Course Project
(Introduction to Reflection Groups)

Classification of finite dimensional semisimple Lie algebra over $\mathbb{C}$

Sarjick Bakshi
April 30, 2013

Abstract

A Lie algebra is an algebraic structure whose main use is in studying geometric objects such as Lie groups and differentiable manifolds. This is an introduction to Simple Lie Algebras and their classifications .

1 Introduction

In this report we study an example of simple Lie algebra, $sl_n(\mathbb{C})$ in detail and observe that its properties generalizes to any simple Lie algebra over $\mathbb{C}$.Corresponding to a simple Lie algebra L we have a Cartan decompostion and so we have a root system. We associate a matrix called Cartan matrix corresponding to a root system of $H_{\mathbb{R}}^*$ where H is a Cartan subalgebra of L and a diagram(graph) which turns out be connected , the quadratic form associted to it is positive definite and the number of bonds between any two nodes is atmost 4.Then we find all the diagrams having above properties called Dynkin diagram and we associate unique simple Lie algebra L upto isomorphism to each Dynkin diagram.

2 Lie algebra

Definition 2.1. *A **Lie Algebras** is a vector space L over K equipped with a Lie Bracket $[,] : L \times L \to L$, which satisfies*

Bilinearity :

$$[ax + b, z] = a[x, z] + b[y, z] \ a, b \in K$$

$$[x, ay + b] = a[x, y] + b[x, b] \ a, b \in K$$

Alternating on L *:*

$$[x, x] = 0 \textit{ for all } x \in L$$

The Jacobi identity:

$$[x, [y, z]] + [z, [x, y]] + [y, [z, x]] = 0 \textit{ for all } x, y, z \in L$$

$M(n, K)$ the set of $n \times n$ matrices can be made into a Lie algebra by defining $[A, B] = AB - BA$ and will denote it by $gl(n, K)$.
In general, any associative K algebra A can be made into Lie algebra by defining $[a, b] = ab - ba$.

Definition 2.2. *Let L be a Lie Algebra over K .A representation of L is a Lie algebra homomorphism $\rho : L \to gl(n, K)$ for some n called the degree of the representation. Two representations ρ, ρ' of degree n said to be equivalent if there is a non-singular $n \times n$ matrix T over K such that $\rho'(x) = T^{-1}\rho(x)T$, for all $x \in L$*

Definition 2.3. *A Lie algebra L is said to be simple if it has no nonzero proper ideal.*

Example 2.1. *Let $sl_n(\mathbb{C})$ be the set of all $n \times n$ matrices of trace $0.sl_n(\mathbb{C})$ is an ideal of $gl_n(\mathbb{C})$,which is clearly nonzero. Thus $gl_n(\mathbb{C})$ is not simple ,while we show that $sl_n(\mathbb{C})$ is simple To see this suppose we have a non-zero ideal I and take a non-zero element in this ideal. By multiplying this element on the left or right by suitable elementary matrix E_{ij} with $i \neq j$ we may simplify its form ,while remaining within the ideal I.*
Eventually we see that I contains some elementary matrix E_{ij},and by further multiplication we see readily that I is the whole $sl_n(\mathbb{C})$.Thus $sl_n(\mathbb{C})$ is simple.

We shall describe certain properties of $sl_n(\mathbb{C})$.Let H be the set of diagonal $n \times n$ matrices of trace 0.Then H is a subalgebra of $sl_n(\mathbb{C})$ of dimensio $n - 1$. Futhermore we have $[H, H]$= 0,so H is abelian.We recall that L may be considered as $L-$module,using $[L, L] \subset L$.We thus have $[H, L] \subset L$ and so we may regard L as a left $H-$module.We may write down a decomposition of L as a direct sum of $H-$submodules:

$$sl_n(\mathbb{C}) = H \oplus \sum_{i \neq j} \mathbb{C}E_{ij}$$

We note that the $1-$dimensional space $\mathbb{C}E_{ij}$ is an $H-$ submodule since,for $x \in H$,we have $x = \begin{pmatrix} \lambda_1 & & & & 0 \\ & . & & & \\ & & . & & \\ & & & . & \\ 0 & & & & \lambda_n \end{pmatrix}$ with $\lambda_1 + \cdots + \lambda_n = 0$

and

$$[xE_{ij}] = (\lambda_i - \lambda_j)E_{ij}.$$

This $H-$ module gives a $1-$ dimensional representation of H

$$\begin{pmatrix} \lambda_1 & & & & 0 \\ & \cdot & & & \\ & & \cdot & & \\ & & & \cdot & \\ 0 & & & & \lambda_n \end{pmatrix} \longrightarrow \lambda_i - \lambda_j$$

