Course project
(Introduction to Reflection Groups)

Reflections on Manifold

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Abstract

Here I will consider the group of diffeomorphisms (see page 4 of ref. 4) generated by separating reflections on a connected differentiable manifold(see page 2 of ref. 4). We call them reflection group for brevity. Later I will define the analogous definition of hyperplane arrangement , walls , chambers , galleries for a manifold with a beautiful example. I'll also show that the reflection group on manifolds have exactly similar properties with that of a reflection group on Affine n-space.

Introduction

A n-dimensional manifold is a topological space whose each point has a neighborhood which is homeomorphic to an open subset of $\mathbb{R}^n$. A differentiable manifold is a manifold with a global differential structure(intuitively a structure where we can use differential calculus). A reflection s on M is a diffeomorphism $s : M \to M$ such that $s^2 = 1$.
$M_s = \{x \in M | s(x) = 1\}$. M_s has co-dimension 1 in M. s is called separating if $M - M_s$ is disconnected. A reflection group W acting on M is a discrete group of diffeomorphism of M generated by separating reflections.

Geometry of manifolds

Proposition 1 :

Let s be a reflection of M. Then $M - M_s$ has at most two connected components.
Proof : Let x_0 and x_1 in $M - M_s$ and let $x(t)$ be a piecewise path from x_0 and x_1 that intersects M_s transversally. Let $x(t_1), ...x(t_N)$ be the points of intersection. Consider the new path $\widetilde{x}(t)$ such that $\widetilde{x}(t) = x(t)$ for $0 \leq t \leq t_1$; $\widetilde{x}(t) = sx(t)$ for $t_1 \leq t \leq t_2$; $\widetilde{x}(t) = x(t)$ for $t_2 \leq t \leq t_3$
Deform the path $\widetilde{x}(t)$ such that in a small neighborhoods of $x(t_1), \dots, x(t_N)$ to make it come off M_s.(see fig. 1)

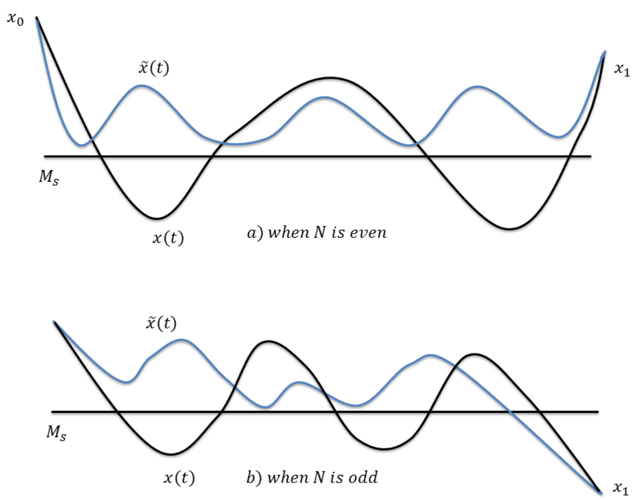

Figure 1

The resulting path $\widetilde{x}(t)$ does not intersect M_s at all if N is even and intersects once if N is odd. Thus any $x, y \in M_s$ can be joined by a continuous path intersecting M_s at most once. Assume $M - M_s$ has three connected components X, Y, Z and choose three points x, y, z in X, Y, Z respectively. Then there are paths $\gamma, \widetilde{\gamma}$ from x to y and from y to z respectively intersecting M_s once. The composite path $\gamma\widetilde{\gamma}$ from x to z intersects M_s twice. This leads to a contradiction.■
From this proposition we can easily conclude that any continuous path between any two points in M intersects with M_s even number of times if and only if they lie in the same component.

Example 1:

(a) Let $M = S^1 \times S^1$ be the two dimensional torus. Then the reflection about it's diagonal is not separating.

(b) Let $M = S^1$ be the unit circle at origin. Then any reflection about it's diameter is separating.

Fact 1 :

If M is simply connected[1] then any reflection of M is separating.

