Course Project
(Introduction to Reflection Groups)

Specht method for constructing irreducible representations of groups of type A_n and B_n

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Abstract

Representation theory of finite groups is an important branch of mathematics. In the context of reflection groups one can see a rich interplay between algebra and combinatorics. In this talk I shall focus on 2 classes of reflection groups; the symmetric groups (type A) and the octahedral groups (type B). I will explain, in detail, the Specht method of constructing all irreducible representations of these groups upto equivalence .

1 Introduction

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I will assume that reader is familier with the representation theory of finite groups.For the details of representation of finite group reader may refer reference [1].A representation of a group G is a group homomorphism $G \to GL(V)$ where V is finite dimensional vector space over $\mathbb{C}$ which is also equivalent to a module over $\mathbb{C}[G]$,which can be easily seen.An irreducible representation correspond to simple $\mathbb{C}[G]$ module. A general strategy to construct irreducible representation of G will be:

1. Determine the conjugacy class of G,
2. For each conjugacy class λ,construct an irreducible representation V_λ in such a way that V_λ is not equivalent to V_μ for $\lambda \neq \mu$.

In Specht method we construct explicitly simple $\mathbb{C}[G]-$module ,as suitable subspaces of the polynomial algbera in n variables over $\mathbb{C}$

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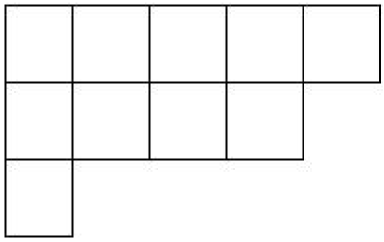

Figure 2.1: Young Diagram for (5,4,1)

2 Type A_n

We have seen in class that Coxeter group corresponding the Coxeter graph of type A_n is isomorphic to S_n which is symmetric group on n symbols.

2.1 The conjugacy class of S_n

Partition of n: A decreasing sequence of positive intergers $\lambda = (\lambda_1, ..., \lambda_r)$ is called partition of n if $\sum \lambda_i = n$ and is denoted by $\lambda \vdash n$.

Theorem: The set of all conjugacy classes of S_n is naturally bijective with the set of all partitions of n.

2.2 Young diagrams and Tableaux

Young diagram: Given a partition $\lambda = (\lambda_1, ..., \lambda_r) \vdash n$, by a Young diagram T_λ of shape λ,we mean a left and top aligned frame of empty boxes having r rows and λ_1 columns in such a way i^{th} row has λ_i boxes for $1 \leq i \leq r$. See figure 2.1 for an example.

Conjugate partitions: Given a partition $\lambda = (\lambda_1, ..., \lambda_r) \vdash n$, the partition $\lambda^{'} = (\lambda_1^{'}, ..., \lambda_s^{'}) \vdash n$ is called the conjugate of λ where $\lambda_i^{'}$ is the number of boxes in the i^{th} column of Young diagram T_λ of shape λ.

Young Tableaux: Given a partition $\lambda = (\lambda_1, ..., \lambda_r) \vdash n$, by a Young tableaux ,we mean a Young diagram T_λ of shape λ whose boxes are filled with the intergers between 1 to n without repetition.See figure 2.2 for example.

There are exactly $n!$ Young tableaux of any given shape λ.Given $\lambda \vdash n$ and $\sigma \in S_n$,the Young tableaux $T_\lambda(\sigma)$ of shape λ is the diagram T_λ which is filled with entries $\sigma(1), ..., \sigma(n)$ down the coulmns starting with the first column ,left to right.

2.3 Specht modules for S_n

We shall give the Specht construction of the irreducible representations of S_n. Let $K = \mathbb{C}[x_1, ..., x_n]$ be the polynomial algebra over $\mathbb{C}$ in the n

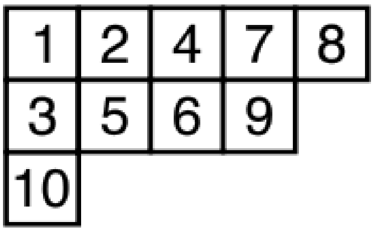

Figure 2.2: Young Tableaux

variables $x_1, ..., x_n$. $S_n \curvearrowright K$ by permuting the variables i.e.