Note that there are $n(n-1)1-$ dimensional representation of H arising in this way.They are called the **roots** of $sl_n(\mathbb{C})$ with respect H.Let Φ denote the set of all roots which lies in the dual space H^* of H.Note that if $\alpha \in \Phi$ then $-\alpha \in \Phi$ which gives that Φ is not linearly independent, while we will see that it spans H^*.For this define $\alpha \in H^*$ by

$$\alpha_i(x) = \lambda_i - \lambda_{i+1}.$$

Then $\Pi = \{\alpha_i, ..., \alpha_{n-1}\}$ is linearly independent and form a basis of H^* as dim$H^* = (n-1)$,Π is called a set of **fundamental roots ,or simple roots**. We consider the way in which the roots are expressed as linear combinations of the fundamental roots.The root $x \longrightarrow \lambda_i - \lambda_j$ is equal to

$$\alpha_i + \alpha_{i+1} ... + \alpha_{j-1} \text{ if } i < j$$

and to

$$-(\alpha_j + \alpha_{j+1} + ... + \alpha_{i-1}) \text{ if } i > j$$

Thus each root in Φ is a linear combination of fundamental roots with coefficients in $\mathbb{Z}$ which are either all non-negative or all non-positive. Thus we may write $\Phi = \Phi^+ \cup \Phi^-$ where Φ^+ (respectively Φ^-)consists of positive(negative) combinations of Π.

Definition 2.4. *A subalgebra H of L is called a* ***Cartan subalgebra*** *if H is nilpotent and $H = I(H)$,where $I(H) = \{x \in L : [yx] \in H$ for all $y \in H\}$.*

Remark: $I(H)$ is a subalgebra of L containing H, and that H is ideal of $I(H)$.Moreover if H is an ideal of some other subalgebra M of L then $M \subset I(H)$.

Theorem 2.5. *:Every finite dimensional Lie algebra L over $\mathbb{C}$ has a cartan subalgebra.Moreover given any two cartan subalgebra H_1 and H_2 of L there exists an automorphism θ of L such that $\theta(H_1) = H_2$.*

Proof. Refer [1]. □

Example 2.2. *Let $L = sl_n(\mathbb{C})$ and H be the subalgebra of diagonal matrices in L.Then H is a cartan subalgebra of L.Since $[H, H] = 0$,H is clearly nilpotent.To show $H = I(H)$*
let $\sum_{i,j} a_{ij}E_{ij}$ be any element of $I(H)$.Choose $p, q \in \{1, ..., n\}$ with $p \neq q$.Then $E_{pp} - E_{qq} \in H$,hence

$$[\sum_{i,j} a_{ij}E_{ij}, E_{pp} - E_{qq}] \in H$$

This gives

$$\sum_i a_{ip}E_{ip} - \sum_i a_{iq}E_{iq} - \sum_j a_{pj}E_{pj} - \sum_j a_{qj}E_{qj} \in H$$

Since this matrix is diagonal we deduce,by considering the coefficient of E_{pq}*, that* $a_{pq} = 0$*.Since this is true for all* p, q *with* $p \neq q$ *we have* $\sum a_{ij}E_{ij} \in H$*.Thus* $H = I(H)$*.*

3 Cartan decomposition

Let L be a simple non-trivial Lie algebra over $\mathbb{C}$ and H be a Cartan subalgebra of L.Then $[H, H] = 0$ being proper ideal of L.We can write

$$L = H \bigoplus \sum_\alpha \mathbb{C}e_\alpha$$

,where $\mathbb{C}e_\alpha$ is $1-$dimensional $H-$module,thus we have $[xe_\alpha] = \alpha(x)e_\alpha$ $\alpha(x) \in \mathbb{C}$ for all $x \in H$. $\alpha \in H^*$.The $1-$dimensional representation α of H arising in the Cartan decomposition are called **roots of L with respect to H**.The set of roots will be denoted by Φ. Φ has the same properties as we have observed while dicussing $sl_n(\mathbb{C})$.

