Vector spaces

$V$ is a set over a field $F$ such that

$+:V \times V \to V$

$\cdot:F \times V \to V$ such that

1. $(V, +)$ is a grp under $+$
2. $1.v = v$
3. $(ab).v = a.(b.v)$
4. $(a + b).v = (a.v + b.v)$
5. $a.(v + w) = a.v + a.w$

(we will use $+$ for both scalar addition and vector addition as it will be clear from context (duh) and we will drop $\cdot$ altogether)

some properties

- $0v = 0_v$ as $0v = (0 + 0)v = 0v + 0v$. So cancelling we get ans
- $c0_v = 0_v$ as $c0_v = c(0_v + 0_v) = c0_v + c0_v$ etc
- $(-1)v = -v$ as $0 = 0v = (1 + (-1))v = -1v + v = 1v - v$

$F$ over $F$ is also a vector space

Subspaces: a subset of vector space which is also a vector space

$S \subseteq V$ is a subspace iff $S$ is closed under $+$,

Proof: cond necc clearly
the other way
if $\alpha \in S$, then $-\alpha \in S$, so $\alpha + -\alpha = 0 \in S$
comm , ass holds
so done

Linear combination of a set of vectors

$A = \{v_1, v_2, \ldots, v_n\}$

$\sum_{i=1}^{k} a_i v_k$ is a linear combination $(1 \leq k \leq n)$
Some statements - easy proof

- set of all finite linear combinations of a non void set of vectors is a space

- \( S + J \) = set of all vectors \( \alpha + \beta, \alpha \in S, \beta \in J \) is a subspace

- \( S \cap J \) is a subspace

- \( S \cup J \) neednt be a subspace as \( \alpha \in S, \beta \in J \ \alpha + \beta \) neednt belong to union

- \( M = V \) if \( M + N = V \) AND \( M \cap N = \{0\} \) (ring sum)

- \( V = M \) iff \( \forall \epsilon \in V, \epsilon = \mu + \nu \) where \( \mu \in M, \nu \in N \) (uniquely can be expressed in this form)
  
  sketch of proof:
  
  \[ u + v = u_1 + v_2 \Rightarrow u - u_1 = v_2 - v \]
  
  but \( M \cap N = \{0\} \). so \( u = u_1, v = v_2 \)

  the other way
  
  let \( \alpha \in M \cap N \)
  
  \( \epsilon = \alpha + 0 = 0 + \alpha \)
  
  as unique rep, so \( \alpha = 0 \)

- let \( v_1, v_2, \ldots, v_n \) be nonzero linearly dependent vectors. then for some \( i \), we can express \( v_i \) as linear comb of \( v_1, v_2, \ldots, v_{i-1} \)
  
  sketch of proof
  
  choose \( i \) as smallest no. such that \( v_1, v_2, \ldots, v_i \) is dep
  
  so \( \sum_{j=1}^{i} a_j v_j = 0 \)
  
  claim: \( a_i \neq 0 \) (as otherwise \( i \) is smallest such no :) )
  
  so divide by \( a_i \) etc

- Converse: any superset of a set of linearly dep vectors is also linearly dep

- If \( \tau \) is a subspace spanned by set \( A \), then there exists a subset of \( A \) which is linearly independent which also spans \( \tau \)

- subset of a linearly independent set is independent

- let \( S \) be an independent set , and \( \tau \) its span. \( S \cup \{\epsilon\} \) is independent iff \( \epsilon \) not in \( \tau \)
Basis
is max independent set of a vector space

- any vector can be uniquely represented in terms of basis
  if \( \epsilon = \sum_{i=1}^{n} a_i \gamma_i = \sum_{i=1}^{n} b_i \gamma_i \)
  \( \Rightarrow \sum_{i=1}^{n} (a_i - b_i) \gamma_i = 0 \)
  Since they are independent, \( a_i = b_i \)

- every basis for a finite dim vector space has same no. of elements
  let \( A = \{a_i|1 \leq i \leq k\} \) and \( B = \{b_i|1 \leq i \leq m\} \)
  add \( a_1 \) to \( B \)
  it becomes dependent
  scanning from the begining, remove the vectors which make it dependent
  clearly \( a_1 \) still remains in the set
  add \( a_2, a_3 \) and so on
  when \( a_k \) is added, all \( b_i \)'s shd have been removed
  and for each \( a_i \) added, atleast one \( b_i \) is removed
  so \( k \leq m \)
  likewise \( m \leq k \)
  so \( m = k \)

No. of vectors in a basis is called \textit{dimension}

- any independent set of vectors can be extended to a basis

- \( V \) has dim \( n \) and \( A = \{a_i|1 \leq i \leq n\} \) then,
  1. \( A \) is basis iff it is independent
  2. \( A \) is basis iff it spans \( V \)

- let \( S, T \) be subspaces of finite dim \( V \). Then \( \dim(S + T) + \dim(S \cap T) = \dim(S) + \dim(T) \)
  let \( S \cap T \) have basis \( A = \{a_i|1 \leq i \leq k\} \)
  extend \( A \) to a basis of \( S \). the extra vectors added are \( b_{k+1}, b_{k+2}, \ldots b_{k+m} \)
  and extend \( A \) to basis of \( T \). the extra vectors added are \( c_{k+1}, c_{k+2}, \ldots c_{k+n} \)
  then \( PT(A \cup \{b_{k+1}, b_{k+2}, \ldots b_{k+m}\} \cup \{c_{k+1}, c_{k+2}, \ldots c_{k+n}\} \) form a basis
  for \( S + T \)
they clearly span $S + T$
if they are dependent, then scanning from beginning, choose first vec-
or which makes it dependent
it has to be $c_{k+i}$ for some $i$ (as $A \cup b_i$s is a basis)
so $c_{k+i} = \text{linear comb of } a_i$’s and $b_j$’s
transfer $a_i$’s to LHS
so some comb of $c_i$’s and $a_i$’s = some comb of $b_j$’s
so LHS is in $T$ and RHS in $S$
so they lie in $S$
so RHS = some comb of $a_m$’s = some comb of $b_j$’s so we get basis of $S$
is dependent (contradiction)

- a basis of a subspace can be extended to a basis of the vector
  space in many ways
Let \( \{u_1, u_2, \ldots u_m\} \) be basis of subspace \( U \), and extend it to basis of \( V \)
by adding vectors \( \{v_1, v_2, \ldots v_n\} \)
now \( S = \{u_1, u_2, \ldots , u_m, v_1+u_1, v_2, v_3, \ldots v_n\} \) form a basis of \( V \) as well

\(^{1}\text{onto one one maps , examples, etc etc}\)

- In finite dimension vector spaces, one one \( \iff \) onto

- In infinite dim vector spaces, this is not true
  ex: in space of polynomials, look at differentiation, integration

- \( \text{Im } S.T \subseteq \text{Im } S \)

- \( \text{Ker } S.T \supseteq \text{Ker } S \)

- \( \text{Im } T^p = \text{Im } T^{p+1} \Rightarrow \text{Im } T^p = \text{Im } T^{p+k} \ k \geq 1 \)

- \( \text{Im } T^p = \text{Im } T^{p+k} \ \text{Ker } T^p = \text{Ker } T^{p+k} \ k \geq 1 \)

\(^{1}\text{This section is a stub .You can help the author by not reading it}\)
a few results with slight sketches of proofs - duals, maps

- **dual space** $V' = \text{dual space of } V \text{ over field } F = \{ f : V \rightarrow F | f \text{ linear maps } \}$

- **dim of $V'$ = dim of $V$**
  construct a basis of $V'$ where each basis vector $f_i'$ is a function which sends $v_j$ to 1 and rest to zero where $\{v_i + 1 \leq i \leq n\}$ is basis for $V$
  *this basis for $V'$ is called the dual basis*

- $V'' = \text{dual of } V' = \text{double dual of } V$

- **there is a natural\(^2\) isomorphism bet $V$ and $V''$**
  let $L : V \rightarrow V''$ be the isomorphism
  $L(u) = u''$ where
  $u'' : V' \rightarrow F$ is defined as $u''(f) = f(u) \forall f \in V'$

- **dim $L(U,V) = \text{dim (U) dim (V)}$ where $L = \{ f : U \rightarrow V | f \text{ linear map } \}$**
  $\{u_1, u_2, ..., u_m\}$ basis for $U$
  $\{v_1, v_2, ..., v_n\}$ basis for $V$
  basis of $L = \text{maps which map } u_i \text{ to } v_j \text{ and rest of basis of } U \text{ to } 0$
  $(1 \leq i \leq m, 1 \leq j \leq n)$

