COMBINATORICS 1
ASSIGNMENT 6
(DUE DATE: 14/11/2016)

- The characteristic polynomial of a hyperplane arrangement $\mathcal{A}$ is defined as
$$\chi(\mathcal{A}, x) := \sum_{s \in L(\mathcal{A})} \mu(s) x^{\dim s}.$$
- Braid arrangement in $\mathbb{R}^n$ is given by $A_{n-1} = \{x_i - x_j = 0 \mid 1 \leq i < j \leq n\}$.
- Consider the following arrangement in $\mathbb{R}^n$
$$\mathcal{C}_n = A_{n-1} \cup \{x_i - x_j = \pm 1 \mid 1 \leq i < j \leq n\}.$$

---

(1) (10 points) Draw $\mathrm{ess}(\mathcal{C}_3)$ in $\mathbb{R}^2$.

(2) (10 points) Use the finite field method to show that
$$\chi(\mathcal{C}_n, x) = x(x - n - 1)(x - n - 2) \cdots (x - 2n + 1).$$
(Hint: Refer to the calculation we did for the Shi arrangement. You will have to use the same idea, but you might have to modify the definition of weak order partitions).

(3) (4 points) Use Zaslavsky's formula to compute $r(\mathcal{C}_n)$ and $b(\mathcal{C}_n)$.

(4) (16 points) Use the following steps to give a bijective method to count the number of regions.

  (a) Let $r_0(\mathcal{C}_n)$ denote the number of chambers contained in the $x_1 > x_2 > \cdots > x_n$ chamber, call it R_0, of A_{n-1}. Using the obvious action of the symmetric group prove that
  $$r(\mathcal{C}_n) = n! r_0(\mathcal{C}_n).$$
  Let $R_0(\mathcal{C}_n)$ be the set of chambers of $\mathcal{C}_n$ that are contained in the region R_0. It is enough to count their number.

  (b) Characterize the (type of) inequalities that define a chamber in $R_0(\mathcal{C}_n)$.

  (c) Denote by $\mathsf{PI}(n)$ the poset of all intervals $[i, j]$ of $[n]$ such that $i < j$, ordered by inclusion. For example, $\mathsf{PI}(3) = \{[1, 3], [1, 2], [2, 3]\}$ with ordering such that the first element is the largest and the remaining two are incomparable. Denote by $\mathsf{PI}^*(n)$ the antichains in $\mathsf{PI}(n)$ (including the empty antichain). Find a bijection between $\mathsf{PI}^*(n)$ and $R_0(\mathcal{C}_n)$.

(5) (Bonus 10 points): Find a formula for the cardinality of $\mathsf{PI}^*(n)$.