Course Project
(Introduction to Reflection Groups)

Construction and Properties of the Icosahedron

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Abstract

The icosahedron is one of the most important platonic solids in Euclidean geometry, admitting as it does a highly rich group of symmetries. The construction of the icosahedron is also an interesting topic and this paper gives an outline of Taylor's and Kepler's methods of construction. Some nice geometrical properties of the icosahedron and a few interesting facts about its symmetry group are also discussed, with special emphasis on the orientation-preserving or rotational symmetries.

1 Introduction

In Euclidean geometry, a regular convex polyhedron is referred to as a platonic solid. Such solids have an equal number of congruent, regular, polygonal faces meeting at each vertex. There are five such platonic solids in existence and the icosahedron, the construction and properties of which we are about to discuss, is one of the most important. It has 20 equilateral triangles as its faces, with five such triangles meeting at each vertex. Clearly, it follows that it has 12 vertices and 30 edges. The icosahedron is the dual of the dodecahedron, another platonic solid having three regular pentagonal faces meeting at each vertex. Thus, while the dodecahedron is represented by the Schlăfli symbol [5,3], the icosahedron is represented by [3,5], and, being duals, the symmetry groups of the two polytopes are also the same.
A regular icosahedron has a symmetry group of order 120, out of which 60 are orientation-preserving or rotational symmetries. This rotational symmetry

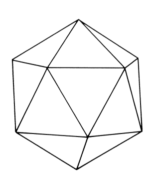


Figure 1: A Regular Icosahedron

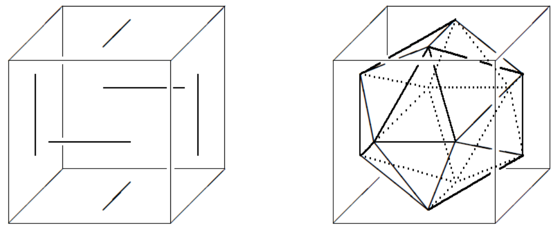


Figure 2: An illustration of Taylor's method

group is isomorphic to A_5, the alternating group of five letters, but we must carefully note that the full icosahedral symmetry group, (involving both rotational and reflectional symmetries), is not isomorphic to S_5 , but to $A_5 \times C_2$, though both S_5 and the icosahedral symmetry group I_h have order 120 and have A_5 as a subgroup of index 2.

2 Construction of the Icosahedron

Perhaps one of the easiest methods of constructing an icosahedron is due to H.M. Taylor. Suppose that we wish to construct an icosahedron of edge length, say, 2. For that, we take a cube of edge length 2ϕ centred at the origin, where $\phi = \frac{\sqrt{5}+1}{2}$ is the golden ratio. (In general, for constructing an icosahedron of edge length p, we take a cube of edge length $\phi \times p$.)

Thereafter, we mark the six line segments joining the twelve points $(0, \pm 1, \pm\phi)$, $(\pm 1, \pm\phi, 0)$, $(\pm\phi, 0, \pm 1)$ in alternating fashion and join each pair of these twelve points by a line segment. The polytope obtained by this construction is what we choose to call an icosahedron, though we are far from knowing whether our construction gives a unique polytope or not. However, what we can easily observe is that the polytope just constructed has each of its sides of equal length and at each of its vertices, five equilateral triangles meet. So, it at least satisfies the basic geometrical properties of the icosahedron. Also note that our construction at least proves the existence of the icosahedron, because it follows from the principle of continuity (the distance between two points is a continuous function of their coordinates and therefore assumes all intermediate values) that by varying the lengths of the line segments drawn on the cube, we can get all edges of the inscribed polytope to be equal at a certain length of the segments.

The second common method of constructing an icosahedron is due to Kepler. In this method, we first construct a pentagonal biprism, the existence of which is guaranteed by the principle of continuity, and then add two pentagonal pyramids with all edges equal-one at the top and the other at the bottom.