- **matrix representation of a linear map**
  $f$ be element of $L$ (as def above)
  the matrix of $f$ is a $n \times m$ matrix

\[
\begin{pmatrix}
  x_{11} & x_{12} & \ldots & x_{1m} \\
  x_{21} & x_{22} & \ldots & x_{2m} \\
  \vdots & \vdots & \ddots & \vdots \\
  x_{n1} & x_{n2} & \ldots & x_{nm}
\end{pmatrix}
\]

where $f(u_i) = \sum_{k=1}^{n} x_{ki}v_k$
*so matrix addition represents addition of linear maps, multiplication*

\(^2\)not involving basis


represents composition

- consider $f : U \to V$ . $\dim(\ker f) + \dim(\text{img } f) = \dim U$
  consider basis of kernel\(^3\) and preimages of basis of image\(^4\). they form basis of $U$

- annihilator of $S = S^o = \{f \in \mathcal{V} | f(v) = 0 \forall v \in S\}$ where $S$ is a subset of $V$

- $S^o$ is a subspace $= (\text{span of } S)^\circ$

- $\dim(U^o) = \dim(V) - \dim(U)$ where $U$ is a subspace of $V$

- $S^{oo} = \text{span } S$

**Transpose of a matrix**

$$A : U^{(m)} \to V^{(n)}$$

Now we want to define a map

$$A' : V'^{(n)} \to U'^{(m)}$$

$f \in V'$

so $f : V \to F$

and $A : U \to V$

we need $A'(f) : U \to F$

so clearly, $A'(f) = f.A$ (*)

Verify that $A'$ is linear\(^5\)

Now let $A$ be the matrix representation of linear map $A$ according to the

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\(^3\)vectors which go to zero under $f$

\(^4\)vectors which lie in range of $f$

\(^5\)by verifying on $A'(f)(x)$
foll bases
Basis of $U$ is $\{u_1, u_2, \ldots, u_m\}$, Basis of $V$ is $\{v_1, v_2, \ldots, v_n\}$
choosing dual bases for the dual spaces, what is the matrix of $A'$?
we claim that it is the transpose
To see why, let $A = (a_{ij})$ and $A' = (a'_{ij})$
So $A(u_j) = a_{1j}v_1 + \ldots + a_{ij}v_i + \ldots + a_{nj}v_n$
to get $a_{ij}$, we need to act $v'_i$ on $A(u_j)$ (as $v'_i(u_j) = \delta_{ij}$)
So $v'_i(A(u_j)) = a_{ij}$
similarly to get the $ij^{th}$ entry of $A'$,
$$u''_j(A'(v'_i)) = a'_{ji}$$
where $u''$ is defined using the natural isomporphism bet $U$ and $U''$
that is $u''(f) = f(u)$
so
$$a'_{ji} = u''_j(A'(v'_i)) = (A'(v'_i))(u_j) = (v'_i.A)(u_j) = (v'_i)(A(u_j)) = a_{ij}$$
so $A' = A^T$

Change of basis
in a vector space
Here $V$ is a vector space and $B_1 = \{u_i|1 \leq i \leq n\}$ and $B_2 = \{v_i|1 \leq i \leq n\}$
be two ordered bases
Let us define $P: V \rightarrow V$ which takes $B_1$ to $B_2$
that is $P(u_i) = v_i$
and let the matrix of $P$ with respect to $B_1, B_2$ be $(p_{ij})$
Questions you can ask
* Given $v \in V, v = a_1u_1 + a_2u_2 + \ldots + a_nu_n = b_1v_1 + \ldots + b_nv_n$, what is the relation between $a_i$ and $b_i$?
Now $P(v_1) = p_{11}u_1 + \ldots + p_{n1}u_n$
This can be written as

$^6$see *
\[
\begin{pmatrix}
p_{11} & p_{12} & \cdots & p_{1n} \\
p_{21} & p_{22} & \cdots & p_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
p_{n1} & p_{n2} & \cdots & p_{nn}
\end{pmatrix}
\begin{pmatrix}
1 \\
0 \\
\vdots \\
0
\end{pmatrix}
= 
\begin{pmatrix}
p_{11} \\
p_{21} \\
\vdots \\
p_{n1}
\end{pmatrix}
\]

here the second term in LHS is interpreted in terms of \( B_2 \)
RHS is interpreted in terms of \( B_1 \)
This is done for the std basis , and so works for any vector
So \( P_{B_1}(\bar{v}_{B_2}) = (\bar{v}_{B_1}) \)

* given a matrix \[
\begin{pmatrix}
x_1 \\
x_2 \\
\vdots \\
x_n
\end{pmatrix}
\]
, let \( v_1, v_2 \) be the vectors (interpreted acc to \( B_1, B_2 \)) respectively .What is the relation bet the two vectors?
so \( v_1 = x_1u_1 + x_2u_2 + \ldots x_nu_n \)
and \( v_2 = x_1v_1 + x_2v_2 + \ldots x_nv_n \)
So \( P(v_1) = v_2 : ) \)

In linear maps

let \( L : V \to V \) be a linear map ,(rest of terms as defined above)

Questions you can ask
* Given \( L, L_{B_1} = (a_{ij}), L_{B_2} = (b_{ij}) \) the matrices according to \( B_1, B_2 \) respectively, what is the relation between \( L_{B_1} \) and \( L_{B_2} \) ?
Ans : \( PL_{B_2} = L_{B_1}P \)

* given a matrix \( X = (x_{ij}) \) , let \( L_1, L_2 \) be the maps (interpreted acc to \( (B_1,B_1), (B_2,B_2) \)) respectively .What is the relation bet the two maps?

\( ^7 \) (hint : calculate \( L(v_i), L(P)(u_i) \) in terms of the matrices and bases and equate the two)
\[ L_1(u_i) = x_1 u_1 + \ldots x_n u_n \]
So \( P(L_1)(u_i) = x_1 v_1 + \ldots x_n v_n \)
Now \( L_2(v_i) = x_1 v_1 + \ldots x_n v_n \)
So \( P(L_1)(u_i) = L_2(v_i) \)
But \( v_i = P(u_i) \)
So \( P(L_1)(u_i) = L_2(P(u_i)) \)
So \( PL_1 = L_2P \)

*Two matrices are said to be similar if they represent the same linear map with respect to bases. Two maps are said to be similar if they have the same matrix representation with two bases.*

Some problems (easy)

1. **If \( F \) is of characterestic 0**, \( PT \) \( AB - BA \neq I \)
   
   Hint: \( \text{Trace}(AB) = \text{Trace}(BA) \), so \( \text{Trace}(AB - BA) = 0 \), \( \text{Trace}(I) \neq 0 \)

2. **\( A_k \) is nilpotent of index \( k \)**, \( PT \) \( A_k + I \) is invertible
   
   Hint: \( PT \) it is one one , if \( (A_k + I)(x) = (A_k + I)(y) \), multiply by \( A_k^{-1} \)
   etc

3. **\( A : V \rightarrow W \)** . \( PT \)
   
   - \( (A + B)' = A' + B' \)
   - \( (AB)' = B'A' \)
   - \( (I_V)' = I_V \)
   - \( 0' = 0 \)
   - **if \( A \) is invertible**, then \( (A^{-1})' = (A')^{-1} \)

   Hint: \( A'(f) = f.A \) , and \( A.A^{-1} = I \) , apply \( \cdot \) to both sides etc

4. **\( \ker(A') = (\text{Image } A)^{\circ} \)**
   
   Hint : \( f \in \ker(A') \)
   \( \Leftrightarrow A'(f) = 0 \)

\( ^8 A^k = 0 \)
\[ \Leftrightarrow f.A = 0 \]
\[ \Leftrightarrow f(Av) = 0 \forall v \in V \]
\[ \Leftrightarrow f(\text{Image } A) = 0 \]
\[ \Leftrightarrow f \in (\text{Image } A)^{\circ} \]

5. \textbf{Image}(A') = (\ker A)^{\circ}

let \( A : V^{(n)} \to V^{(n)} \)
\[ f \in \text{Image}(A') \]
\[ \Rightarrow A'(g) = f \]
\[ \Leftrightarrow g.A = f \]
\[ \Leftrightarrow g(Av) = f(v) \forall v \in V, \text{ in particular } v \in \ker(A) \]
\[ \Rightarrow g(Av) = 0 = f(v) \forall v \in \ker(A) \]
\[ \Rightarrow f \in (\ker A)^{\circ} \]

dimension of \((\ker A)^{\circ} = n \cdot \dim(\ker(A)) = \dim(\text{Image } A)\)

dimension of \((\text{Image } A') = n \cdot \dim(\ker(A')) = n \cdot \dim(\text{Image } A)^{\circ} \) (by above prob) \(= n - (n - \dim(\text{Image } A) = \dim(\text{Image } A)\)

so both dimensions equal

6. \( T(T(\alpha) = 0 \Rightarrow T(\alpha) = 0 \Leftrightarrow \text{Image}(T) \cap \ker(T) = \{0\} \)

Hint: Just let \( \gamma \in \text{Image}(T) \cap \ker(T), \) apply LHS, we get it to be zero etc

