Course Project
(Introduction to Reflection Groups)

W-Permutahedron And Matrix Mutation

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Abstract

For a reflection group W, the associated W-permutahedron is the convex hull of the W-orbit of a generic point. I shall first describe the properties of a W-permutahedron associated to a classical root system. Then, the definition of matrix mutation will be given and the operations of it will be shown using a concrete example. Later, by proper example it will be sketched how diagonal flip is related with matrix mutation.We shall then introduce the concept of exchange relation and that goes hand in hand with matrix mutation.

1 W-Permutahedron

1.1 Introduction

The *permutahedron* of order n is an $(n-1)$-dimensional polytope embedded in an n-dimensional space, the vertices of which are formed by permuting the co-ordinates of the vector $(1, 2, ..., n)$. The name '*permutahedron*'(or rather its French version '*permutoedre*') comes from the fact that the vertices of an $A_n - permutahedron$ are obatined by permuting the co-ordinates of a generic point in $\mathbb{R}^{n+1}$.

1.2 Definition and Properties

Definition 1.1. *Let W be a finite coxeter group and u be a point in the interior of the fundamental chamber. We write* $u = \sum_{s\in S} u_s w_s$ *with* $u_s \in \mathbb{R}_{>0}$. *We define the* $W - Permutahedron$*,* $\mathrm{Perm}^u(W)$ *to be the convex hull of the orbit of u under W, whose combinatorical properties are determined by that of the coxeter group W.*

W-permutahedron of order n has the following properties.

1. Number of vertices is $n!$.
2. Each vertex is adjacent to $(n-1)$ others. So, number of edges is $\dfrac{(n-1)n!}{2}$. Each edge has length $\sqrt{2}$.
3. The permutahedron has one panel for each non-empty proper subset S of $\{1, 2, ..., n\}$, consisting of the vertices in which all co-ordinates in positions in S are smaller than all co-ordinates in positions not in S. So, number of panels is $2^n - 2$.

