Reflection Groups

Homework 4

Due date: 14/03/2013

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1. Prove that the reflection groups of type A_3 and D_3 are isomorphic by exhibiting an explicit isometry between the corresponding root systems.

2. Draw the hyperplane arrangement corresponding to the root system BC_2 and label its chambers by the elements in the dihedral group of order 8.

Use the following information for the remaining problems. Φ is a root system in $\mathbb{R}^n$, Σ is the corresponding arrangement of reflecting hyperplanes, W is the group generated by reflections which acts transitively on $\mathcal{C}$, the chambers of the reflection arrangement. For two chambers C and D let $\mathcal{S}(C, D)$ denote the set of all hyperplanes in Σ that separate them (i.e., the set of all those hyperplanes such that C and D lie on their opposite sides). Recall that for a chamber C the chamber opposite to it is denoted by $-C$.

3 Prove that for chambers C_1, C_2, C_3 the following is true:

$$\mathcal{S}(C_1, C_3) = [\mathcal{S}(C_1, C_2) \setminus \mathcal{S}(C_2, C_3)] \bigcup [\mathcal{S}(C_2, C_3) \setminus \mathcal{S}(C_2, C_1)].$$

4 Given a face F and a chamber C of Σ prove that there is a unique chamber C_F satisfying the following:

- $F \subseteq \overline{C_F}$,
- $\mathrm{gd}(C, C_F) = \min\{\mathrm{gd}(C, D) \mid D \in \mathcal{C}, F \subseteq \overline{D}\}$.

Moreover, conclude that if $F \subseteq \overline{G}$ then $(C_F)_G = C_G$ and that if $F \subseteq \overline{C}$ then $C_F = C$.

5 Prove that $\mathrm{gd}(C, D) = |\mathcal{S}(C, D)|$.

6 For a chamber C prove the following assertions:

(a) $\mathrm{gd}(C, -C) = \mathrm{gd}(C, D) + \mathrm{gd}(D, -C)$ for every $D \in \mathcal{C}$.

(b) $\mathrm{gd}(C, -C) = |\Sigma|$.

(c) $\mathrm{gd}(C, D) = \mathrm{gd}(-C, -D)$ for every $D \in \mathcal{C}$.