The next question is whether the constructed icosahedron is unique, at least up to isometry. The two methods that we described above are quite different from each other and yet we have chosen to call the polyhedrons constructed in

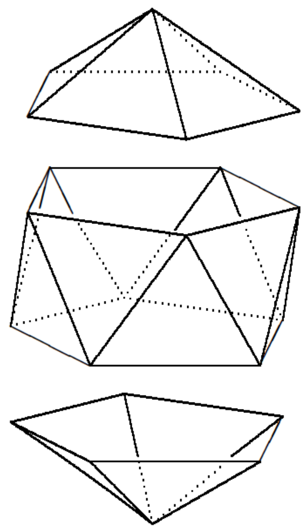


Figure 3: Kepler's method of Construction

each case by the common name of 'icosahedron'. That this is justified can be established by Cauchy's rigidity theorem, which we discuss next.

3 Cauchy's Theorem and the Uniqueness of our Construction

Now, we prove the uniqueness of our construction using Cauchy's theorem. We first give the definition of a polytopal map.

Definition. *Let Δ and Δ' be two convex polytopes and F, F' the sets of their faces. A map $\alpha : F \to F'$ is said to be polytopal if :*

• *α takes vertices into vertices, edges into edges, and faces into faces.*

• *α preserves the adjacency of faces, i.e., the common edge of two neighbouring faces is mapped to the common edge of the images of the faces.*

Now Cauchy's theorem can be stated as follows :

Theorem. *Let Δ and Δ' be two convex polytopes and F, F' the sets of their faces. Let $\alpha : F \to F'$ be a polytopal map such that for every face F of Δ, there is an isometry $I_F : F \to \alpha(F)$ which agrees with the map α on all edges of F,i.e, if E is an edge then the image $I_F(E)$ of E coincides with $\alpha(E)$.*

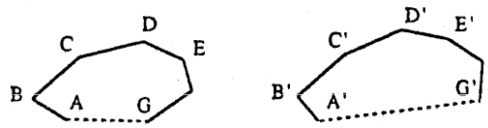


Figure 4: The so-called Arm Lemma

Then there is an isometry $I : \Delta \to \Delta'$ that agrees with α, that is, if F is an arbitrary face of Δ, then the image $I(F)$ of F coincides with $\alpha(F)$.

Stated in a simpler form, the theorem states that any two convex polyhedra consisting of the same number of equal similarly placed faces are isometric.

The proof of Cauchy's theorem is in 3 parts and below we give a bare outline of the idea behind the proof. The first part of the proof is a lemma, commonly referred to as the Arm lemma, which we state below without proof.

Lemma 1. *Let $ABC...G$ and $A'B'C'...G'$ represent two convex planar polygons satisfying $AB = A'B, BC = B'C', ...FG = F'G'$ and, for the corresponding angles, $\angle ABC \leq \angle A'B'C', \angle BCD \leq \angle B'C'D', ...\angle EFG \leq \angle E'F'G'$. Then $AG \leq A'G'$, with equality holding iff the polygons are congruent. (see Fig. 4)*

A result which follows from the Arm Lemma is the following :

Lemma 2. *Let P and P' be two convex planar polygons with an equal number of sides and with the length of the corresponding sides equal. We assign + and - signs for those vertices of P where the internal angle increases or decreases, respectively, in going from P to P'. (In case there is no change, no sign is assigned.) Then there are at least four changes in sign as we go around P or there is no sign attached to any vertex, which happens iff P and P' are congruent.*

Proof. Firstly, we note that if there is a change of sign at all, there can't be an odd number of sign changes, and so, if there are fewer than four, the number of sign changes can be two or zero and, in the latter case, the polytopes are easily seen to be congruent. So, if possible, let there be two changes of sign. We then choose a line segment with its endpoints lying on the two edges of P seperating the two signs. The Arm lemma then forces that the length of this segment increases as we go from P to P' and also when gong from P' to P, which is impossible as the two length changes must be opposite for the length of the segment to be preserved in P. This contradiction forces that there must be at least four sign changes, if at all. ☐

We now define a triangulation of a sphere and state another lemma, commonly referred to as Cauchy's combinatorial lemma.