Analog of hyperplane arrangement in manifolds

In the rest of the paper I'll consider only separating reflections and groups generated by them. By Fact 1, if M is simply connected then the assumption is automatically satisfied.

Now I'll establish some terminology.

• *Half space :* The closures ${M_s}^{\varepsilon}, \varepsilon = \pm 1$, of connected components of $M - M_s$ are the two closed half-spaces. If $A \subset M$ intersects only one component of $M - M_s$ then we denote the corresponding half-space by $M_s(A)^+$ and the other by $M_s(A)^-$

• *Wall and Chamber :* The sets $M_s, s \in R$ are called the (reflecting)walls of M and the closure of the connected components of $M - \cup_{s \in R} M_s$ are the chambers of M. Since a wall M_s defines s uniquely so one can identify elements of R with the corresponding walls.

• *Face :* Faces of a chamber C are the elements of the set $C \cap \cup_{s \in R} M_s$

• *Adjacent chamber :* Two chambers $C \neq D$ are adjacent if they have a common face. Let M_r be the unique wall containing this face then $D = rC$.

• *Gallery and minimal gallery :* A sequence of chambers $C_0, C_1, \ldots, C_N$ of chambers is a gallery of length N going from C_0 to C_N if for $i = 1, 2, \ldots, N$ the chambers C_{i-1}, C_i are adjacent. If $C_i = r_i C_{i-1}$ for all i then the corresponding sequence of reflections is $r_1, r_2, \ldots, r_N$. The distance between C and D denoted by $d(C, D)$ is the length of the minimal sequence of reflections $r_1, r_2, \ldots, r_N$. A minimal gallery from C to D is the minimal gallery $C_0, C_1, \ldots, C_N$ such that $C = C_0$ and $D = C_N$. A wall M_s separates C and D if $C \subset {M_s}^{\varepsilon}$ and $D \subset {M_s}^{-\varepsilon}$. The set of reflections separating C and D is denoted by $R(C, D)$.

Example 2 :

In this figure $\{\{(1,0),(-1,0)\}, \{(1/2, \sqrt{3}/2), (-1/2, -\sqrt{3}/2)\}, \{(-1/2, \sqrt{3}/2), (1/2, -\sqrt{3}/2)\}\}$ is the hyperplane arrangement. $\{C_0, C_1, C_2, C_3, C_4, C_5\}$ is the set of chambers. Faces of C_0 are $(1/2, -\sqrt{3}/2)$ and $(-1/2, -\sqrt{3}/2)$. Here C_0, C_1, C_2, C_3, C_4 is gallery from C_0 to C_4. But the minimal gallery from C_0 to C_4 is C_0, C_5, C_4. So, $d(C_0, C_4) = 2$

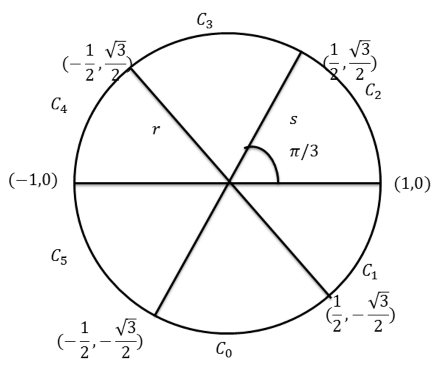

Figure 2

[1]A simply connected space is a topological space which is path connected and has trivial fundamental group(equivalently where every loop can be shrunk to a point).e.g. S^n for $n \geq 2$, $\mathbb{R}^n$ for $n \geq 1$.

