COMBINATORICS 1
ASSIGNMENT 5
(DUE DATE: 24/10/2016)

- All posets are finite and graded.
- An arrangement of hyperplanes in $\mathbb{R}^n$ is a finite collection $\mathcal{A} = \{H_1, \dots, H_k\}$ of hyperplanes.
- The characteristic polynomial of a hyperplane arrangement $\mathcal{A}$ is defined as
$$\chi(\mathcal{A}, x) := \sum_{s \in L(\mathcal{A})} \mu(s) x^{\dim s}.$$
- Braid arrangement in $\mathbb{R}^n$ is given by
$$A_{n-1} = \{x_i - x_j = 0 \mid 1 \leq i < j \leq n\}.$$
- The type B arrangement is given by
$$B_n = A_{n-1} \cup \{x_i + x_j = 0 \mid 1 \leq i < j \leq n\} \cup \{x_i = 0 \mid 1 \leq i \leq n\}.$$
- For a natural number a, define $[-a, a]^n_{\mathbb{Z}} = [-a, a]^n \cap \mathbb{Z}^n$.

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(1) (10 points) Let W be a k-subspace of $\mathbb{R}^n$. Then prove that
$$|W \cap [-a, a]^n_{\mathbb{Z}}| = |[-a, a]^k_{\mathbb{Z}}|$$
if and only if $W \in L(B_n)$.

(2) (15 points) For a hyperplane $H \subset \mathbb{R}^n$ given by the linear form $f(x) = \alpha$ define the cone over it as the hyperplane cH in R^{n+1} given by the linear form $f(x_1, \dots, x_n) - \alpha x_{n+1} = 0$. Let $\mathcal{A}$ be a non-central arrangement then the cone over this arrangement is
$$c\mathcal{A} = \{cH_1, \dots, cH_k, x_{n+1} = 0\}.$$
Prove that
$$\chi(c\mathcal{A}, x) = (x - 1)\chi(\mathcal{A}, x).$$

(3) (15 points) Let G be a graph and $\mathcal{A}_G$ be the associated graphic arrangement. Suppose that G has m-element clique, i.e., m vertices such that any two are joined by an edge. Show that $m! \mid r(\mathcal{A}_G)$.

(4) (10 points) An arrangement of type D is defined as
$$D_n = A_{n-1} \cup \{x_i + x_j = 0 \mid 1 \leq i < j \leq n\}.$$
Find the number of chambers of D_n.