Definition. *A triangulation of a sphere S is a partition of S into spherical triangles (regions bounded by three arcs such that the internal angle at each vertex is less than π) by the non-intersecting diagonals of an inscribed spherical polygon.*

Lemma 3. *Suppose + and - signs are associated with some of the edges of a triangulated sphere so that at each vertex with some labeled edge, there are at least four changes of sign as one goes around the vertex. Then none of the edges are labeled.*

The proof of the rigidity theorem can now be easily completed. If P and P' are two convex polyhedra, we label each edge of P with a + or a - sign depending on whether the dihedral angle at the edge increases or decreases. By taking a small sphere centred at each of P's vertices, we conclude that there must be at least four sign changes around each vertex with at least one labeled edge coming into it. Cauchy's combinatorial lemma then forces that there is actualy no sign change at all, which can only happen if P and P' are isometric.

Now, let's see how Cauchy's theorem helps us in proving the uniqueness of the constructed icosahedron. In both the methods of construction we described, the ways the polytopes are assembled from equilateral triangles are the same. The rigidity theorem then automatically implies that as soon as we ensure that the edge lenghths for the polytopes are equal for both the methods, the polytopes will turn out to be isometric. So, Kepler's and Taylor's polytopes are identical, at least up to isometry and we call it an icosahedron. In fact, for most methods of constructing an icosahedron, a simple application of Cauchy's theorem will establish that the constructed polytope is isomtric to, say, Taylor's polytope having the same edge length.

4 The Icosahedral Symmetry Group

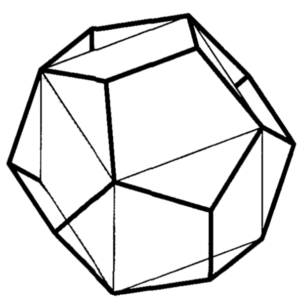
Let us recollect how we constructed the icosahedron by Taylor's method. We already saw that this construction is the same as any other, since Cauchy's theorem guarantees uniqueness in any case. We can easily verify that the symmetry group $\mathrm{Sym}(\Delta)$ of the constructed polytope acts transitively on the set of vertices of Δ. Actually, the group of symmetries of the ambient cube has a subgroup of order 24 that preserves Δ and acts transitively on the set of vertices. In fact, it's true that $\mathrm{Sym}(\Delta)$ acts transitively not only on the set of vertices, but also on the edges and faces of the icosahedron.

Now, let's analyze the symmetry group of the icosahedron a bit more, concentrating first on the rotational or orientation-preserving symmetries. One can easily check that rotating an icosahedron by angles of $\frac{2\pi}{5}$, $\frac{4\pi}{5}$, $\frac{6\pi}{5}$ or $\frac{8\pi}{5}$ about any axis joining the extreme opposite vertices , or by angles of $\frac{2\pi}{3}$ or $\frac{4\pi}{3}$ about any axis joining the centres of opposite faces or by an angle of π about any axis joining the midpoints of opposite edges preserves symmetry. We also note that the number of such axes of rotation are 6 for the first type, 10 for the second and 15 for the third. Including the identity, the order of this rotation group thus comes out to be

$$6.4 + 10.2 + 15.1 + 1 = 60 \tag{1}$$

Theorem. *If I is the rotational symmetry group of an icosahedron, $I \simeq A_5$.*

Proof. The first observation we make is that I is a simple group. This is because the order of a proper normal subgroup of I divides 60 and additionally, it must also be the sum of some of the terms on the right side of the class equation (1), including the term 1, which is the order of the conjugacy class of the identity element. Since there is no integer satisfying both these conditios, there can't be any such proper normal subgroup and hence I must be simple.