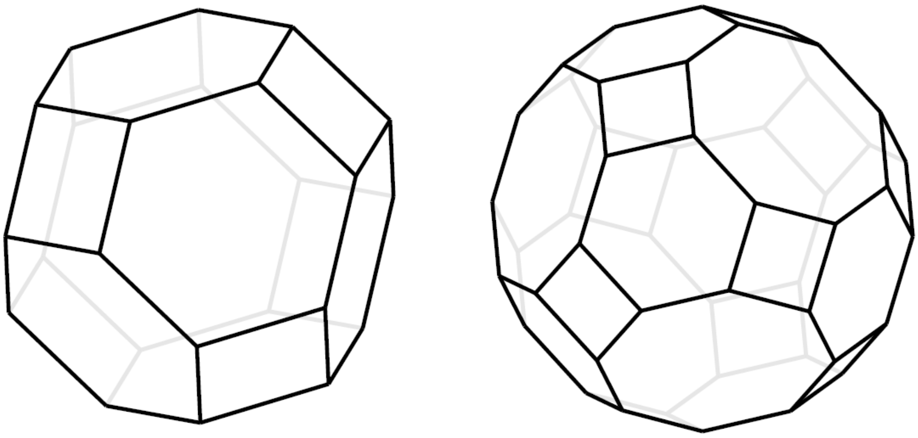


Figure 1.1: The permutahedra of type A_3 and B_3 respectively

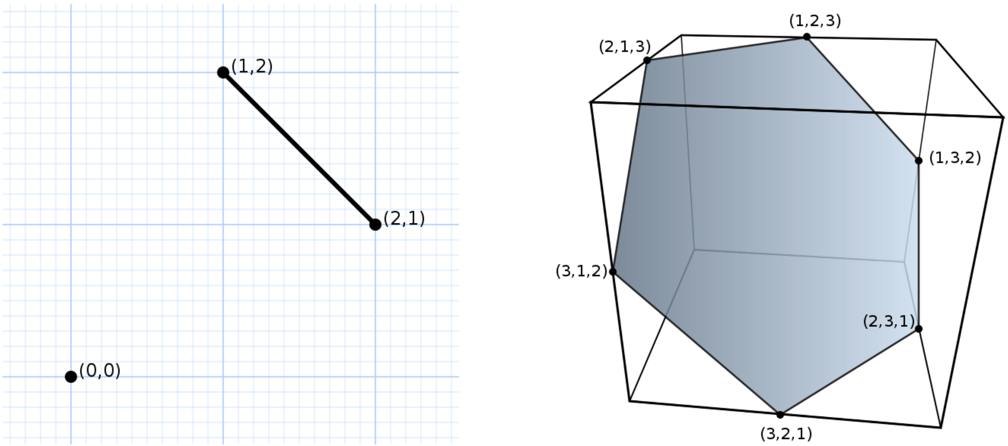


Figure 1.2: The permutahedra of order 2 and 3 respectively

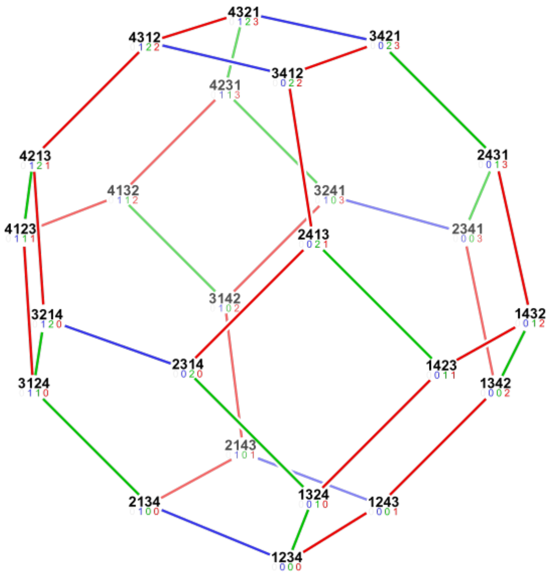
1.3 Examples

The A_2, B_2 and G_2 permutahedra are respectively a hexagon, an octagon and a dodecagon and under the choice of a generic point, these polygons are regular. Figure 1 show the permutahedra of types A_3 and B_3. Each of these realizations derives from a choice of $x \in \mathbb{R}_1$ which makes the permutahedron an *Archimedean solid* (*i.e.* a non-regular polytope whose all facets are regular polygons, and whose symmetry group acts transitively on vertices.). The non-crystallographic H_3-permutahedron is also an Archimedean solid.

We can write each element $w \in W$ as a product of elements of S *i.e.* $w = s_{i_1}s_{i_2}...s_{i_k}$. A shortest factorization of this form is called a *reduced word* for w. The number of factors k is called the *length* of w.

Any finite Coxeter Group has a unique element w_0 of maximal length. In the symmetric group S_{n+1} ($\simeq A_n$), this is the permutation w_0 that reverses the order of the elements of the set $\{1, 2, ..., n+1\}$. For example, in Figure 1, the bottom vertex can be associated with the identity element $1 \in W$, so that the top vertex is w_0.

Realization of Perm(A_3)**:** Let $W = S_4$ be the Coxeter group of type A_3. The standard choice of simple reflection yields $S = \{s_1, s_2, s_3\}$, where s_1, s_2 and s_3 are the transpositions which interchange 1 with 2, 2 with 3 and 3 with 4, respectively. Since $1 \in W$, the top vertex is w_0. A reduced word for w corresponds to a path



from 1 to w which moves up in a monotone fashion. There are 16 such paths from 1 to w_0 in the A_3-permutahedron. The word $s_1s_2s_1s_3s_2s_3$ is a non-reduced word for the permutation that interchanges 1 with 3 and 2 with 4. This permutation has two reduced words $s_2s_1s_3s_2$ and $s_2s_3s_1s_2$. An example of a reduced word for the w_0 is $s_1s_2s_1s_3s_2s_1$. In the adjacent figure, the bottom vertex$(1, 2, 3, 4)$ can be associated with the identity$(0, 0, 0, 0)$ and the top most vertex$(4, 3, 2, 1)$ with (0,1,2,3). There are 16 such distinct paths between these two points.

