Identical Particles in Quantum Mechanics

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December 26, 2010

1 Identical Particles

1.1 Describing a two state system

Consider the system comprising of 2- two state systems. For example, consider a system of two electrons. The state of the two state system is given by the tensor product of the individual states of the system. Let vectors $|k_1\rangle$ and $|k_2\rangle$ represent the individual states of the systems (1) and (2) respectively. (From now, it is implicit that the subscript denotes the particle number.) The state of this composite system is given by $|k_1k_2\rangle$ or, equivalently, $|k_2k_1\rangle$. Physically, we see no reason as to why we must prefer one to the other, but however mathematically they are orthogonal states.

$$\langle k_1k_2|k_2k_1\rangle = \delta_{k_1,k_2} \quad (1)$$

So, it is now evident that if we are given a state of the system, we do not know a priori whether the system is in state $|k_1k_2\rangle$ or in $|k_2k_1\rangle$. (In other words, if we are told that the state of the composite system is $|a,b\rangle$ then, we do not know whether the state of the first system is $|a\rangle$ or $|b\rangle$.) More generally, by the principle of superposition, the state of the two state system can be any linear combination of the states $|k_1k_2\rangle$ and $|k_2k_1\rangle$:

$$|\psi\rangle = c_1|k_1k_2\rangle + c_2|k_2k_1\rangle \quad (2)$$

Now, when a measurement (given by some measurement operator) is performed on this composite system given by $|\psi\rangle$, the eigenvalues produced by the two states $|k_1k_2\rangle$ and $|k_2k_1\rangle$ will be identical (since eigenvalues are just numbers and $k_1k_2 = k_2k_1$). So, now different eigenkets of $|\psi\rangle$ will have the same eigenvalues, thereby introducing degeneracy into the system. This is called the exchange degeneracy.

1.2 Permutation operator

In the previous subsection, we said that we could describe the same composite system using two orthogonal states. If the state of the system is $|k_1k_2\rangle$, and we interchange the particles 1 and 2, then we get the state $|k_2k_1\rangle$. So, the system is physically the same as before. To do this exchange of particles, we define an operator called the permutation operator, with the following property:

$$P_{21}|k_1k_2\rangle = |k_2k_1\rangle \quad (3)$$

From the above definition, it is evident that $P_{21} \equiv P_{12}$. Also,

$$P_{21}P_{21}|k_1k_2\rangle = P_{21}|k_2k_1\rangle \Rightarrow |k_1k_2\rangle : \ (P_{21})^2 = I$$
Also since \((P_{12})^2 = \mathbb{I}\), and the eigenvalue of \(\mathbb{I}\) is 1, we may say that the eigenvalues of \(P_{12}\) are \(\pm 1\). \(P_{12}\) is also Hermitian.

Hence, the permutation operator changes the state of particle 1 to \(|k_2\rangle\) and that of particle 2 to \(|k_1\rangle\). Let us now take some operator \(T\) where \(T = T_1 \otimes T_2\). The action of \(T\) is defined as:

\[
T_1|t_1t_2\rangle = t_1|t_1t_2\rangle \quad (4)
\]

\[
T_2|t_1t_2\rangle = t_2|t_1t_2\rangle \quad (5)
\]

Now, applying \(P_{12}\) on both sides of (eq. 4), we have:

\[
P_{12}T_1|t_1t_2\rangle = t_1P_{12}|t_1t_2\rangle
\]

Since \(P_{12}\) is unitary (very easy to check from above assumptions), we have:

\[
P_{12}T_1P_{12}^\dagger P_{12}|t_1t_2\rangle = t_1P_{12}|t_1t_2\rangle
\]

\[
P_{12}T_1P_{12}^\dagger |t_2t_1\rangle = t_1|t_2t_1\rangle
\]

Now, on comparing the above equation with (eq. 5), we obtain the relation:

\[
P_{12}T_1P_{12}^\dagger = T_2
\]

This shows that the permutation operator \(P_{12}\), can permute the particle number of the operators as well.