4 The Killing form

We consider the map $L \times L \longrightarrow \mathbb{C}$ defined by $< x, y >$=trace($adxady$).If L is non-trivial simple Lie algebra over $\mathbb{C}$. Then Killing form satisfies following properties:

1. It is symmetric bilinear form on L
2. It is non-degenerate on L
3. Its restriction to a Cartan subalgebra H is non-degenerate

Thus we may define a map $H \longrightarrow H^*$ given by $x \longrightarrow f_x$,where

$$f_x(y) =< x, y > \text{ for all } y \in H$$

Since the Killing form is non-degenerate on H this map is an isomorphism. Thus each element of H^* has form f_x for unique $x \in H$.We may define a map $H^* \times H^* \longrightarrow \mathbb{C}$ by

$$< f_x, f_y >=< x, y > \text{ for } x, y \in H.$$

We may restrict this bilinear form to the real vector space $H^*_\mathbb{R}$.It can be shown that its values then lie in $\mathbb{R}$ .Then we have a map

$$H^*_\mathbb{R} \times H^*_\mathbb{R} \longrightarrow \mathbb{R}$$

This scalar product is positive definite on $H^*_\mathbb{R}$.Therefore $H^*_\mathbb{R}$ is Euclidean space.This Euclidean space contains the set of roots Φ.

Example 4.1. *Let* $L = sl_2(\mathbb{C})$*.Then* $dimH = 1$*.Let* $\Pi = \{\alpha_1\}$*.Then* $\Phi = \{\alpha_1, -\alpha_1\}$*.The configuration formed by* Φ *us the* $1-$*dimensional Euclidean space* $H^*_\mathbb{R}$*.*

Now let $L = sl_3(\mathbb{C})$*.Then* $dimH = 2$*.Let* $\Pi = \{\alpha_1, \alpha_2\}$*.Then we have* $\Phi = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2, -\alpha_1, -\alpha_2, -\alpha_1 - \alpha_2\}$ *The configuration formed by* Φ *is the* $2-$*dimensional Euclidean space* $H^*_\mathbb{R}$

5 The Weyl group

For each $\alpha \in \Phi$ let $s_\alpha : H^*_{\mathbb{R}} \longrightarrow H^*_{\mathbb{R}}$ be the map defined by

$$s_\alpha(\lambda) = \lambda - 2\frac{<\alpha,\lambda>}{<\alpha,\alpha>}\alpha$$

It can be easily seen that s_α is the reflection in the hyperplane orthogonal to α.Let W be the group generated by the maps s_α for all $\alpha \in \Phi$.W is called the **Weyl group**. W has following properties :

1. It permutes the roots
2. W is a finite
3. Given any $\alpha \in \Phi$ there exists $\alpha_i \in \Pi$ and $w \in W$ such that $\alpha = w(\alpha_i)$.In short $\Phi = W(\Pi)$
4. W is generated by the s_{α_i} for $\alpha_i \in \Pi$. The importance the Weyl group is that it enables us to reconstruct the full root system Φ given only the set Π.

An example when $L = sl_3(\mathbb{C})$ is follows
Given α_1, α_2 the remaining roots are obtained by reflecting successively by $s_{\alpha_1}, s_{\alpha_2}$. We note that

$$s_{\alpha_i}(\alpha_j) = \alpha_j - 2\frac{<\alpha_i,\alpha_j>}{<\alpha_i,\alpha_i>}\alpha_i$$

We define $A_{ij} = 2\frac{<\alpha_i,\alpha_j>}{<\alpha_i,\alpha_i>}$ and call them **Cartan integers** and $A = (A_{ij})$ is called the **Cartan matrix**. It can be shown that A_{ij} are non-positive integers if $i \neq j$ and is equal 2 if $i = j$
Let θ_{ij} be the angle between α_i, α_j.Then using the cosine formula we obtain $4\cos^2(\theta_{ij}) = A_{ij}A_{ji} = n_{ij}(\text{say})$. The we see that $n_{ij} = \{0, 1, 2, 3\}$. We shall encode this information about the Π in terms of graph.