7. \( \dim(\text{Image } T) = \dim(\text{Image } (T^2)) \cdot \text{PT } \text{Image}(T) \cap \ker(T) = \{0\} \)

\( \ker(T) \in \ker(T^2) \)
\( \dim(\text{Image } T) = \dim(\text{Image } (T^2)) \)
so \( \dim(\ker T) = \dim(\ker(T^2)) \)
so \( \ker(T) = \ker(T^2) \)
apply above prob, done

8. \( (M \cap N)^{\circ} = M^{\circ} + N^{\circ} \)

9. \( (M + N)^{\circ} = M^{\circ} \cap N^{\circ} \)

10. \( (\frac{V}{W})' \cong W^{\circ} \)

hint: \( T : W^{\circ} \to (\frac{V}{W})' \)
\[ T(f) = g \text{ where } g(v + W) = f(v) \]

11. \( \frac{V'}{W'} \cong W' \)
   
   **Hint:** \( T : \frac{V'}{W'} \rightarrow W' \)
   
   \( T(v' + f) = g \text{ where } g = v'|_W \)

12. \( W_1 = W_2 \text{ iff } W_1^? = W_2^? \)

13. \( T : V_{m \times n} \rightarrow W_{p \times n} \text{ is a linear map such that } T(A) = BA \). \( PT \)

   \( T \text{ is invertible iff } p = m \text{ and } B \text{ is a } m \times m \text{ invertible matrix} \)

   **Hint:** If \( B \) is invertible and \( p = m \), then \( T^{-1} \) is defined by \( T^{-1}(A) = B^{-1}A \)

   If \( T \) is invertible, then \( V \cong W \) (isomorphism)

   so \( \dim W = \dim V \), so \( p = m \)

   \( T^{-1}(T(A)) = XBA = A \) where \( X \) is matrix rep of \( T^{-1} \)

   \( (XB)A = A \)

   so \( XB = I \) (choosing \( A = e_{ij} \forall i, j \) etc)

14. \( \text{given } f \in V' , f \neq 0, S = \{v \in V | f(v) = 0\} \), \( PT \) \( S \) is a \( n - 1 \) \( \dim \) subspace of \( V \)

   **Hint:** proving its a subspace is trivial

   \( S = \ker(f) \)

   Image \( (f) = 1 \)

   hence done

15. \( \text{Let } U : V \rightarrow W \text{ be an isomorphism}. \ PT \ L(V, V)^{10} \cong L(W, W) \)

   **Hint:** The isomorphism is given by

   \[ X : L(V, V) \rightarrow L(W, W) \]

   \[ X(T) = UTU^{-1} \]

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9 \( v' \) restricted to acting only on \( W \)

10 \( L(A, B) \) is the space of linear maps from \( A \) to \( B \)
16. Let \( V_n/F \) be a vector space and \( B = \{ \alpha_i \mid 1 \leq i \leq n \} \). \( \text{PT} \) \( \exists \) unique \( T : V \rightarrow V \) such that \( T(\alpha_i) = (\alpha_{i+1}) \) \( \forall i \leq n - 1 \) and \( \text{PT} \) \( T^n = 0 \) iff \( i \geq n \)

Hint: heh

17. Let \( S : V \rightarrow V \) such that \( S^i = 0 \) iff \( i \geq n \). \( \text{PT} \) \( \exists \) \( B \) (basis) for \( V \) such that matrix of \( S \) with respect to \( B \) is

\[
\begin{pmatrix}
0 & 0 & \cdots & 0 \\
1 & 0 & \cdots & 0 \\
0 & 1 & 0 & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1 & 0
\end{pmatrix}
\]

Hint: \( \ker S \subset \ker S^2 \ldots \ker S^n = V \)

It is a strictly increasing sequence (for if at some point two kernels are equal, from then on they will continue being equal)

So dimension increases by 1 each time, (that is a new vector is added to the kernel each time)

Let \( \{ v_i \mid 1 \leq i \leq n \} \) be the basis of \( V \) got from above method

Work out that \( S(v_i) \) is a linear comb of \( v_j \)s with \( j \leq i - 1 \)

And choose basis vectors \( u_j \) to be \( S(u_{j-1}) \) with \( u_1 = v_n \) (or something like that)

So if there are 2 maps such that \( M^n = N^n = 0 \) but \( M^{n-1} \neq N^{n-1} \), then they are similar as acc to some 2 bases, they represent the above matrix

18. \( M^n = N^n = 0 \) and \( M^{n-1} \neq 0 \), \( M^{n-1} \neq 0 \) where \( M, N \) are matrices. \( \text{PT} \) they are similar

Hint: Let \( S^* \) denote the matrix in the above problem (the one with 1s and 0s)

\( M, S^* \) are similar as they rep some \( T \) where \( T^n = 0 \) etc

So \( S^* = P^{-1}MP \)

Similarly \( S^* = Q^{-1}NQ \)

So \( N = QS^*Q^{-1} = QP^{-1}MPQ^{-1} = XMX^{-1} \)

19. \( W = \{(x_1, \ldots, x_n) \mid x_i \in F, \sum x_i = 0\} \) \( \text{PT} \) \( W^o \) consists of all linear functionals of form \( f(x_1, \ldots, x_n) = c \sum x_i \)

Hint: if \( f \) is of given form, clearly \( f \in W^o \)
Let \( f \in W^o \), and \( f(0,0,\ldots,1,0,\ldots,0) = c_i \)
so \( f(x_1,\ldots,x_n) = \sum c_i x_i \)
look at \( f(x,-x,0,\ldots,0) \)
this is \((c_1 - c_2)x = 0\) .
so \( c_1 = c_2 \) etc

\[ \sum_{i=1}^{m} c_i x_i, \sum c_i = 0 \]

\[ f(x_1,\ldots,x_n) = \sum c_i x_i, \sum c_i = 0 \]

20. **PT dual space** \( W' \) can be naturally identified with linear functionals

\[ f(x_1,\ldots,x_n) = \sum c_i x_i, \sum c_i = 0 \]

\[ f(x_1,\ldots,x_n) = \sum c_i x_i, \sum c_i = 0 \]

21. \( g,f_1,f_2,\ldots,f_k \) be elements of \( V' \) and \( \ker g = N \) , \( \ker f_i = N_i \)

**Hint:** \( g \) is a linear comb of \( f_i \)'s iff \( \cap N_i \subseteq N \)

**Hint:** if \( g \) is a linear comb of \( f_i \)'s, the result is trivial
if \( \cap N_i \subseteq N \) , choose max independent set of \( f_i \)'s and extend to basis
choose the dual basis of \( V \)
the dual basis vectors of basis vectors \( f_k \)'s which are not part of the
given \( f_i \)'s indeed lie in \( \cap N_i \)
however if \( g \) contains such components , then such dual basis vectors
\( \not\in N \)
Contradiction

22. \( W \) is a subspace of \( V \) , \( \{g_1,\ldots,g_r\} \) is the basis of \( W^o \) . \( N_i = \ker g_i \) . **PT** \( W = \cap N_i \)

hint : just work it out

23. **Let** \( S \) be a set , \( F \) be a field. \( V(S,F) \) be space of all functions from \( S \) to \( F \), \( W \) be any \( n \) dim subspace. **PT** \( \exists x_1,\ldots,x_n \in S \) and
\( f_1,\ldots,f_n \in W \) such that \( f_i(x_j) = \delta_{ij} \)
Vague hint: choose basis of \( W = \{f_i\} \)
\( f_1(x_1) = 1 \) , if \( f_2(x_1) = u_1 \) , \( f_2' = f_2 - u_1 f_1 \) etc
Some nice problems

1. **PT if** $V$ **is a vector space over infinite field** $F$, it can never be written as union of finitely many proper subspaces of $V$
   
   Proof:
   
   claim: if $V_1 \cup V_2 = V$, then $V_1 \subseteq V_2$ or vice versa
   
   if not let $\alpha \in V_1$ and not in $V_2$
   
   and $\beta \in V_2$, not in $V_1$
   
   $\alpha + \beta \in V$ but it can’t be in either $V_1$ or $V_2$ (else $\beta$ will belong to $V_1$ etc etc)
   
   $\Rightarrow V_i = V$ so not proper subspace
   
   Induction hypotheses: assume $V$ can’t be written as union of $n$ proper subspaces
   
   if possible let $V = V_1 \cup V_2 \ldots V_n \cup V_{n+1}$
   
   $V_{n+1}$ can’t belong to $V_1 \cup V_2 \ldots V_n$ (due to hypothesis)
   
   also $V_1 \cup V_2 \ldots V_n$ can’t belong to $V_{n+1}$
   
   so let $\alpha \in V_{n+1}$, not in $V_1 \cup V_2 \ldots V_n$
   
   let $\beta \in V_1 \cup V_2 \ldots V_n$, not in $V_{n+1}$
   
   Let $S = \{\alpha + c\beta | c \in F, c \neq 0\}$
   
   $S$ is infinite set
   
   and no element of $S$ can belong to $V_{n+1}$ as if it belongs, $\Rightarrow \beta \in V_{n+1}$
   
   so $\alpha + c\beta \in V_1 \cup V_2 \ldots V_n$
   
   also if $\alpha + c_1\beta \in V_i$ and $\alpha + c_2\beta \in V_i$
   
   $\Rightarrow \alpha \in V_i$
   
   so one element of $S$ can belong to only one of $V_i$ but $S$ has infinite elements
   
   and $S \subseteq V$ (so contradiction)