Properties of reflection groups

Proposition 2 :

$C = C_0, C_1, \ldots, C_N = D$ is a minimal gallery from C to D if and only if $\{r_1, r_2, \ldots, r_N\} = R(C, D)$

Proof : I'll prove this proposition by showing $\{r_1, r_2, \ldots, r_N\} \subseteq R(C, D)$ and $\{r_1, r_2, \ldots, r_N\} \supseteq R(C, D)$ by an intuitive idea.
Let $r \in R(C, D)$. So, $C \subset {M_r}^{\varepsilon}$ and $D \subset {M_r}^{-\varepsilon}$. A gallery from a chamber C to another chamber D can be visualized as a path from C to D. As, C and D lies in the different half-space so, the path must cuts M_r. So, there are two chambers C_i and C_{i+1} such that $C_i \subset {M_r}^{\varepsilon}$ and $C_{i+1} \subset {M_r}^{-\varepsilon}$ and $C_{i+1} = rC_i$. Then $r \in \{r_1, r_2, \ldots, r_N\}$. So, $\{r_1, r_2, \ldots, r_N\} \supseteq R(C, D)$
If $r \in \{r_1, r_2, \ldots, r_N\}$ and $r \notin R(C, D)$ then C and D lies in the same half-space of $M - M_r$. So, the path from C to D cuts M_r even number of times.
By a similar construction as I did in proposition 1 I can construct a new path which does not intersect with M_r. This new path is equivalent to a new gallery from C to D. It is obvious from the following figure(see figure 3) that the length of the new gallery is less than the length of the previous gallery. This contradicts with the assumption that $C = C_0, C_1, \ldots, C_N = D$ is a minimal gallery from C to D.
So, $\{r_1, r_2, \ldots, r_N\} \subseteq R(C, D)$ ■

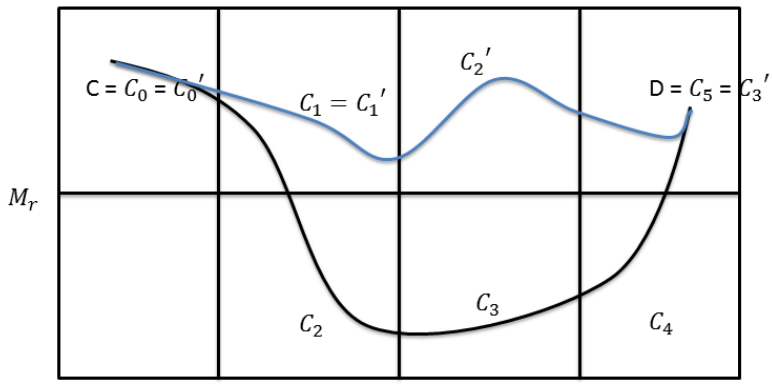

Figure 3

We've a gallery G from C to D $C_0 = C, C_1, C_2, C_3, C_4, C_5 = D$. We construct new gallery $\acute{G}$ from C to D $\acute{C_0} = C, \acute{C_1}, \acute{C_2}, \acute{C_3} = D$. length of $\acute{G}$ = length of $G - 2$.

Corollary 1 :

Let $D \neq C$ be two chambers, let M_s (resp. M_r) be a wall of C (resp. D) such that $r, s \in R(C, D)$. Then there exists a minimal gallery $C = C_0, C_1, \ldots, C_N = D$ such that $C_1 = sC$ and $C_{N-1} = rD$.

Proof :
I'll apply induction to $d(C, D)$. If $d(C, D) = 1$ then obviously $r = s$. So, the assertion is trivial. If $t \in R$ and $t \neq s$ then t can not separate sC from C. Moreover, if $t \in R(C, D)$ then $C, sC \subseteq M_t(D)^-$ and if $t \notin R(C, D)$ then $C, sC \subseteq M_t(D)^+$. Therefore $R(sC, D) = R(C, D)\backslash\{s\}$ and $d(sC, D) = d(C, D) - 1$.
This proves the corollary. ■

Let W be the reflection group acting on M and let $R \subset W$ be the set of reflections in W. The group W acts on the set R by conjugations $r \to grg^{-1}$ which I'll denote $g.r$ for my convenience. The group W acts on the set of chambers of M. As in a manifold there is no analogue of root system,then without loss of generality we can choose any chamber C_+ to be fundamental chamber. We denote the set of reflections in the walls of C_+ by S_{C_+}. $s \in S_{C_+}$ are called simple reflections of M.