6 The Dynkin diagram

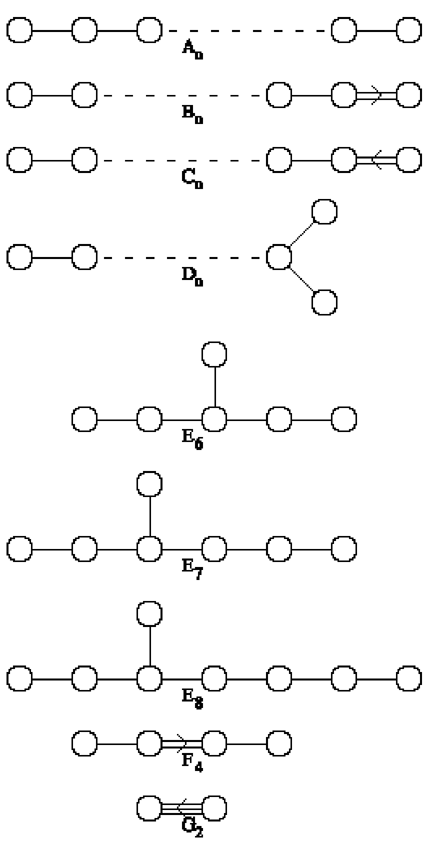
The Dynkin diagram Δ of L is the graph with nodes labelled $1, ..., l$ where l =dim$H$,where H is Cartan subalgebra of L and l is also called the rank of L;in bijective correspondence with element of Π such that nodes i, j with $i \neq k$ are joined by n_{ij} bonds.

Example 6.1. *: Let* $L = sl_3(\mathbb{C})$.*Then* $\Pi = \{\alpha_1, \alpha_2\}$ *and* $s_{\alpha_1}(\alpha_2) = \alpha_1 + \alpha_2$ *and* $s_{\alpha_2}(\alpha_1) = \alpha_1 + \alpha_2$. *Thus* $A_{12} = -1$, $A_{21} = -1$ *and so* $n_{12} = 1$.

Remark The Dynkin diagram is uniquely determined by L.The choice of the Cartan subalgebra doest matter as any two Cartan subalgebra are related by some automorphism of L.the choice of fundamental systems Π_1, Π_2 have the property that $\Pi_2 = w(\Pi_1)$ for some $w \in W$.

The Dynkin diagram has following properties :

1. Δ is connected graph if L is non-trivial simple Lie algebra.
2. Any two nodes are joined by at most 3 bonds.
3. Also let $Q(x_1, ..., x_l)$ be the quadratic form



$$Q(x_1, ..., x_n) = 2\sum_{1\le i\le l} x_i^2 - \sum_{i\neq j} \sqrt{n_{ij}} x_i x_j$$

This quadratic form is determined by the Dynkin diagram and is positive definite.

Theorem 6.1. *:Consider graphs Δ with the following properties:*

1. *Δ is connected,*
2. *The number of bonds joining any two nodes is $0, 1, 2, 3, 4$,*
3. *The quadratic form Q determined by Δ is positive definite.*

Then Δ must be one of the graphs on the above list:

The graphs on this list will be called Dynkin diagrams.

Proof. Refer [2]. □

We now consider to what extent the Dynkin diagram determines the matrix of Cartan integers.We recall that

$$n_{ij} = A_{ij}A_{ji} \; i \neq j$$

and that A_{ij}, A_{ji} are integers ≤ 0.Moreover,$A_{ij} = 0$ iff $A_{ji} = 0$. If $n_{ij} = 0$ then $A_{ij} = 0 = A_{ji}$.If $n_{ij} = 1$ then $A_{ij} = -1 = A_{ji}$.If