Theorem 1.1. *The number of reduced words for w_0 in the reflection group A_n is*

$$\frac{\binom{n+1}{2}!}{1^n 3^{n-1} 5^{n-2} \dots (2n-1)^1}.$$

This formulla was given by R. Stanley. We can simply calculate the number of reduced words for $w_0 \in A_3$ to be 16.

Definitions 1.1.

- *If $u = 1 := \sum_{w \in \triangle} W$, we say that the W-permutahedron* $\mathrm{Perm}^u(W)$ *is balanced and it is denoted by* $\mathrm{Perm}(W)$.
- $\mathrm{Perm}^u(W)$ *is said to be fairly balanced if* $W_0(u) = -u$ *i.e.* $u_s = u_\phi(s)\ \forall\ s \in S$.
- *The classical permutahedron is the convex hull of all permutations of 0,1,2,...,n, regarded as vectors in $\mathbb{R}^{n+1}$. According to the fundamental weights , we get $\sum_{w \in \triangle} W$ = 0,1,2,...,n. So the classical permutahedron coincides with the balanced A_n-*permutahedron $\mathrm{Perm}(A_n)$.

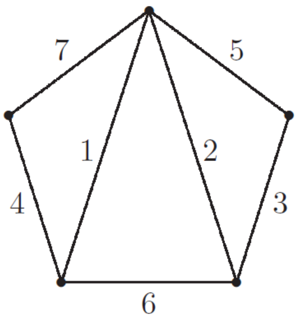
Remark 1.1. The permutahedron of types A_3, B_3 and H_3 are also known as the *truncated octahedron, great rhombicuboctahedron* and *great rhombicosidodecahedron,* respectively.

2 Matrix Mutation

2.1 Introduction

Definition 2.1. *A triangulation T is a collection of n triangles satisfying the following requirements :*

- *The interiors of the triangles are pairwise disjoint.*
- *Each edge of a triangle in T is either a common edge of two triangles in T or else it is on the boundary of the union of all the triangles.*



$$\tilde{B} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \\ 1 & -1 \\ 1 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

Figure 2.1: the edge-adjacency matrix and principal matrix

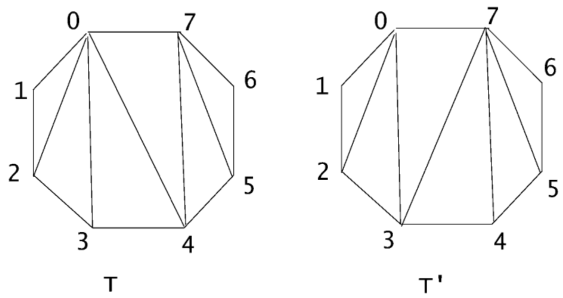
Fix a triangulation T of the (n+3)-gon. Label the n diagonals of T arbitrarily by the numbers 1, 2,...., n and label the n+3 sides of T arbitrarily by the numbers n+1, n+2,...., 2n+3. The combinatorics of T can be encoded by the *edge-adjacency matrix* or *signed adjacency matrix* B̃.

Definition 2.2. *The edge-adjacency matrix is a* $(2n+3) \times n$ *matrix* $\tilde{B} = (b_{ij}) \ni$

$$b_{ij} = \begin{cases} 1 & \text{if } i \text{ and } j \text{ label two sides in some triangle of } T \text{ so that } j \text{ follows} \\ & i \text{ in the clockwise traversal of the triangle's boundary;} \\ -1 & \text{if the same holds, with the counter-clockwise direction;} \\ 0 & \text{otherwise.} \end{cases}$$

Note that the first index i is a label for a side or a diagonal of the (n + 3)-gon, while the second index j must label a diagonal. The *principal part* of B̃ *i.e. principal matrix* is an $n \times n$ submatrix B = $(b_{ij})_{i,j \in n}$ that encodes the signed adjacencies between the diagonals of T. Here is an example of the edge-adjacency matrix and principal matrix.

Let $v_i v_j$ be an edge of a planar triangulation T and $\{v_i,v_j,v_k\}$ and $\{v_i,v_j,v_l\}$ be the vertices of the faces of G containing $v_i v_j$ on their boundaries.



