Introduction to Manifolds

Assignment 2
Due Date: 4/09/2017

Problem 1: In each of the following questions show that the given function f is one-to-one on the given set A. Sketch A and $B = f(A)$. For $p \in B$ find $Df^{-1}(p)$.

1. $f(x, y) = (x^2 - y^2, 2xy)$, $A = \{(x, y) \mid x > 0\}$ and $p = (0, 1)$.
2. $f(x, y) = (e^x \cos y, e^x \sin y)$, $A = \{(x, y) \mid 0 < y < 2\pi\}$ and $p = (0, 1)$.

Problem 2: Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be given by the equation $f(x) = \|x\|^2 \cdot x$.

1. Show that f is a smooth function.
2. Show that f maps the unit ball onto itself in a one-to-one fashion.
3. Is the inverse function differentiable at the origin ? Why ?

Problem 3: Let $g : \mathbb{R}^2 \to \mathbb{R}^2$ be given by $g(x, y) = (2ye^{2x}, xe^y)$ and $f : \mathbb{R}^2 \to \mathbb{R}^3$ be given by the equation $f(x, y) = (3x - y^2, 2x + y, xy + y^3)$.

1. Show that there is a neighborhood of $(0, 1)$ that g carries in a one-to-one fashion onto a neighborhood of $(2, 0)$.
2. Find $D(f \circ g^{-1})$ at $(2, 0)$.

Problem 4: Let $U \subset \mathbb{R}^n$ be open; let $f : U \to \mathbb{R}^n$ be a smooth function; assume $Df(x)$ is non-singular for $x \in U$. Show that even if f is not one-to-one on U the image is open in $\mathbb{R}^n$.

Problem 5: The function $f : \mathbb{R}^2 \to \mathbb{R}$ is given by

$$f(x, y) = x^2 + y^2 - 5.$$

1. Explain why there exist a unique single variable continuous function g such that $y = g(x)$ in a neighborhood of 1 ? Find g.
2. Draw a picture depicting the neighborhood and its image.