2. **If** $S \subset G$ (**subfield of** $G$ **where** $G$ **is finite**), $\Rightarrow |S|^n = |G|$
   
   Consider $G$ as a vector space over $S$
   
   $G$ has finite dim
   
   let basis be $\{\alpha_i | 1 \leq i \leq n\}$
   
   there are $|S|$ choices for coeffs of each basis vector
   
   so total no. of vectors = $|S|^n$

3. **PT any finite field** $F$ **has prime power number of elements**
   
   characteristic of a field = $k$ where $k$ is the least number such that
   
   $1 + 1 + \ldots 1$ ($k$ times) = 0.
   
   claim: $k$ is prime
   
   if $k = pq$, then $(1+1+\ldots 1(p \text{ times}))(1+1+\ldots 1(q \text{ times})) = 1+1+\ldots 1$
(k times)
so done
so let p be characteristic of field
Z_p \subseteq F
hence by above problem , done

4. \( f, g : V \to F \) . \( \ker(f) \) strictly contained in \( \ker(g) \). what can you say abt \( f, g \)
\[
\dim(\ker f) + \dim(\text{img } f) = \dim(V)
\]
\[
\dim(\ker f) + \dim(\text{img } f) = \dim(V)
\]
\[
\dim(\ker f) < \dim(\ker g)
\]
and \( \dim(\text{img } f) = 0 \) or 1 , likewise for \( g \) as \( F \) is the codomain
hence conclude that \( g \) is zero map

5. how many invertible \( n \times n \) matrices over \( Z_p \) where \( p \) is a prime are there
interpreting matrices as linear maps from \( V \to V \) over \( F \) , we need invertible maps
so let \( \{ e_1, e_2, \ldots, e_n \} \) be basis of \( V \)
so there are \( p^n \) vectors totally
\( e_1 \) can go to any vector except 0 (so \( p^n - 1 \) choices).
let \( v_i \) denote image of \( e_i \)
now \( e_2 \) can go to any vector which forms an independent set with \( v_1 \)
so it shd not go to any multiple of \( v_1 \)
there are \( p \) such multiples
so \( p^n - p \) choices for \( e_2 \)
now \( e_3 \) shd skip all linear combinations of \( v_1, v_2 \) (there are \( p^2 \) such combinations)
so \( p^n - p^2 \) choices for \( e_3 \) and so on..
\[
\prod_{i=0}^{n-1}(p^n - p^i)
\]
is the answer

6. Let \( F \) be a subfield of finite field \( G \) . a set of vectors are independent in \( F^n/F \) iff they are independent in \( G^n/G \)
If a set of vectors are independent in \( G^n/G \) , then clearly it is independent in \( F^n/F \)
TPT if \( v_1, v_2, \ldots, v_k \) are independent in \( F^n/F \) , they are independent in \( G^n/G \)
\[ TPT v_1, v_2, \ldots, v_k \] are dependent in \( G^n/G \) , they are dependent in \( F^n/F \)
wlog assume \( k = n \)
now each \( v_i \) is an \( n \) tuple, say \( (v_{i1}, v_{i2}, \ldots, v_{in}) \)
Let them be dependent over \( G \).
so \( \exists \alpha_i (i = 1, 2, \ldots, n) \) not all zero such that \( \sum_{i=1}^n \alpha_i v_i = 0 \)
that is
\[
\begin{pmatrix}
v_{11} & v_{12} & \cdots & v_{1n} \\
v_{21} & v_{22} & \cdots & v_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
v_{n1} & v_{n2} & \cdots & v_{nn}
\end{pmatrix}
\begin{pmatrix}
\alpha_1 \\
\alpha_2 \\
\vdots \\
\alpha_n
\end{pmatrix}
= 
\begin{pmatrix}
0 \\
0 \\
\vdots \\
0
\end{pmatrix}
\]
So by cramers rule, det of the left most matrix is zero (over \( G \))
But all entries are in \( F \) (as each row is a vector over \( F^n \))
so zero determinant over \( F \), too
so again by cramer's rule, we can find \( \beta_i \)s such that these are dependent

7. Let \( F \) be any field. Let \( m, n \) be positive integers. Let \( F_1, f_2, f_3, \ldots, f_n \) be elements of \( (F^n)' \). Consider the map
\[
L : F^n \rightarrow F^m x \rightarrow Lx = (f_1(x), \ldots, f_m(x))
\]
PT \( L \) is linear. Conversely show that every linear map from \( F^n \) to \( F^m \) arises this way
proving it is linear is straightforward
let \( \{e_i | 1 \leq i \leq n\} \) be std basis for \( F^n \), and \( \{d_i | 1 \leq i \leq m\} \) the std basis for \( F^m \)
\( L(e_i) = \sum_{j=1}^m a_{ji}d_j \)
so \( L(e_i) = (a_{i1}, \ldots, a_{im}) \)
we need \( f_j(e_i) = a_{ji} \)
let dual basis for \( F^m \) be \( \{g_i | 1 \leq i \leq m\} \)
so \( f_i = g_iL \)
so this works for basis, shd work for any other vector

8. In prob 4, if \( \ker f = \ker g \), what happens?
So \( \dim(\text{Im } f) = \dim(\text{Im } g) \)
if it is zero, then both are zero maps
If both are 1, so \( \dim(\ker f) = n - 1 \)
extend it to a basis by adding another vector say \( w \)
as $\ker f = \ker g$, only $g(w) \neq 0$

So $g = \frac{g(w)}{f(w)} f$

9. Let $V$ be $n$ dimensional. PT there is a one one correspondence between $m$ dim subspaces and $n - m$ dim subspaces of $V$
annihilator of $W$ will be $n - m$ dim
by some isomorphism bet the dual space and the actual space, the annihilator will go to a subspace of $V$

10. Let $F$ be an infinite field and $V$ be finite dim vector space. Let $x_1, x_2, \ldots, x_T$ be any finite collection of nonzero vectors. PT there is an $f$ in $V'$ such that each $f(x_i) \neq 0$
Proof 1:
Suppose , no such $f$ exists , then $\forall f \in V'$, $f(x_i) = 0$ for some $i$
Put $W_i = \{ f | f(x_i) = 0 \}$
Each $W_i$ is a proper subspace of $V'$ (as you can always produce a functional which takes $x_i$ to 1)
So $V' = \bigcup W_i$
By prob 1 , this cant be done .

Proof 2 (lengthy):
We proceed by induction
Base case:
Let the finite collection of elements have a single vector
Then obviously , we can extend the vector to a basis, ans allow $f$ to go to 1 on the vector alone , and rest to 0
Induction hypothesis:
Assume that a suitable $f$ is found for a finite set of cardinality $k$
now let the given set be $\{ x_1, x_2, \ldots, x_k, x_k + 1 \}$
Find out the maximal independent set of the first $k$ of $x_i$s , say $Y = \{ 1, y_2, \ldots, y_d \}$
let $f_{old}$ be the suitable functional required

Claim : If $Y$ is extended to a basis of $V$, and $G$ be the corr dual basis, then we can find a $f^*$ which can be written as a linear combination of the first $d$ vectors of the dual basis alone
because if \( f_{\text{old}} \) has a component of say \( f_{d+j} \), the latter functional will not have any effect on the \( x_i \)s. So we can just remove those components and get our \( f^* \). 

So we can now assume \( f_{\text{old}} \) is a linear combination of first \( d \) vectors of dual basis.

**Case 1:**

If \( x_{k+1} \) doesn’t lie in span of \( Y \), extend \( Y \cup \{ x_{k+1} \} \) to a basis. Let \( G \) be the corresponding vector in dual basis.

**Claim:** my required \( f = f_{\text{new}} = f_{\text{old}} + g_{d+1} \) (where \( g_{d+1} \) is the corresponding vector in \( x_{k+1} \)).

As \((f_{\text{old}} + g_{d+1})(x_i) = f_{\text{old}}(x_i) + g_{d+1}(x_i) = a + 0(a \neq O) = a \) (i ≤ \( k \)),

as \( f_{\text{old}} + g_{d+1} \) is \( f_{\text{old}}(x_{k+1}) + g_{d+1}(x_{k+1}) = 0 + 1 = 1 \) (as \( f_{\text{old}} \) is a linear comb of only the first \( d \) vectors).

Hence done.

**Case 2:**

If \( x_{k+1} \in \text{span} Y \), then all \( x_i \)s can be written as a linear comb of vectors in \( Y \).

