V, \alpha_1 \mapsto -\alpha_1, \alpha_i \mapsto \beta_i \text{ for } 2 \leq i \leq r$$

With respect to the basis B^{K_0}, σ is a matrix of the form

$$\begin{pmatrix} -1 & c_2 & \dots & c_r \\ 0 & & & \\ \vdots & & A & \\ 0 & & & \end{pmatrix}$$

for some $c_2, ..., c_r \in \mathbb{N}_0$, A is a matrix with entries in $\mathbb{N}_0$.By interchanging the roles of K_0 and K ,we will have A_{-1} has entries in $\mathbb{N}_0$, and hence A has to be a permutation matrix. □

Let $K,K^{'}$ be adjacent chambers and $B^K \cap -B^{K'} = \{\alpha\}$.By lemma there exist unique $c_\beta \in \mathbb{N}_0$,$\beta \in B^K \setminus \{\alpha\}$ such that

$$\phi_{K,K'} : B^K \to B^{K'}\alpha \mapsto -\alpha, \beta \mapsto \beta + c_\beta\alpha$$

is a bijection.
Fix a chamber K_0 and fix an ordering $B^{K_0} = \{\alpha_1, \alpha_2, ..., \alpha_r\}$ For any sequence $\mu_1, ..., \mu_m \in \{1, ..., r\}$ we get a unique chain of chambers $K_0, ..., K_m$ such that K_i and K_{i+1} are adjacent and $B^{K_i} \cap -B^{K_{i+1}} = \{\phi_{K_i,K_{i-1}}...\phi_{K_0,K_1}(\alpha_{\mu_1})\}$ for all $i \in \{1, 2, ..., m\}$. Define $\sigma^{K_i}_{\mu_{i+1}} : B^{K_i} \to B^{K_{i+1}}\alpha \mapsto \phi_{K_i,K_{i+1}}(\alpha)$
By lemma $\sigma^{K_i}_{\mu_{i+1}} \in Aut(V^*); (\sigma^{K_i}_{\mu_{i+1}})^2 = id$
We denote by $\sigma_{\mu_m}...\sigma_{\mu_2}\sigma^{K_0}_{\mu_1} = \sigma^{K_{m-1}}_{\mu_m}...\sigma^{K_1}_{\mu_2}\sigma^{K_0}_{\mu_1}$.
We construct a Weyl Groupoid for a crystallographic arrangement $(\mathcal{A}, R)$ of rank r.
Let $I = \{1, ..., r\}$.Fix a chamber K_0 and fix an ordering $B^{K_0} = \{\alpha_1, \alpha_2, ..., \alpha_r\}$.
Consider Â=$\{(\mu_1, ..., \mu_m) | m \in \mathbb{N}, \mu_i \in I\ \forall i\}$. We write $a\cdot \nu$ for $(\mu_1, ..., \mu_m, \nu)$ where $a = (\mu_1, ..., \mu_m)$
Have a map π :Â$\to End(V^*)(\mu_1, ..., \mu_m) \mapsto \sigma_{\mu_m}...\sigma_{\mu_2}\sigma^{K_0}_{\mu_1}$
This yields an equivalence relation $\sim$ on Â via $v \sim w \iff \pi(v) = \pi(w)$
Define A=Â/$\sim$.Each $a \in A$ defines a unique map ϕ_a by $\phi_a = \phi_{K_m,K_{m-1}}...\phi_{K_0,K_1}$ where $K_0, ..., K_m$ is the sequence corresponding to $a = (\mu_1, ..., \mu_m)$.We write $K^a = K_m$.
We define a generalized cartan matrix C^a as follows: Let $i, j \in I$, let K' be the chamber adjacent to K^a with $B^{K^a} \cap -B^{K'} = \{\phi_a(\alpha_i)\}$ By lemma there exist integers c_{ij} such that

$$\phi_{a\cdot i}(\alpha_j) = -c_{ij}\phi_a(\alpha_i) + \phi_a(\alpha_j)$$

We set $C^a = (c_{ij})_{1\le i,j\le r}$. We note that $C^a = C^b$ if $a \sim b$ for $a, b\ \in$Â.Hence we get a well defined matrix for each element a of A,which we denote by C^a.One can check that this is a generalized cartan matrix.
We define maps $\hat{\rho_i} : \hat{A} \to \hat{A},\ a \mapsto a\cdot i$
Since $\pi(a) = \pi(b)$ implies $\phi_a = \phi_b$ so this induces a well defined map

$$\rho_i : A \to A, \overline{a} \mapsto \overline{a\cdot i}$$

Theorem1:Let $(\mathcal{A}, R)$ be a crystallographic arrangement of rank r. Then $I, A, (\rho_i)_{i\in I}, (C^a)_{a\in A}$ defined as above forms a Cartan scheme $\mathcal{C}$ which we denote by $\mathcal{C}(\mathcal{A}, R, K_0)$,which gives rise to a Weyl groupoid $\mathcal{W}(\mathcal{A}, R, K_0)$.

As one would expect, choosing a different chamber or a different ordering of B^K gives rise to equivalent Cartan schemes.In fact one has:

Theorem2:Let $\mathfrak{U}$ denote the set of all crystallographic arrangements and $\mathfrak{C}$ denote the set of all connected simply connected Cartan schemes for which the real roots form a finite root system.Then the map

$$\mathfrak{U}/\sim\to\mathfrak{C}/\sim, \overline{(\mathcal{A},R)}\mapsto\overline{\mathcal{C}(\mathcal{A},R,K)}$$

where K is any chamber of $\mathcal{A}$ is a bijection.

We say a crystallographic arrangement is **irreducible** if the corresponding Cartan scheme is irreducible ,that is if the real roots form a irreducible root system.

5 Classification

Theorem3: There are exactly three families of connected simply connected Cartan schemes for which the real roots form a finite irreducible root system:
(1) The family of Cartan schemes of rank two parametrized by triangulations of a convex n-gon by non-intersecting diagonals.
(2) For each rank $r > 2$, the standard Cartan schemes of type A_r, B_r, C_r and D_r , and a series of $r - 1$ further Cartan schemes.
(3) A family consisting of 74 further sporadic Cartan schemes (including those of type F_4, E_6, E_7 and E_8)of rank r, $3 \leq r \leq 8$

Because of the bijection given in Theorem2 and the classification of Theorem3, one obtains a classification for crystallographic arrangements:

Theorem4:There are exactly three families of irreducible crystallographic arrangements:
(1) The family of rank two parametrized by triangulations of a convex n-gons by non-intersecting diagonals.
(2) For each rank $r > 2$, arrangements of type A_r, B_r, C_r and D_r , and a further series of $r - 1$ arrangements.
(3) Further 74 sporadic arrangements of rank r, $3 \leq r \leq 8$.

References

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