To show that I is isomorphic to A_5, we first of all need to find a set of 5 elements on which I operates. It can be shown that the set of 5 cubes that can be inscribed in the dodecahedron (one such cube is shown in the figure) is such a set and we can take I to act on it by the operation of permutation. Corresponding to the permutation operation on the cubes inscribed in the dodecahedron, there is an operation on the inscribed cubes of the icosahedron as the polytopes are duals.

Now, let $\phi : I \to S_5$ be the permutation representation corresponding to this operation. The kernel of ϕ is a normal subgroup of I and, I being simple, $ker(\phi)$=$\{1\}$ or the whole group I. Since the operation is not trivial, $ker(\phi)$ is not I, i.e, $ker(\phi)$=$\{1\}$. This means that ϕ is injective and defines an isomorphism from I to its image in S_5.

Now, consider the sign homomorphism $\sigma : S_5 \to \{\pm 1\}$ and compose it with ϕ to obtain the homomorphism $\sigma\phi : I \to \{\pm 1\}$. If this homomorphism is surjective, its kernel will be a proper normal subgroup of I, which is impossible.

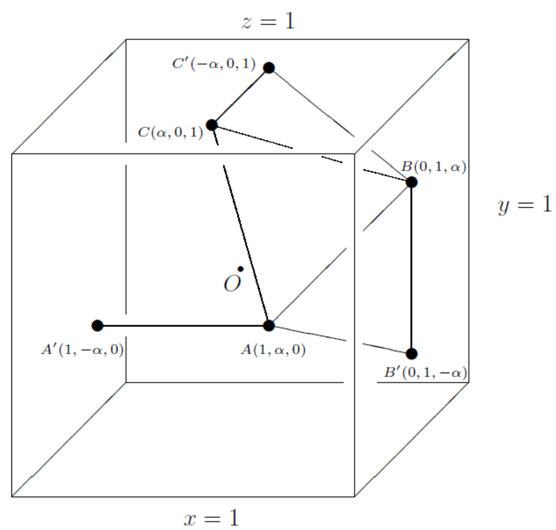


Figure 5: The icosahedron inscribed in the cube $[-1, 1]^3$ of edge length 2. The icosahedron has edge length 2α. The coordinates of the endpoints of the segments marked on the cube are shown for the $x = 1, y = 1$ and $z = 1$ faces.

So the restriction is the trivial homomorphism, implying that the image of ϕ is contained in the kernel of σ, which is A_5. Since both I and A_5 have order 60 and ϕ is injective, we conclude that $I \simeq im(\phi) \simeq A_5$. $\square$

So, we have more or less analysed the group I of rotational symmetries of the icosahedron. If, however, we consider both rotational and reflectional symmtries, the analysis is more cumbresome. To simplify the discussion, we yet again take the help of Cauchy's theorem.

We define a flag of an icosahedron Δ to be a triple (V,E,F) such that the vertex V is an end-point of the edge E and E itself is a side of the face F. Then there is a map α from one flag (V,E,F) to another (V',E',F') sending V to V', E to E' and F to F'. It can be shown that this map is unique also. Cauchy's theorem then implies that there is an unique isometry of Δ that sends (V,E,F) to (V',E',F'), i.e, Sym(Δ) acts transitively on the set of flags.

So, if the number of flags is N, $|Sym(\Delta)| = N$.

Now, let's calculate N. Let's fix a vertex V. The number of ways this can be done is $\binom{12}{1}$=12 since there are 12 vertices to choose from. For each such vertex, E can be chosen in $\binom{5}{1}$=5 ways because 5 edges propagate from a vertex and since the number of faces meeting at an edge is 2, F can be chosen in $\binom{2}{1}$=2 ways once V and E are fixed.

Thus, N=$12\times5 \times 2 = 120$ and so the icosahedral symmetry group has order 120. It can be verified, however, that it's not isomorphic to S_5, but to $A_5\times C_2$. However, it has a subgroup of order 60 (the group of rotational symmetries I we already saw) that has index 2 and, as we saw, that subgroup is isomorphic to A_5.

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