Reflection Groups

Homework 1

Due date: 18/01/2013

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1 Isometries of $\mathbb{A}^n$

1. Let $t_\alpha \in \text{iso}(\mathbb{A}^n)$ denote the translation through the vector $\overrightarrow{\alpha}$ and let $A \in \text{iso}(\mathbb{A}^n)$ be an orthogonal transformation then prove that

$$At_\alpha A^{-1} = t_{A\alpha}.$$

Definition 1.1. A map $f : \mathbb{A}^n \to \mathbb{A}^n$ is called a **similarity** if and only if for all points $w, x, y, z \in \mathbb{A}^n$ we have

$$d(w, x) = d(y, z) \Longleftrightarrow d(f(w), f(x)) = d(f(y), f(z));$$

i.e., f preserves equal distances.

The set of all similarities of $\mathbb{A}^n$ form a group under composition.

Definition 1.2. A map $f : \mathbb{A}^n \to \mathbb{A}^n$ is called an **elation** if and only if there exists a non-zero scalar k such that for all points $w, x \in \mathbb{A}^n$ we have

$$d(f(w), f(x)) = k \cdot d(w, x).$$

The constant k is called the *coefficient of elation* (or the *stretching factor*) of f.

The set of all elations of $\mathbb{A}^n$ form a group under composition.

2. (**Bonus**) Prove that every similarity is an elation.
(It should be clear that elations are similarity; you may take an extra week to submit the answer to this particular question.)

3. Prove that similarity preserves angles (i.e., $\angle xyz = \angle f(x)f(y)f(z)$).

4. Prove that elations of $\mathbb{A}^2$ can be characterized as maps that send straight lines to straight lines and preserve perpendicularity.

5. Prove that the group of elations is isomorphic to $(\mathbb{R}^n \rtimes \mathbb{O}_n) \rtimes \mathbb{R}_{>0}$. In fact, it is isomorphic to $\mathbb{R}^n \rtimes (\mathbb{O}_n \times \mathbb{R}_{>0})$