Reflection Groups

Homework 3

Due date: 07/03/2013

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1. Prove that any finite group of rotations of the Euclidean plane $\mathbb{R}^2$ about the origin is cyclic. Prove that, in general, if it is a finite group of orthogonal transformations then it could be a dihedral group.

2. Prove that if r is a rotation of $\mathbb{R}^2$ and s is a reflection then sr is a reflection. Deduce from this that $srs^{-1} = r^{-1}$.

3. If D_{2n} is a dihedral group of order $2n$ then prove that D_{2n} has one conjugacy class of reflections if n is odd, and two if n is even.

4. Prove that in a planar root system, the lengths of roots can take at most two values. In particular conclude that in a root system of D_{2n}, with n odd, all roots have the same length.

Let V_1 and V_2 be two orthogonal subspaces of a vector space V and let Φ_i be a root system in V_i, $i = 1, 2$. Then $\Phi_1 \bigcup \Phi_2$ is a root system in V denoted by $\Phi_1 \oplus \Phi_2$.

5 Make a sketch of the root systems $A_1 \oplus A_1$ in $\mathbb{R}^2$, $A_1 \oplus A_1 \oplus A_1$ in $\mathbb{R}^3$, D_3 and B_3 in $\mathbb{R}^3$.

6 Prove that the reflection group of type BC_2 is isomorphic to D_8.

Bonus Prove that there exists a group, unique up to isomorphism, generated by two involutions such that their product has infinite order.