So let \( x_i = a_{i1}y_1 + a_{i2}y_2 + \ldots + a_{id}y_d \) for \( 1 \leq i \leq k + 1 \),

and let \( f_{\text{old}} = b_1g_1 + b_2g_2 + \ldots b_dg_d \).

So in matrix notation

\[
\begin{pmatrix}
a_{11} & a_{12} & \cdots & a_{1d} \\
a_{21} & a_{22} & \cdots & a_{2d} \\
\vdots & \vdots & \ddots & \vdots \\
a_{k1} & a_{k2} & \cdots & a_{kd}
\end{pmatrix}
\begin{pmatrix}
b_1 \\
b_2 \\
\vdots \\
b_d
\end{pmatrix}
= 
\begin{pmatrix}
r_1 \\
r_2 \\
\vdots \\
r_k
\end{pmatrix}
\]

where none of \( r_i \) is 0 (because its \( f_{\text{old}} \)).

We need

\[
\begin{pmatrix}
a_{11} & a_{12} & \cdots & a_{1d} \\
a_{21} & a_{22} & \cdots & a_{2d} \\
\vdots & \vdots & \ddots & \vdots \\
a_{k1} & a_{k2} & \cdots & a_{kd}
\end{pmatrix}
\begin{pmatrix}
b_1 \\
b_2 \\
\vdots \\
b_d
\end{pmatrix}
= 
\begin{pmatrix}
r_1 \\
r_2 \\
\vdots \\
r_k
\end{pmatrix}
\]

where none of \( r_i \) s is 0.

Procedure to find \( f_{\text{new}} \).

Try \( f_{\text{old}} \), if \( r_{k+1} \neq 0 \), then \( f_{\text{new}} = f_{\text{old}} \). Done.

If it is zero, that is

\[
a_{(k+1)1}b_1 + a_{(k+1)2}b_2 + \ldots + a_{(k+1)(d-1)}b_{d-1} + a_{(k+1)d}b_d = 0 - (1)
\]
choose a $b_{d+1}$, not equal to any of the prev $b_i$s from the infinite field. Clearly

$$a_{(k+1)1}b_1 + a_{(k+1)2}b_2 \ldots + a_{(k+1)(d-1)}b_{d-1} + a_{(k+1)d}b_{d+1} \neq 0 - (2)$$

(otherwise comparing 1, 2 we get $b_d = b_{d+1}$)

But it may happen that, for some $i$

$$a_{i1}b_1 + a_{i2}b_2 \ldots + a_{i(d-1)}b_{d-1} + a_{id}b_{d+1} = 0 - (3)$$

Now choose a new $b_{d+2}$ not equal to any of prev $b_i$s.

The trick is, once a particular $b_{d+j}$ satisfies eq (3) above for some $i$, then for that $i$, we can plug in any other value instead $b_{d+j}$, and the LHS will not go to zero.

So as there are only finitely many $x_i$s and infinitely many $b_i$s, we can keep choosing and throwing out $b_i$s, till we get an appropriate $b_x$

so $f_{new} = b_1g_1 + b_2g_2 + \ldots b_{d-1}g_{d-1} + b_xg_d$  \(^{11}\)

Every vector space has a basis

Cantor’s lemma : $\#(S) < \#(P(S))$

That is $\exists$ no bijection bet a set and its powerset

Proof: if possible, let $f$ be bijection bet $S$ and $P(S)$

$x \in S \rightarrow f(x)$ where $f(x) \subseteq S$

so we can ask if $x \in f(x)$ ?

let $S_0 = \{x | x \in S, x \not \in f(x)\}$

$S_0 \subseteq S$, so $S_0 \in P(S)$

as $f$ is a bijection, look at preimage of $S_0$ say $x_0$

so $f(x_0) = S_0$

if $x_0 \in S_0$, then by def of $S_0$, $x_0 \not \in f(x_0) = S_0$

so $x_0 \in S_0$ and not in $S_0$ (:P)

if $x_0$ not in $S_0$, then $x_0 \not \in f(x_0) = S_0$ and we get a contradiction

\(^{11}\)all along, assumed $b_d$ is non zero, otherwise choose the max index $i \leq d$ where $b_i$ is nonzero, instead of $d$ and keep replacing it
Continuum hypothesis  axiom of choice

Let $A_i \neq \phi$, $i \in I$ where $I$ is index set
Then $\prod_{i \in I} A_i \neq \phi$ Axiom of choice is same as well ordering principle$^{12}$ which is same as Zorn’s lemma

Some definitons

**Partially ordered set** not every 2 elements are related and $\exists$ a maximal element $x$ such that if $y \in S$ and $y \geq x$, then $y = x$

**chain** subset of $S^{13}$ which is ordered.

**upperbound** let $C$ be chain of $S$. $x \in S$ is upperbound of $C$ if $x > y \forall y(\neq x) \in C$

**Zorn’s lemma**

If every chain of $S$ has an upper bound, $S$ has a maximal element

**every vector space has a basis**

Proof:
let $V$ be the given vector space
define $S_V = \{(W, B_W) | W$ subspace of $V$ which has a basis $B_W \}$
let the partial ordering be $\leq$ on $S$ where
$(W, B_W) \leq (Y, B_Y)$ iff $W \subseteq Y$ and $B_W \subseteq B_Y$
claim: every chain has an upperbound
let $\{(W_i, B_{W_i}) | i \in I \}$ be a chain where $I$ is index set
then $(\bigcup_{i \in I} W_i, \bigcup_{i \in I} B_{W_i})$ is an upperbound.$^{14}$
so by Zorn , $S_V$ has a maximal element say $(Z, B_Z)$
claim : $Z = V$
if not let $v \in V - Z$
look at subspace $T$ spanned by $Z \cup \{v\}$, its basis is $B_T = B_Z \cup \{v\}$
and $(T, B_T) > (Z, B_Z)$ and the latter is maximal element
so contradiction

$^{12}$any set can be well ordered implies every non empty set has a least element.so induction works
$^{13}$partially ordered
$^{14}$as $W_{i_1} \subset W_{i_2}$, the union is also a vector space etc
Determinant of a matrix

$k$ linear maps

$f : U_1 \oplus U_2 \oplus \ldots U_k \to W$ is $k$ linear if $f$ is linear in each variable \(^{15}\)

set of $k$ linear maps on $U_1 \oplus U_2 \oplus \ldots U_k \to W$ is a vector space of dimension $= \prod_{i=1}^k \dim U_i$

Proof:
checking its a vector space is boring and easy
let $f$ be a $k$ linear map, expand $f(a_1,a_2,\ldots,a_k)$ by writing $a_i$s in terms of bases.
we get it to be $\sum_{i_1,i_2,\ldots,i_k=1}^{n_1,n_2,\ldots,n_k} X_i f(a_{i_1},b_{i_2},\ldots x_{i_k})$ where $n_i$s are dims of $U_i$s and the $a_i$s, $b_i$s $\ldots x_i$s are the chosen bases
so the basis of this space is
$\{f_{i_1,i_2,\ldots,i_k}$ which sends $(i_1,i_2,\ldots,i_k)$ to 1 and rest to 0 $|1 \leq i_j \leq n_j\}$

Types of $k$ linear maps

permutation acting on a map $\sigma(f)(a_1,a_2,\ldots,a_k) = \sigma f(a_1,a_2,\ldots,a_k) = f(a_{\sigma(1)},a_{\sigma(2)},\ldots,a_{\sigma(k)})$ where $\sigma$ is a permutation on $[k]$

symmetric $k$ linear map if $\sigma f = f \forall \sigma \in S_k$

$ex$: $\sum_{\sigma \in S_k} \sigma f$

skewsymmetric maps if $\sigma f = (\text{sign})(\sigma)f$

$ex$: $\sum_{\sigma \in S_k} \text{sign}(\sigma)\sigma f \forall \sigma \in S_k$

alternating maps $f(a_1,a_2,\ldots,a_k) = 0$ whenever $a_i = a_j$ for some $i,j$

Alternating $\Rightarrow$ Skewsymmetric

$f(\ldots,u+v,\ldots,u+v,\ldots) = 0$
expanding and as terms like $f(\ldots,u,\ldots,u,\ldots) = 0$

\(^{15}\) $f(a_1,a_2,\ldots,a_i + xb,\ldots,a_k) = f(a_1,a_2,\ldots,a_k) + xf(a_1,a_2,\ldots,a_k)$
\[ f(\ldots, u, \ldots, v, \ldots) = -f(\ldots, v, \ldots, u, \ldots) \]

so for transpositions, the sign gets changed

as any perm is a product of transpositions, we find that this is skew symmetric

we can reverse this procedure if the char of field is not 2

as \( f(\ldots, u, \ldots, u, \ldots) = -f(\ldots, u, \ldots, u, \ldots) \) (transposition acting on the 2 u’s)

so \( 2f(\ldots, u, \ldots, u, \ldots) = 0 \)

so if field char is not 2, then \textbf{skew symmetric} \Rightarrow \textbf{Alternating}

