COMBINATORICS 1
ASSIGNMENT 2 (DUE DATE: 22/08/2016)

Each problem is worth 10 points.

(1) Let n, l be fixed positive integers. Denote by $f(l, n)$ the number of sequences $(A_1, \dots, A_l)$ of subsets of the set $[n]$ such that $\bigcap_{i=1}^{l} A_i = \emptyset$. Find a recurrence relation for $f(l, n)$.

(2) Let $F_l(x)$ be the exponential generating function for the above numbers $f(l, n)$ given by:

$$F_l(x) = \sum_{n \geq 0} f(l, n) \frac{x^n}{n!}.$$

First show that $F_l(x) = e^x F_{l-1}(2x)$. Then use this identity to conclude that

$$f(l, n) = (2^l - 1)^n.$$

(3) Let n, k be fixed positive integers. Denote by $c(n, k)$ the number of permutations in S_n that have exactly k cycles (this number is also known as the signless Stirling number of the first kind). Then prove that

$$c(n, k) = (n-1)c(n-1, k) + c(n-1, k-1),$$

with the initial conditions $c(n, k) = 0$ if either $n < k$ or $k = 0$ except $c(0, 0) = 1$.

(4) Let n, k be fixed positive integers. Denote by $P(n, k)$ the number of partitions of an n-set as a disjoint union of k nonempty subsets ($k!P(n, k)$ is known as the Stirling number of the second kind). Prove the following recurrence:

$$\mathcal{P}(n, k) = kP(n-1, k) + P(n-1, k-1),$$

with the initial conditions $P(n, k) = 0$ if either $k = 0$ or $n < k$ and $P(n, k) = 1$ if either $k = 1$ or $k = n$.

(5) Find a closed form formula for the number of k-dimensional subspace of $\mathbb{F}_q^n$ where q is a prime power and $0 \leq k \leq n$.