Proposition 3 :

(i) Any $r \in R$ is conjugate to some $s \in S$.

(ii) S generates W.

Proof :
Let $r \in R$ and let C be such that M_r is a wall of C. Let $\widetilde{W}$ be the subgroup of W generated by S. There is a $w \in \widetilde{W}$ such that $w^{-1}C = C_+$ (as C_+ is the fundamental chamber). Thus $w^{-1}M_r$ is a wall of C_+.
So, $w^{-1}M_r = M_s$ for some $s \in S$, therefore $r = wsw^{-1}$. This proves (i). From (i) we get that $R \subseteq \widetilde{W}$. The group W is generated by R and $R \subseteq \widetilde{W}$. Thus $W = \widetilde{W}$. This proves (ii). ■

Proposition 4 :

(i) W acts simply transitively on the set of chambers. i.e. for any two chambers C_i and C_j there exists an unique $g \in W$ such that $C_i = gC_j$.

(ii) Let $g \in W$ and let $g = s_1 s_2 \ldots s_N$ be a decomposition of g into simple reflections. Then the sequence $C_0 = C_+, C_1 = s_1 C_+, \ldots, C_i = s_1 s_2 \ldots s_i C_+, \ldots, C_N = s_1 s_2 \ldots s_N C_+$ is a gallery. This establishes a one to one correspondence between the word in s_i and galleries starting from C_+.

Proof :
This prove is exactly same as that we've done in our course for hyperplane arrangement in Affine n-space.(see page 86,87 of ref. 1.) □

Choose a fundamental chamber C_+ from the set of chambers and let S be the corresponding set of simple reflections. S generate W. A decomposition of $g = s_1 s_2 \ldots s_N, s_i \in S$ of $g \in W$ is called minimal if it is the shortest possible decomposition. Then we denote the length of g to be $d(g) = N$. The distance $d(g, h)$ is defined by $d(g, h) = d(g^{-1}h)$. We denote $M_r(C_+)^{\varepsilon}$ by just ${M_r}^{\varepsilon}$ and $R(C_+, gC_+)$ by $R(g)$.

Corollary 2 :

(i) For any $g, h \in W$, $d(g,h) = d(gC_+, hC_+)$ and $d(g) = |R(g)|$.
(ii) $R(g) = \{r \in R | g^{-1}{M_r}^{\varepsilon} = {M_{g^{-1}.r}}^{-\varepsilon}\}$

Proof :
(i) follows immediately from Proposition 2 and 4.
$R(g) = \{r \in R | gC_+ \subset {M_r}^-\}$.
Since $g^{-1}gC_+ = C_+$ we have $g^{-1}{M_r}^- = M^+_{g^{-1}r}$. On the other hand if $r \notin R(g)$ then $gC_+ \subset {M_r}^+$ therefore $g^{-1}{M_r}^+ = {M_{g^{-1}r}}^+$. This proves (ii). ■

For $x \in M$ define isotropy subgroup W_x of W by $W_x = \{g \in W | g(x) = x\}$ and $R_x = \{r \in R | r(x) = x\}$

Proposition 4 :

(i) Let $x, y \in C, g \in W$ and let $gx = y$. Then $x = y$ and $g \in W_x$.
(ii) For any $x \in M$ the group W_x is generated by reflections $r \in R_x$.

Proof :
Let C, D be such chambers such that $C \cap D \neq \Phi$. Since any wall that separates C and D, contains $C \cap D$. A minimal gallery $C = C_0, C_1, \ldots, C_N = D$ going from C to D crosses only the walls $M_r \in R(C,D)$, so every chamber $C_1, \ldots, C_{N-1}$ contains $C \cap D$.
So the corresponding sequence on reflections $\{r_1, r_2, \ldots, r_N\}$ leave $C \cap D$ fixed point-wise. Now $g = r_N r_{N-1} \ldots r_1$. As $\{r_1, r_2, \ldots, r_N\}$ leave $C \cap D$ fixed so is g.
So, $x = g.gx = gy = y$. Hence (i) holds.
For $x \in M$ let C be the chamber containing it. By the same argument as before any $g \in W_x$ is a product of $r_i \in R_x$ which proves (ii). ■

Corollary 3 :

The natural mapping $\varphi : C_+ \to M/W$ is an isomorphism.[2].