\textit{sum of 2 symmetric/skewsymmetric maps is symmetric/skewsymmetric} 16

so the set of all symmetric/skewsymm \( k \) linear maps form a subspace

\begin{align*}
\text{Finding dimensions of symm/skewsymm } k \text{ linear maps from } V \oplus V \oplus \ldots V \text{ (k such spaces)} & \\
\end{align*}

\[ f : V \oplus V \oplus \ldots \rightarrow F \]

let \( a_1, a_2, \ldots, a_n \) be basis of \( V \)

\textbf{symmetric}

Let \( f \) be symmetric

so if \( f \) acting on \((a_1, a_1, a_2, \ldots, a_k)\) is specified, it is known on any permutation of it

so just need to specify how many \( a_1 \)s, \( a_2 \)s chosen etc..

that is just \( n_1 + n_2 \ldots + n_n = k \)

this is just permutation of \( k \) 1’s and \( n - 1 \) 0’s

which is \( C_{n+k-1}^k \)

\textbf{skew symmetric}

verify that if \( f \) is skewsymm, then if it acts on a tuple which is linearly dependent, it is 0

so if \( k > n \), then the vector tuple will be dependent, so everything will take everything to 0

if \( k \leq n \), then we have to specify how \( f \) acts on any \( k \) basis vector tuple

so \( C_k^n \)

this space is denoted as \( \Omega^k(V) \)

if \( k = n \), its a 1 dim space 17

so this 1 dim space will be denoted as \( \Omega^n(V) \)

16 work it out

17 important
more abt this 1 dim subspace

volume of an n dimensional object is a n linear map (as if one parameter doubles, volume doubles etc)
and if the n vectors are dependent, the volume shd be 0
so volume is a n linear alternating map
and because this space is a 1 dim space, volume of an object can be defined only in 1 way (except scale difference)

And finally the determinant

Let $A$ be a linear map
$A : V \to V$
now let $\bar{A} : \Omega^n(V) \to \Omega^n(V)$
such that $\bar{A}(w)(v_1, v_2, \ldots, v_n) = w(A(v_1), \ldots, A(v_n))$
PT $\bar{A}$ is linear , alternating
PT $\bar{A}$ is linear map
so $\bar{A}(w) = kw$ as we are in a 1 dim space
this $k$ is the determinant of $A$

some properties of determinants

(just by writing down the maps we get the foll results)

1. determinant of $I = 1$
2. if $A$ takes a basis to a dependent set, then $\det(A) = 0$
3. $\det(AB) = \det(BA)$

The last of determinants

a small lemma: Let $w \in \Omega^n(V)$ be non zero , $V$ is n dim . then $w(v_1, v_2, \ldots, v_n) = 0$ iff these are dependent

\footnote{best to work out for 2X2 or 3X3 matrices}
if vectors are dependent, then \( w \) takes the tuple to 0 (as it is alternating)
if \( w((v_1, v_2, \ldots, v_n) = 0 \), and these are independent, then these form a basis for \( V \)

now expand \( w \) acting on any other tuple in terms of this basis and you will find that everything goes to 0
so \( w \) is zero (contradiction)

\[ A \text{ is not invertible iff } \det(A) \text{ is zero} \]

let \( v_1, v_2, \ldots, v_n \) be a basis
let \( A \) be non invertible
TPT \( \det(A) \) is 0
as the map is non invertible, then \( A(v_1), A(v_2), \ldots, A(v_n) \) is dependent
as \( (\det A)w(v_1, v_2, \ldots, v_n) = w(A(v_1), A(v_2), \ldots, A(v_n)) \) and RHS is zero
so \( \det(A) = 0 \)

TPT \( \det(A)= 0 \Rightarrow A \) is not invertible
TPT \( A \) is invertible \( \Rightarrow \) \( \det(A) \) is nonzero
\( AA^{-1} = I \)
so \( \det(A)\det(A^{-1}) = 1 \)
so \( \det(A) \) is nonzero

**Trace of a matrix**

Let \( A: V \rightarrow V \) be a linear map on \( V \), an \( n \) dim vector space, over \( F^{19} \). Let \( \Omega^n(V) \) be the space of alternating \( n \) forms on \( V \). Define

\[ T(A) : \Omega^n(V) \rightarrow \Omega^n(V) \]

\[ w \rightarrow T(A)w \text{ by} \]

\[ T(A)(w)(x_1, x_2, \ldots, x_n) = w(Ax_1, x_2, \ldots, x_n) + w(x_1, Ax_2, \ldots, x_n) + \ldots + w(x_1, x_2, \ldots, Ax_n) \]

\^19 characteristic \( \neq 2 \)
Some properties

- **$T(A)(w)$ is indeed an alternating $n$ form**
  Checking it is $n$ linear is trivial
  Check that $T(A)w(x_1, x_2, \ldots, x_n) = 0$ if 2 entries are equal
  let that be $x_2 = x_n = y$
  $T(A)w(x_1, x_2, \ldots, x_n) = w(x_1, Ay, \ldots, y) + w(x_1, y, \ldots, Ay)$ (as all remaining terms vanish, as $w$ is an alternating form)
  add $w(x_1, Ay, \ldots, Ay) + w(x_1, y, \ldots, y)$ to RHS (as this is zero)
  hence done

- **$T(A)$ is linear. So $T(A)(w) = kw$**

- **$T(0) = 0, T(I) = n$**
  as $T(0)w(x_1, x_2, \ldots, x_n) = w(0, x_2, \ldots, x_n) + w(x_1, 0, \ldots, x_n) + \ldots + w(x_1, x_2, \ldots, 0)$
  and 0 is dependent on any set of vectors, so $w$ vanishes on all $n$ tuples having a 0
  and $T(I)w(x_1, x_2, \ldots, x_n) = w(x_1, x_2, \ldots, x_n) + w(x_1, x_2, \ldots, x_n) + \ldots + w(x_1, x_2, \ldots, x_n) = nw(x_1, x_2, \ldots, x_n)$
  as $T(A)(w) = kw, k = n$

- **$T(A + B) = T(A) + T(B), T(cA) = cT(A)$**

- **$A = (a_{ij}),$ then $T(A)(w) = (\text{Trace of matrix } (a_{ij})w$**
  $T(A)(w)(x_1, x_2, \ldots, x_n) = w(Ax_1, x_2, \ldots, x_n) + w(x_1, Ax_2, \ldots, x_n) + \ldots + w(x_1, x_2, \ldots, Ax_n)$
  $A(x_1) = a_{11}x_1 + a_{21}x_2 + \ldots a_{n1}x_n$
  $w(Ax_1, x_2, \ldots, x_n) = w(a_{11}x_1 + a_{21}x_2 + \ldots a_{n1}x_n, x_2, \ldots, x_n) = a_{11}w(x_1, x_2, \ldots, x_n)$ (just expanding etc)
  similarly expand the other terms, we get
  $T(A)(w)(x_1, x_2, \ldots, x_n) = (a_{11} + a_{22} + \ldots + a_{nn})w(x_1, x_2, \ldots, x_n)$
  since this is a one dim space etc, done

- **$T(AB) = T(BA)$**
  $A = (a_{ij}), B = (b_{ij})$
  Now $AB(x_1) = (\sum_{i=1}^n a_{i1}b_{i1})x_1 + \text{some } x_2s, x_3s \text{ etc etc}$
  and like above we get
\[w(ABx_1, x_2, \ldots, x_n) = (\sum_{i=1}^{n} a_1 b_{i1}) w(x_1, x_2, \ldots, x_n)\]
so we get \(T(AB)(w)(x_1, x_2, \ldots, x_n) = (\sum_i \sum_j a_{ij} b_{ji}) w(x_1, x_2, \ldots, x_n)\)
this is symmetric
so \(T(BA)\) is also the same thing

**Canonical forms**

**Definitions**

given \(V\) a vector space and \(T\) a linear operator

1. if \(T(\alpha) = c\alpha\), then \(c\) is a **characteristic value**
2. \(\alpha\) is **characteristic vector**
3. \(S = \{\alpha | T(\alpha) = c\alpha, \alpha \in V\}\) is a subspace = kernel of \(T - cI\)
   so \(c\) is characteristic iff this kernel is not \(\{0\}\) which is iff \(\det(T - cI) = 0\)
4. dim of the above space = **geometric multiplicity of \(c\)**
   clearly geo mult \(\leq n\) (as subspace dim \(\leq\) original space dim)
5. \(f(x) = \det(xI - A)\) where \(A\) is a matrix is **characteristic polynomial**
   of \(A\) this is of deg \(n\) (\(A\) is \(nXn\))
6. **algebraic multiplicity of \(c\)** in \(f(x) = k\) where \((x - c)^k / f(x)\) and
   \((x - c)^k + 1\) doesn't
7. **algebraic multiplicity of eigen value** \(c\) = algebraic multiplicity of \(c\) in char poly