$$\theta(f(x_1, ..., x_n)) = f(x_{\theta(1)}, ..., x_{\theta(n)})$$

$\forall\ \theta \in S_n, f \in K$. For each positive integer m.Let H_m denote the space of all homogeneous polynomials of degree m in K which is a finite dimensional vector space of K.Each H_m is an S_n-module of K. Given a partition $\lambda \vdash n$,let $\lambda^{'}$ be the conjugate of λ.Let $\lambda = (\lambda_1, ..., \lambda_r)$,$\lambda^{'} = (\lambda_1^{'}, ..., \lambda_s^{'})$ Let

$$m_\lambda = \frac{1}{2}\sum_{j=1}^{s}\lambda_j^{'}(\lambda_j^{'} - 1) \tag{2.1}$$

Let T_λ be a fixed Young tableaux of shape λ.Let $a_{1j}, ..., a_{\lambda_j^{'}j}$ be the entries of the j^{th} column of T_λ.Define

$$\Delta_j = \Delta_j(a_{1j}, ..., a_{\lambda_j^{'}j}) = \begin{vmatrix} 1 & \cdots & 1 \\ x_{a_{1j}} & \cdots & x_{a_{\lambda_j^{'}}} \\ x_{a_{1j}}^2 & \cdots & x_{a_{\lambda_j^{'}}}^2 \\ & \cdots & \\ x_{a_{1j}}^{\lambda_j^{'}-1} & \cdots & x_{a_{\lambda_j^{'}}}^{\lambda_j^{'}-1} \end{vmatrix}$$ Which is a Vandermonde determinant and so we know that

$$\Delta_j = \prod_{1 \leq p < q \leq \lambda_j^{'}} (x_{a_{qj}} - x_{a_{pj}})$$

2.3.1 Specht polynomials:

Let $\Delta(T_\lambda) = \prod_{j=1}^{\lambda_1}\Delta_j$. This is a homogeneous polynomial of degree m_λ,called the Specht polynomial associated to the Young tableaux T_λ. For $\sigma \in S_n$,we have $\sigma\Delta(T_\lambda) = \Delta(\sigma T_\lambda)$.

2.3.2 Specht modules

Given $\lambda \vdash n$,the cyclic S_n-submodule of H_{m_λ} generated by the Specht polynomial $\Delta(T_\lambda)$ is independent of the tableaux T_λ but depends only on shape λ. It is called the Specht module associated to the partition λ and is denoted by W_λ.

Theorem 2.1. *For $\lambda \vdash n$,the Specht modules W_λ is simple S_n- module.*

Proof. Refer [2]. □

Theorem 2.2. *For $\lambda \neq \mu$,the Specht modules W_λ , W_μ are non-isomorphic.*

Proof. We do this by showing that $Ann_{\mathbb{C}[S_n]}W_\lambda \neq Ann_{\mathbb{C}[S_n]}W_\mu$.The details of the proof are in [2]. □

3 Type B_n

Treat S_{2n} as the group of permutation of the $2n$ symbols $\pm 1, \cdots, \pm n$.

3.1 The group B_n:

For an integer $n \geq 2$,$B_n := \{\theta \in S_{2n} | \theta(i) + \theta(-i) = 0\}$.

3.1.1 Positive and Negative Cycles:

An element in B_n which is

1. a product of 2 $l-$cycles in S_{2n} of the form $\theta = (a_1, \cdots, a_l)(-a1, \cdots, -a_l)$ is called postive $l-$cycles.
2. a $2l-$ cycles in S_{2n} of the form $\theta = (a_1, \cdots, a_l, -a_1, \cdots, -a_l)$

Some facts for the group B_n:

1. $B_n \cong C_2^n \rtimes S_n$.See [3].
2. Every element of B_n can be uniquely expressed as a product of disjoint positive and negative cycles.

3.1.2 Complementary partitions:

Two partitions $\lambda \vdash a$ and $\mu \vdash b$ $a, b \geq 0$ are said to be complementary partitions of an integer if $a + b = n$.An ordered pair (λ, μ) of complementary partitions of n is denoted by $(\lambda, \mu) \models n$.

Fact: The set of conjugacy classes of B_n is naturally bijective with the set of pairs of complementary partitions.