We say that $v_i v_j$ is *flippable* if v_k and v_l are not adjacent in T. By flipping $v_i v_j$, we mean the operation of removing it from T followed by the insertion of $v_k v_l$ into T. It is easy to see that this produces a new graph T' which is also a planar triangulation. The operation is called a *diagonal flip* on $v_i v_j$. In the adjacent figure, the edge between the vertices 0 and 4 in the triangulation T is flipped to produce the new triangulation T', where unlike in the triangulation T the vertices 3 and 4 are joined.

In the languages of matrices B̃ and B, diagonal flips can be described as certain transformations called matrix mutations.

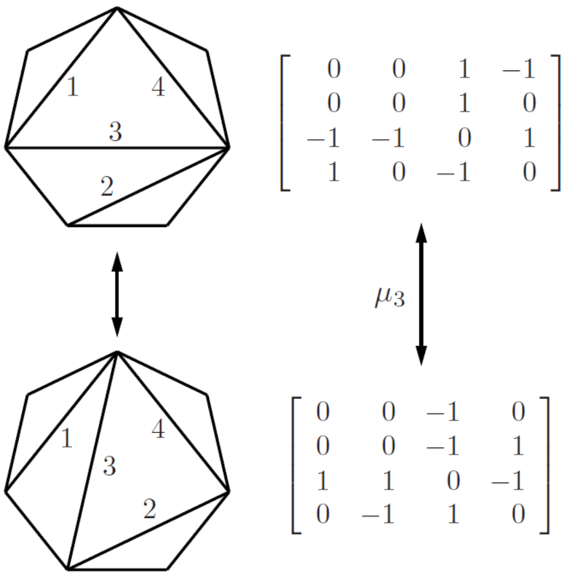


Figure 2.2: A diagonal flip and the corresponding matrix mutation

2.2 Matrix Mutation

Definition 2.3. *Let* $B = (b_{ij})_{i,j \in n}$ *and* $B' = (b'_{ij})_{i,j \in n}$ *be integer matrices. We say that* B' *is obtained from* B *by a matrix mutation in direction* k *i.e.* $B' = \mu_k(B)$*, if*

$$b'_{ij} = \begin{cases} -b_{ij} & \text{if } k \in i, j \\ b_{ij} + |b_{ik}| b_{kj} & \text{if } k \notin i, j \text{ and } b_{ik} b_{kj} > 0 \\ b_{ij} & \text{otherwise} \end{cases}$$

Lemma 2.1. *Assume that* $\tilde{B}$ *and* $\tilde{B}'$ *are the edge-adjacency matrices and* B *and* B' *are their principal parts respectively for two triangulations* T *and* T' *obtained from each other by flipping the diagonal numbered* k*; the remaining labels are the same in* T *and* T'*. Then* $B' = \mu_k(B)$ *(respectively* $B = \mu_k(B')$*).*

The lemma stated above is illustrated in figures 4 and 6. Note that the numbering of the diagonals used in defining the matrices $\tilde{B}$ and B can change as we move along the exchange graph. For instance, the sequence of 5 flips shown in figure results in switching the labels of the two diagonals.

One can similarly define edge-adjacency matrices for centrally symmetric triangulations (those matrices will have entries 0, 1 , -1, +2, -2) and verify that cyclohedral flips translate precisely into matrix mutations.

Corollary 2.1. *Matrix Mutation is an involution i.e.* $\mu_k(\mu_k(B)) = B$*.*

Proof. From the previous lemma, we have that matrix mutation in direction k is equivalent to diagonal flip of the diagonal numbered k. So it's enough to establish that a diagonal flip is an involution.

Let, $\{v_i, v_j, v_k\}$ and $\{v_i, v_j, v_l\}$ be the vertices of the faces of a triangulation T containing $v_i v_j$ on their boundaries and suppose $v_i v_j$ is *flippable* *i.e.* v_k and v_l are not adjacent in T.