* similar matrices have same characteristic polynomial

\[B = PAP^{-1}\]
\[\det(xI - B) = \det(xI - PAP^{-1}) = \det(PxIP^{-1} - PAP^{-1})\]
\[= \det(P(xI - A)P^{-1}) = \det(P)\det(xI - A)\det(P^{-1}) = \det(xI - A)\]

So roots of characteristic polynomials of a linear operator are the characteristic values and field in consideration is important

\[
\begin{pmatrix}
0 & -1 \\
1 & 0
\end{pmatrix}
\]
in $R$, no char values, in $C$, there are char values

algebraically closed fields

any polynomial has at least one root in the field So all roots belong to the field (keep removing one root at a time)

* Alg multiplicity $\geq$ geometric multiplicity

let $\lambda$ be an eigen value with geo mult $= k$
let the basis of space associated with it $v_1, v_2, \ldots, v_k$
extend it to a basis of $V\{v_i|1 \leq i \leq n\}$
let $A$ be the matrix with respect to this basis

$$A = \begin{pmatrix}
\lambda & 0 & . & . & 0 & * & . & *
0 & \lambda & 0 & . & 0 & * & . & *
0 & 0 & \lambda & . & 0 & * & . & *
0 & 0 & 0 & . & \lambda & * & . & *
0 & 0 & . & 0 & . & * & . & *
0 & 0 & . & 0 & . & * & . & *
\end{pmatrix}$$

expanding along first col of $xI - A$, we get $f(x) = (x - \lambda)^k g(x)$
so alg mult of $\lambda \geq k$
(ex) to show geometric mult $= \text{alg mult of } c$
Basis of $V = \{v_1, v_2\}, T(v_1) = c(v_1), T(v_2) = 0$

* If $T(\alpha) = c\alpha$, then $f(T)(\alpha) = f(c)\alpha$

$T(\alpha) = c\alpha$
$\rightarrow T^2(\alpha) = T(c\alpha) = c^2\alpha$
$\rightarrow$ so $T^k(\alpha) = c^k\alpha$
$f(T) = \sum a_i T^i$
$f(T)\alpha = \sum a_i T^i(\alpha) = \sum a_i c^i \alpha = f(c)\alpha$

* $T$ is linear operator on finite dim $V$. $c_1, c_2, \ldots, c_k$ are distinct char values and $W_i$ be the spaces associated with $c_i$. $W = W_1 + W_2 + \ldots + W_k$, then dim $W = \sum \text{dim} W_i$

Note $W$ is a space spanned by all eigen vectors of $V$
Claim : in fact $\{B_1, B_2, \ldots, B_k\}$ is a basis for $W$ where $B_i$ is a basis for $W_i$
TPT : $\sum \beta_i = 0 \rightarrow \beta_i = 0$ where $\beta_i \in W_i$
0 = f(T)(0) = f(T)(\Sigma \beta_i) = \Sigma f(T)(\beta_i) = \Sigma f(c_i)\beta_i \text{ for all polynomials } f

choose \( f_i(x) = \frac{X_i \prod(x-c_i)}{x-c_i} \) where \( X_i \) is a constant such that \( f_i(c_i) = 1 \)

so putting \( f = f_i \), we get \( \beta_i = 0 \).

The following are equivalent

1. \( T \) is diagonalizable

2. char poly = \( f(x) = \prod(x-c_i)^{d_i} \) and \( \dim W_i = d_i \)

3. \( \Sigma \dim W_i = \dim V \)

if \( T \) is diagonalizable, find a basis \( B \) such that the matrix looks like

\[
\begin{pmatrix}
  x_1I_1 & 0 & 0 \\
  0 & x_2I_2 & 0 \\
  & & \ddots & \ddots \\
  0 & 0 & \cdots & x_kI_k
\end{pmatrix}
\]

where \( I_i \) is identity matrix of order \( f_i \times f_i \)

so \( f(x) = \prod(x-x_i)^{f_i} \) and \( x_i \)s are eigen values :)

\( W_i \) has \( \dim f_i \) (as obv from matrix)

so 1 implies 2

now \( \Sigma f_i = n \) by 2. and \( n = \dim V \).

so 2 implies 3

now if 3 holds, \( \Sigma W_i = V \) (by some prev lemma etc)

so 3 implies 1

some exercises from hoffman and kunze -pg 189

1. Let \( A, B \) be \( n \times n \) matrices over field \( F \). PT if \((I - AB)\) is invertible then so is \((I - BA)\) and \((I - BA)^{-1} = I + B(I - AB)^{-1}A\)

\[(I - AB)\) is invertible

\( \rightarrow (I - AB)(x) = 0 \rightarrow x = 0 \)

\( AB(x) = x \rightarrow x = 0 \)

let \((I - BA)(y) = y\)

\( \Rightarrow BA(y) = y \)

\( \Rightarrow AB(A(y)) = A(y) \)

\( \Rightarrow A(y) = 0 \)

\( \Rightarrow BA(y) = 0, \text{ so } y = 0 \) (as \( BA(y) = y \)) so invertible
let \((I - BA)(x) = y\)
So \(x - y = BA(x)\)
\[\text{PT} \ (I - BA)^{-1}(y) = (I + B(I - AB)^{-1}A)(y)\]
\[\text{PT} \ x = y + (B(I - AB)^{-1}A)(y)\]
\[\text{PT} \ x - y = (B(I - AB)^{-1}A)(y)\]
\[\text{PT} \ BA(x) = (B(I - AB)^{-1}A)(I - BA)(x)\]
\[\text{PT} \ BA(x) = (B(I - AB)^{-1})(A - ABA)(x)\]
put \(A(x) = z\)
\[\text{PT} \ B(z) = (B(I - AB)^{-1})(z - AB(z))\]
\[\text{PT} \ B(z) = B(z) \text{ which is true .}\]

2. **PT AB and BA have same eigen values**
just like above we can prove that \((kI - AB)\) is invertible iff \((kI - BA)\)
is invertible
so \(\det(kI - AB) = 0 \iff \det(kI - BA)\)

3. **Let A be a n X n diagonal matrix with char poly \[\prod(x-c_i)^{d_i}.(all\ distinct)\]**
Let \(V\) be the vector space of all matrices \(B\) such that \(AB = BA.\)**
\(\text{PT} \) The dim of \(V\) is \(\sum d_i^2\)
so \(A\) has \(d_i\) columns with a single entry \(c_i\), and rest 0 for \(i = 1, 2, ...k\).
Multiply out \(AB\) \((B\ is\ some\ arbit\ matrix)\)
Multiply out \(BA\)
comparing entries col. wise we get that in each col of \(B\) (if the corr col
in \(A\), the entry is \(c_i\) has to be 0 everywhere except at \(d_i\) places
and there are \(d_i\) such cols for a given \(i\)
so the result

**Annihilating polynomials**

**Definitions**

1. \(F[x] = \) all polynomials with coeffs from field \(F\)

2. ideal of \(F[x] = M = \) subspace of \(F[x]\) such that if \(p \in M\), \(q \in F[x] \Rightarrow pq \in M\)

**Any such** \(M = pF[x]\) **(principal ideal) (p monic and unique)**

**Proof** : look at min deg polynomial. wlog assume it is monic (else multiply
by scalar as ideal ). Call it \(p\)
if \(f \in M\), \(f = qp + r\).
\(\text{deg}(r) < \text{deg}(p)\)
\(\Rightarrow r = 0\)
if \( pF[x] = qF[x] \),
\[ \Rightarrow p = qr, \quad q = ps \]
\[ \Rightarrow p = psr \]
\[ \Rightarrow \deg(s) = \deg(r) = 0, \text{ so } r = s = 1 \text{ (monic)} \]

polynomials which annihilate a linear operator \( T \) form a principal ideal

also if \( T \) acts on finite dim \( V \), then ideal is nonempty
Proof: look at \( I, T, T^2, \ldots T^{n^2} \). Since \( L(V, V) \) has dim \( n^2 \), these are linearly dependent. Hence done
that monic polynomial which generates this annihilating polynomials = minimal polynomial
from above we get minimal polynomial has deg \( \leq n^2 \)

similar matrices have same minimal polynomial
Proof: \( f(PAP^{-1}) = P f(A) P^{-1} \forall f \) polynomials
so if \( f \) annihilates \( A \) iff it annihilates \( PAP^{-1} \)