3.2 Young diagram and Young tableaux for B_n

Given $(\lambda, \mu) \models n$,let $\lambda = (\lambda_1, ..., \lambda_r)$ and its conjugate partition $\lambda^{'} = (\lambda_1^{'}, ..., \lambda_{r'}^{'})$ .Likewise ,let $\mu = (\mu_1, ..., \mu_s)$ and its conjugate $\mu^{'} = (\mu_1^{'}, ..., \mu_{s'}^{'})$.

Young diagram: Given $(\lambda, \mu) \vdash n$,by a Young diagram of shape (λ, μ) or a $(\lambda, \mu)-$ diagram $T_{(\lambda,\mu)}$ of shape (λ, μ) we mean a pair of Young diagrams T_λ and T_μ of shape λ and μ respectively.

Young tableaux: We fill a $(\lambda,\mu)-$Young diagram by filling the entries of T_λ, T_μ by $\{\pm 1, ..., \pm n\}$ in such a way that each i or $-i$ must occur but not both.

3.3 Specht modules for B_n

Consider K as previous . Define $x_{-j} = -x_j$ for all $1 \leq j \leq n$ Now the group B_n acts linealy on the polynomial algbera K by permuting and sign change of variables.

1. For non-zero intergers $a_1, ..., a_l$ between $-n$ and n ,define a Vandermonde type determinant ,namely,
$$\Omega(a_1,...,a_l) = \begin{vmatrix} 1 & ... & 1 \\ x_{a_1}^2 & ... & x_{a_l}^2 \\ & ... & \\ x_{a_1}^{2l-2} & ... & x_{a_l}^{2l-2} \end{vmatrix}$$
we have
$$\Omega(a_1,...,a_l) = \prod_{1 \leq j \leq k \leq l} (x_{a_k}^2 - x_{a_j}^2)$$
2. Given $(\lambda,\mu) \models n$,let $\lambda \vdash l$ and $\mu \vdash m$ with $l + m = n$ .Let $T_{(\lambda,\mu)}(\theta) = (T_\lambda(\theta), T_\mu(\theta))$ be a Young tableaux of shape (λ,μ) filled along a $\theta \in B_n$ where $T_\lambda(\theta)$ is filled with $\{\theta(i)|i \leq l\} = \{a_{jk}\}$ and $T_\mu(\theta)$ filled with $\{\theta(j)|l+1 \leq j \leq n\} = \{b_{jk}\}$.

Let $\Delta_{(\lambda,\mu)}(\theta) = \Gamma_\lambda(\theta)\Omega_\lambda(\theta)\Omega_\mu(\theta)$,where $\Gamma_\lambda(\theta) = \prod_{j=1}^{\lambda_1'}\prod_{k=1}^{\lambda_1} x_{a_{jk}}$

$$\Omega_\lambda(\theta) = \prod_{k=1}^{\lambda_1} \Omega(x_{a_{1k}}, ..., x_{a_{\lambda_k' k}}) \ ,$$
$$\Omega_\mu(\theta) = \prod_{k=1}^{\mu_1} \Omega(x_{a_{1k}}, ..., x_{a_{\mu_k' k}}).$$

3.3.1 Specht polynomials for B_n

Given $(\lambda,\mu) \models n$ and $\theta \in B_n$,the homogeneous polynomial $\Delta_{(\lambda,\mu)}(\theta)$,defined above ,is called the Specht polynomial associated to θ.

3.3.2 Specht modules for B_n

Given $(\lambda,\mu) \models n$ the cyclic B_n- submodule of K generated by the Specht polynomial $\Delta_{(\lambda,\mu)}$ associated to (λ,μ) and is denoted by $W_{(\lambda,\mu)}$.

Theorem: The Specht module $W_{(\lambda,\mu)}$ is simple.
Proof: Refer [2]

Theorem: The family $Irr_{\mathbb{C}}(B_n) = \{W_{(\lambda,\mu)}|(\lambda,\mu) \models n\}$ is a complete set of inequivalent irreducible representations of B_n.
Proof: Refer [2].

References

[1] William Fulton,Joe Harris
Representation theory: a first course

[2] Meinolf Geck ,G6tz Pfeiffer
Characters of Finite Coxeter Groups and Iwahori-Hecke Algebras

[3] Alexandre V. Borovik, Anna Borovik, *Mirrors and Reflections : The Geometry of Finite Reflection Groups*, Springer, 2010