Let us now take a general Hamiltonian describing the two state system:

\[
H = \frac{p_1^2}{2m} + \frac{p_2^2}{2m} + V_{int}(|x_2 - x_1|) + V_1(x_1) + V_2(x_2)
\]

Let us now see the action of the permutation operator, or in other words, the change in the Hamiltonian of the composite system under the exchange of the two particles. So, we have:

\[
P_{12}HP_{12}^\dagger = P_{12}\frac{p_1^2}{2m}P_{12}^\dagger + P_{12}\frac{p_2^2}{2m}P_{12}^\dagger + P_{12}V(|x_2 - x_1|)P_{12}^\dagger + P_{12}V_1(x_1)P_{12}^\dagger + P_{12}V_2(x_2)P_{12}^\dagger
\]

\[
\Rightarrow \frac{p_2^2}{2m} + \frac{p_1^2}{2m} + V_{int}(|x_1 - x_2|) + V_2(x_2) + V_1(x_1) \Rightarrow H
\]

\[
\therefore P_{12}HP_{12}^\dagger = H
\]

Hence, we see that the Hamiltonian of the composite does not change under the exchange of the two particles. Hence, \([H, P_{12}] = 0\), and \(\frac{d}{dt}P_{12} = 0\). Hence, \(P_{12}\) is a constant of time.

### 1.3 Symmetry and Asymmetry in the wave functions

From (eq. 8), we see that, \(P_{12}\) is a constant at all times. This means that if the action of \(P_{12}\) on \(|\psi\rangle\) is known initially, then it is known for all times. We still have the physical requirement that when the state produced on acting \(P_{12}\) on any of the states must not be different from \(P_{12}\). With this physical requirement, we see that only those states which are invariant under the action of \(P_{12}\) are physically consistent. Hence, we must look for the eigenstates of \(P_{12}\). We already know that the eigenvalues are \(\pm 1\). The corresponding eigenstates of \(P_{12}\) are:

\[
|\psi\rangle_+ = \frac{1}{\sqrt{2}} (|k_1k_2\rangle + |k_2k_1\rangle)
\]

\[
|\psi\rangle_- = \frac{1}{\sqrt{2}} (|k_1k_2\rangle - |k_2k_1\rangle)
\]
These two eigenstates are the only possible states that are physically valid out of all the states given in (eq. 2). Therefore, any composite system can only exist in one of the two states.

The eigenstate $|\psi\rangle_+$ is such that, if we exchange the two particles of the composite system (exchange the particle indices $k_1$ and $k_2$), then the state remains the same. In other words, the state of the system is \textit{symmetric} under the exchange of the two particles. On the other hand, the eigenstate $|\psi\rangle_-$ is such that, if we exchange the two particles in the system, the state of the composite system picks up a negative sign. In other words, the system is \textit{asymmetric} under the exchange of the two particles.

1.4 Extending to many state systems

We may now extend the idea of the two state system to a system comprising of $N$ identical particles. The state of the system can now be given by any permutation of:

$$|\zeta\rangle = |k_1k_2k_3 \ldots k_i k_{i+1} \ldots k_j k_{j+1} \ldots k_n\rangle \quad (11)$$

The permutation operators $\{P_{ij}\}$ are defined:

$$P_{ij}|\zeta\rangle = |k_1k_2k_3 \ldots k_j k_{i+1} \ldots k_i k_{j+1} \ldots k_n\rangle \quad (12)$$

(Exchanges the particles $i$ and $j$ in the system.). Here again, the system should be physically the same under any of the permutations and therefore, we must consider the only the eigenstates of the permutation operators as the possible states of the composite system. These eigenstates again are the symmetric and the asymmetric states.

1.5 Bosons and Fermions

Let us take the systems that are described by the asymmetric wave-function $|\psi\rangle_-$. If the two particles in the composite system were in the same state, that is $k_1 = k_2$, then we see that the two terms in the wave-function cancel each other. Hence if the two particles are in the state, the wave-function vanishes. A more physical interpretation of the scenario is that, we cannot expect both the particles to be in the same state. These ”particles” are called \textit{fermions} and the famous rule that no two fermions can be in the same quantum state is called the \textit{Pauli Exclusion Principle}. The statistics used to study fermions is called the \textit{Fermi-Dirac statistics}.

On the other hand, let us examine the systems given by the symmetric wave-function $|\psi\rangle_+$. Here, if the two particles constituting the composite system are in the same quantum state, i.e; $k_1 = k_2$, then we see that $|\psi\rangle_+ = |k_1k_1\rangle = |k_2k_2\rangle$. Therefore, two quantum systems can in fact be in the same state. We can also generalize this (without proof) to the statement that any number of particles can occupy the same quantum state. There ”particles” are called \textit{Bosons} and the statistics used to study them is the \textit{Bose-Einstein statistics}. An important consequence of this is the \textit{Bose-Einstein condensation} where, at absolute zero temperature, all the quantum states of the system come into a single state (of the lowest energy), and as a result, this energy state becomes macroscopically large.
References