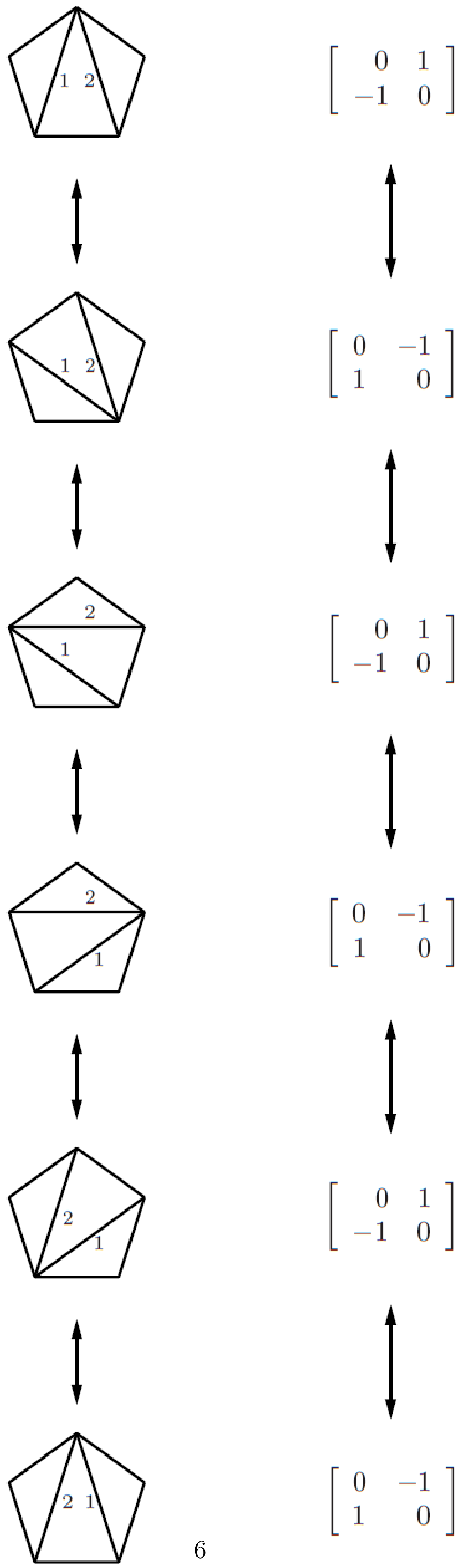


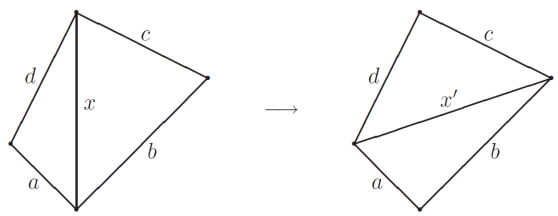
Figure 2.3: Diagonal flip in a pentagon and the corresponding matrix mutations

Doing a diagonal flip, we get a new triangulation T', edges of which are same as edges of T except the edge v_iv_j in T which is flipped to form v_kv_l in T'.

We perform another diagonal flip on T' and now another triangulation(say T'') is formed whose vertices v_i and v_j are joined and v_kv_l is not an edge. As all the others edges remain unchanged, the triangulation T'' is same as the triangulation T. So, performing diagonal flip twice gives the same triangulation *i.e.* diagonal flip is an involution, which implies $\mu_k(\mu_k(B)) = B$. □

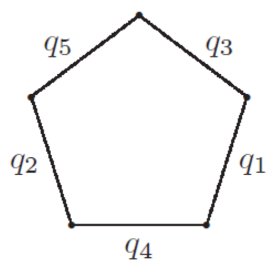
2.3 Exchange Relation

Let us fix an arbitrary *initial triangulation* T_0 of a convex $(n+3)$-gon, and introduce a variable for each diagonal of this triangulation, and also for each side of the $(n+3)$-gon. We now associate a rational function in these $2n+3$ variables to every diagonal of the $(n+3)$-gon (This can be done in a recursive fashion).



Whenever we perform a diagonal flip as shown in the adjacent figure, all but one rational functions associated to the current triangulation remain unchanged. The rational function x associated with the diagonal being removed gets replaced by the rational function x' associated with the new diagonal, where x' is determined from the *exchange relation* $xx' = ac + bd$.