$n_{ij} = 2$ however,there are two possible factorisations of n_{ij}.Either $A_{ij} = -1, A_{ji} = -2$ or we have $A_{ij} = -2, A_{ji} = -1$.Since

$$A_{ij} = \frac{2<\alpha_i,\alpha_j>}{<\alpha_i,\alpha_i>}$$

We have

$$\frac{A_{ij}}{A_{ji}} = \frac{<\alpha_j,\alpha_j>}{<\alpha_i,\alpha_i>}$$

Thus in the first case above we have $<\alpha_i,\alpha_i>><\alpha_j,\alpha_j>$ and reverse the inequality in the second case.We distinguish between these two cases by putting an arrow on the Dynkin diagram pointing towards them long root. Similarly if $n_{ij} = 3$ we get two possible factorisations $n_{ij} = A_{ij}A_{ji}$ which are distinguished by putting an arrow on the given triple bond.
The main theorem on the classification of finite dimensional simple Lie algebras over $\mathbb{C}$ is as follows:

Theorem 6.2. *Let L be a finite dimensional simple non-trivial Lie algebra over $\mathbb{C}$.Then the Cartan matrix of L is one of those on the standard list:*

$$A_l\ l \geq 1\ B_l\ l \geq 2\ C_l\ l \geq 3\ D_l\ l \geq 4,$$
$$E_6, E_7, E_8, F_4, G_2$$

Moreover for any Cartan matrix on the standard list there is just one simple Lie algebra ,up to isomorphism ,giving rise to it.

Proof. Refer [2]. □

The dimensions of the simple Lie algebras may be calculated as follows: The Dynkin diagram determines the configuration formed by the set Π of fundamental roots .i.e., the angles between the fundamental roots and their relative lengths.We may then obtain the full root system Φ by successive reflection by elements of the Weyl group.Also from the Cartan decomposition of L it is clear that dim$L$ =dim$H + |\Phi|$.The dimensions of the simple Lie algebras are given in the following table:

$$\dim A_l = l(l+1)$$
$$\dim B_l = l(2l+1)$$
$$\dim C_l = l(2l+1)$$
$$\dim D_l = l(2l-1)$$

$$\dim G_2 = 14$$
$$\dim F_4 = 52$$
$$\dim E_6 = 78$$
$$\dim E_7 = 133$$
$$\dim E_8 = 248$$

The algebras A_l, B_l, C_l, D_l are called classical Lie algebras.
The simple Lie algebra A_l is isomorphic to the $sl_{l+1}(\mathbb{C})$.

The simple Lie algebra B_l is isomorphic to the Lie algebra $so_{2l+1}(\mathbb{C})$ of all $(2l+1) \times (2l+1)$ skew-symmetric matrices.It is also isomorphic to the Lie algebra of all $(2l+1) \times (2l+1)$ matrices satisfying $TA + AT^t = 0$ where

$$A = \begin{pmatrix} 1 & 0 & . & . & . & 0 \\ 0 & 0 & & & & I_l \\ . & & & & & \\ . & & & & & \\ . & & & & & \\ 0 & I_l & & & & 0 \end{pmatrix}$$

The simple Lie algebra C_l is isomorphic to the Lie algebra of $2l \times 2l$ matrices T satisfying $TA + AT^t = 0$ where

$$A = \begin{pmatrix} 0 & I_l \\ -I_l & 0 \end{pmatrix}$$

The simple Lie algebra D_l is isomorphic to Lie algebra $so_{2l}(\mathbb{C})$ of all $2l \times 2l$ skew-symmetric matrices.It is also isomorphic to Lie algebra 0f $2l \times 2l$ matrices T satisfying the condition

$$TA + AT^t = 0$$

where

$$A = \begin{pmatrix} 0 & I_l \\ I_l & 0 \end{pmatrix}$$

The advantage of this description of L is that the diagonal matrices in L form a Cartan subalgebra H,and the cartan decomposition can be readily obtained.

References

[1] James E. Humphyreys, *Introduction to Lie Algebras and Representation Theory*, Springer

[2] Carter ,Roger W. *Lectures on Lie algebras and Lie groups*, Cambridge University Press

[3] Alexandre V. Borovik, Anna Borovik, *Mirrors and Reflections : The Geometry of Finite Reflection Groups*, Springer, 2010