Main Theorem

Coxeter group :

Definition :
A coxeter group is a group W with a finite set S of generators and a presentation
$W = \langle S | (sr)^{m(s,r)} = 1 \forall r, s \in S \rangle$.
Where the function $m : S \times S \to \{1, 2, \ldots, \infty\}$
Example :

(a) Coxeter group of type A_{n-1} is $W = \langle S | (s_i)^2 = 1 \forall 0 \leq i \leq n-1; s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \forall 1 \leq i \leq n-2; s_i s_j = s_j s_i \forall |i-j| = 1 \rangle \cong S_n$.
(b) Coxeter group of type B_n is $W = \langle S | (s_i)^2 = 1 \forall 0 \leq i \leq n-1; s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \forall 1 \leq i \leq n-2; s_i s_j = s_j s_i \forall |i-j| > 1; (s_0 s_1)^2 = (s_1 s_0)^2 \rangle \cong {S_n}^B$ the group of signed permutation.

Main Theorem on representation of reflection groups :

Let W be a reflection group acting on M, let C_+ be a fundamental chamber, let $S \subset R$ be the corresponding set of simple reflections and for $s, r \in S$ let $m(s,r)$ be the order of sr. Then W is a coxeter group with the presentation

$$W = \langle S : (sr)^{m(s,r)} = 1 \rangle$$

[2]We denote by M/s the quotient of M by the action of s endowed with natural topology. e.g. for a) of Example 1 M/s is the Möbius band.

Main Example

Here I'll consider S^1 and it's reflection group W generated by finite number of it's dissecting reflections.(reflections w.r.t it's diameters)
We know that a group of orthogonal transformations in $\mathbb{R}^2$ consisting of at least one reflection is a dihedral group. Moreover, if the generator s, r meets at an angle of π/m then the reflection group is the coxeter group of order $2m$ with the presentation

$$D_m = \langle s, r : s^2 = r^2 = (sr)^m = 1 \rangle$$

So, if we consider the example that we've seen in example 2 there each walls are meeting at $\pi/3$. So, the reflection group is D_6.

Labeling of chambers of Example 2 :

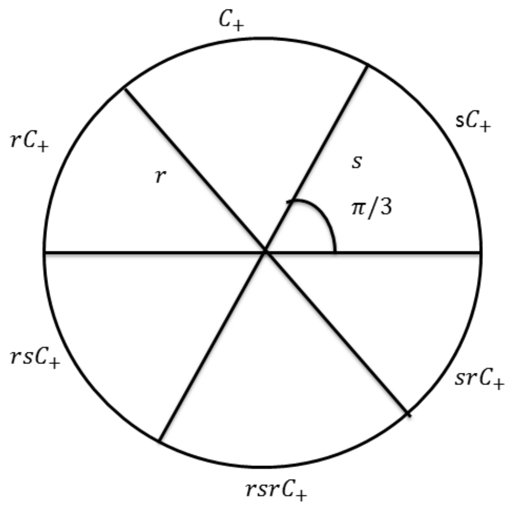

Figure 4

Here choose fundamental chamber C_+ to be C_3. So, the set of simple reflection $S = \{s, r\}$
So, reflection group $W = \langle s, r : s^2 = r^2 = (sr)^3 = 1 \rangle$.
Figure 4 is showing the labeling of S^1 by the elements of W.

Conclusion

So, we've seen that reflections in manifold is more generalized than reflections in euclidean space and both of them satisfy almost similar properties. In figure 4 we've found that the reflection group generated by two dissecting reflection of S^1 is coxeter group of order 6. This illustrates the main theorem.

References

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