Consider a triangulation of a pentagon *i.e.* $n = 2$. We label the sides of the pentagon (see adjacent figure) by the variables q_1, q_2, q_3, q_4, q_5. We then then label the diagonals incident to the top vertex by the variables y_1 and y_2. Thus, in the figure 5, the initial triangulation T_0 appears at the top. The rational functions y_3, y_4, y_5 associated with the remaining three diagonals are then computed from the exchange relations associated with the flips shown there.



Starting from the top of Figure 5 and moving clockwise, we recursively express y_3, y_4, y_5 in terms of y_1, y_2 and q_1, q_2, q_3, q_4, q_5 hereby.

- $y_3 = \dfrac{q_2y_2 + q_4q_5}{y_1}$
- $y_4 = \dfrac{q_3y_3 + q_5q_1}{y_2} = \dfrac{q_3q_2y_2 + q_3q_4q_5 + q_5q_1y_1}{y_1y_2}$
- $y_5 = \dfrac{q_4y_4 + q_1q_2}{y_3} = ... = \dfrac{q_3q_4 + q_1y_1}{y_2}$

And from the above equations, we get

- $y_1 = \dfrac{q_5y_5 + q_2q_3}{y_4} = ... = y_1$
- $y_2 = \dfrac{q_1y_1 + q_3q_4}{y_5} = ... = y_2$

Remark 2.1. Under the specialization $q_1 = q_2 = q_3 = q_4 = q_5 = 1$, the phenomenon we just observed is nothing else but the *5-periodicity of the pentagon recurrence.*

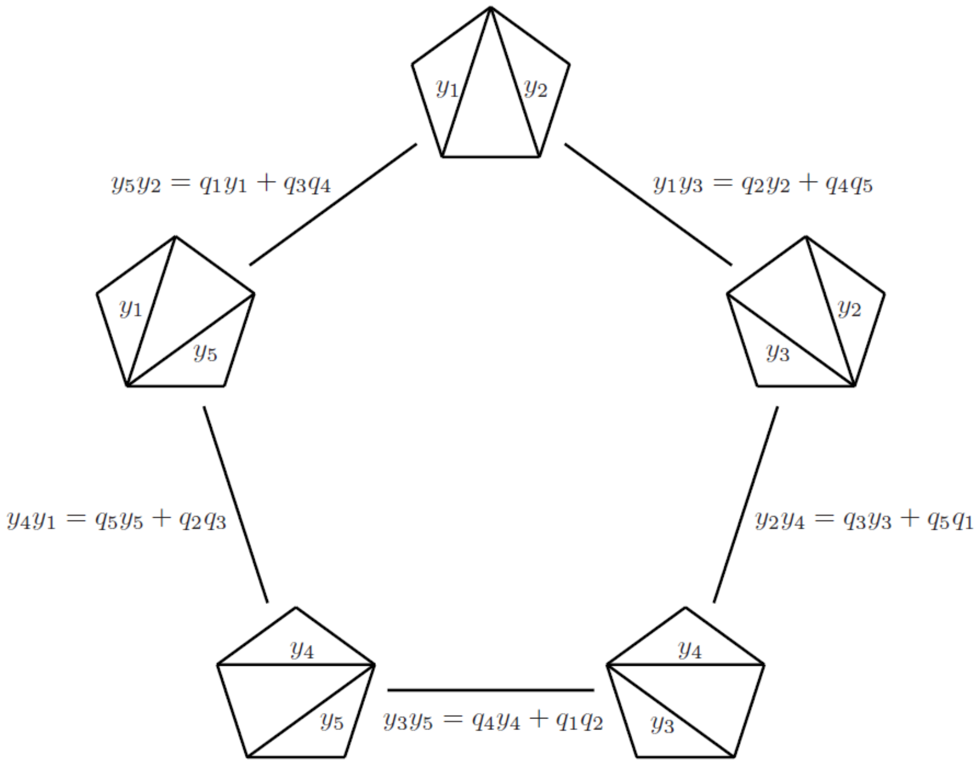


Figure 2.4: Exchange relations for the flips in a pentagon

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