If \( A \) is a \( nXn \) matric on \( F \) which is a subfield of \( G \), then \( A \) over \( F \) has same minimal polynomial as \( A \) over \( G \)
let \( f \) be minimal polynomial of \( A \) over \( F \), clearly \( f \) is an annihilating polynomial of \( A \) over \( G \)
let \( f \) be minimal poly of \( A \) over \( G \) of deg \( k \).
\[ f(A) = A^k + a_1 A^{k-1} + a_2 A^{k-2} + \ldots + a_0 I = 0 \]
\( A \) has \( n^2 \) entries. so evaluating \( f(A) \), we get \( n^2 \) equations for \( a_i \)s.
we already know that these \( a_i \)'s exist and unique
so the coeffs of these \( n^2 \) polynomials lie in \( F \) (as \( A \) entries lie in \( F \))
so \( \det \neq 0 \) in \( G \)
but as all entries in \( F \), so \( \det \neq 0 \) in \( F \)
so solution to \( a_i \)s exist in \( F \)
as solution is unique, same minimal polynomial
$T : V \rightarrow V$, \dim V = n. Characteristic polynomial and minimal polynomial have same roots except for multiplicities

let $p$ be minimal polynomial and $p(c) = 0$
\Rightarrow $p(x) = (x - c)q$

as $p$ is minimal polynomial, then $q(T) \neq 0$
so choose $\beta$ such that $q(T)(\beta) = \alpha \neq 0$
$p(T)(\beta) = 0 = (T - cI)q(T)(\beta) = (T - cI)(\alpha)$ where $\alpha \neq 0$
so $c$ is eigen value

let $c$ be an eigen value
\Rightarrow $T(\alpha) = c(\alpha)$

from prev lemmas $p(T)(\alpha) = p(c)\alpha$
but $p(T)(\alpha) = 0$
So $p(c)\alpha = 0$
$\alpha \neq 0$, $\Rightarrow p(c) = 0$

If $T$ is diagonalizable, and $c_i$ s where $i \in \{1, 2, \ldots k\}$ are eigen values, then minimal polynomial = $p = \prod_{i=1}^{k}(x - c_i)$

$p(T)(\alpha) = 0$ for any eigen vector $\alpha$
but the basis consists of eigen vectors as $T$ is diagonalizable
so $p(T) = 0$

Cayley hamilton theorem: the minimal polynomial divides the characteristic polynomial

Exercises Pg 197-8

1. $T : V \rightarrow V$, \dim V = n. $T^k = 0$ for some +ve int $k$. $PT^n = 0$
if $k \leq n$, then done
else
now char poly has deg $n$ and is div by min poly = $p(x)$, so deg of $p(x) \leq n$
also $p(x)|x^k$, so $p(x) = x^l$
\Rightarrow $l \leq n$

hence done
2. *let P be a space of all poly of deg ≤ n. and D be diff operator. what is the minimal poly?*

\[ p(x) = x^{n+1} \]

\[ A = \text{the matrix of D is a } n + 1 \times n + 1 \text{ matrix} \]

\[
\begin{pmatrix}
0 & 1 & 0 & \ldots & 0 \\
0 & 0 & 2 & \ldots & 0 \\
0 & 0 & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & n
\end{pmatrix}
\]

so char poly is \( x^{n+1} \)

each time a power is taken of \( A \), kernel dim increases by 1

so min poly is \( p(x) = x^{n+1} \)

3. *Let V be vector space of n X n matrices over field F. Let A be a fixed n X n matrix. Let T be linear operator such that \( T(B) = AB \) . PT A, T have same min poly*

let Basis of \( T = \{ e_{ij} | 1 \leq i, j \leq n \} \) where \( e_{ij} = 1 \) in the \( ij^{th} \) pos and rest 0.

so matrix of \( T = B \) looks like this

\[
\begin{pmatrix}
A & 0 & 0 & \ldots & 0 \\
0 & A & 0 & \ldots & 0 \\
0 & 0 & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & A
\end{pmatrix}
\]

and \( B^k \) like this

\[
\begin{pmatrix}
A^k & 0 & 0 & \ldots & 0 \\
0 & A^k & 0 & \ldots & 0 \\
0 & 0 & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & A^k
\end{pmatrix}
\]

hence done
Invariant subspaces

- $T(W) \subseteq W \Rightarrow W$ is an invariant subspace under $T$

some examples

1. kernel, image, zero space, the full space

2. if $U$ is a map such that $TU = UT$, then kernel, image of $U$ are invariant under $T$
   proof:
   let $\alpha \in$ kernel of $U$
   $\Rightarrow U(\alpha) = 0$
   $TU(\alpha) = 0$
   so $U(T(\alpha)) = 0$, so $T(\alpha)$ in kernel of $U$
   similarly we can do for image of $U$\footnote{example of $U$ is any polynomial of $T.$}

3. space spanned by all characteristic vectors is invariant (: )

- if $V$ has an invariant space $W$, then $T$ can be represented by foll matrix

\[
\begin{pmatrix}
A & C \\
0 & D
\end{pmatrix}
\]

\footnote{by extending basis of $W$ to basis of $V$}

- $T$ restricted to $W = T_W$

- char poly, min poly of $T_W$ divides char poly, min poly of $T$
  proof: char poly - obv from matrix
  min poly : matrix raised to $k$th power is
  \[
  \begin{pmatrix}
  A^k & C_k \\
  0 & D^k
  \end{pmatrix}
  \]
  so we can conclude the result
• **T conductor of** $\alpha$ **into** $W = \text{all polynomials } g \text{ such that } g(T)(\alpha) \in W$

• **special term : T annihilator of** $\alpha$ **if** $W = \{0\}$

• **T conductor of** $\alpha$ **into** $W$ **is an ideal in** $F[x]^{\geq 2}$
  its also a principal ideal (as ideal of $F[x]$)
  so **T conductor of** $\alpha$ **into** $W$ **also sometimes refers to the monic generator**

• **T conductor of** $\alpha$ **into** $W$ **divides min poly of** $T$
  proof : min poly of $T = p$ say
  then $p(T)(\alpha) = 0 \in W$
  so $p$ in ideal

• $p = \prod_{i=1}^{k}(x - c_i)^{r_i}$ **is the minimal polynomial of** $T$ **a linear operator over** $V$. **let** $W \neq V$ **be an invariant subspace. then** $\exists \alpha$ **not in** $W$ **such that** $(T - c_i)(\alpha) \in W$ **where** $c_i$ **is an eigen value**
  let $\beta$ **not in** $W$.
  so $g$ divides $p$ and it is not constant
  so some $x - c_j$ divides $g$
  $g = (x - c_j)h$
  $h(T)(\beta) = \alpha$ **is not in** $W$
  so $(T - c_j)(\alpha) \in W$
  hence done

$T$ **is triangulable iff min poly is linearly factorizable**

choose $W = \{0\}$
apply lemma to get $\alpha_1$
$W_1 = \text{subspace spanned by } \alpha_1$
repeat procedure ($W_i$s so formed are invariant)
the other way - in the triangular matrix , char poly is factorizable into linear

\[22\text{work it out}\]
factors
so as min poly divides char poly, the result follows

so over any algebraically closed field, any operator is triangulable

\( T \) is diagonalizable iff min poly is \( p = \prod_{i=1}^{k}(x - c_i) \) where all \( c_i \) are distinct

one way proved before
let min ploy be of this form
\( W \) be space spanned by all eigen vectors and \( \neq V \)
so by some lemma, there is \( \alpha \) not in \( W \) such that \( (T - c_j I)(\alpha) = \beta \in W \)
let \( p = (x - c_j)q \)
we will arrive at a contradiction by saying \( x - c_j \) divides \( q \) (so a non distinct root)
\( q - q(c_j) = (x - c_j)h \)
so \( q(T)(\alpha) - q(c_j)(\alpha) = h(T)(T - c_j I)(\alpha) = h(T)(\beta) \in W \)
also \( 0 = p(T)(\alpha) = (T - c_j I)q(T)(\alpha) \)
so \( q(c_j)(\alpha)inW \)
so \( q(c_j) = 0 \)

note that cayley hamiltonian theorem can be proved using this (it is proved for matrices over algebraically closed field, and any field is a subfield of algebraically field ) so hence done

any matrix over algebraically closed field is similar to triangular matrix
let the matrix be
\[
\begin{pmatrix}
c_1 & * & * & * \\
0 & c_2 & * & * \\
. & . & . & . \\
0 & 0 & . & c_k
\end{pmatrix}
\]
look at \( \prod_{i=1}^{k}(T - c_i I)^{r_i} \) acting on any of this particular basis vector
it will become 0
hence done
Exercises-Pg 205-6

1. Let $T$ be a linear operator on finite dim $V$ over closed field $F$. Let $f$ be a polynomial. PT $c$ is an eigen value of $f(T)$ iff $c = f(t)$ where $t$ is an eigen value of $T$
if $T(\beta) = t\beta$
$\Rightarrow f(T)(\beta) = f(t)(\beta)$ so done

the other way
As $T$ is triangular, let matrix be the following

$$
\begin{pmatrix}
c_1 & * & * & * \\
0 & c_2 & * & * \\
. & . & . & . \\
0 & 0 & . & c_k
\end{pmatrix}
$$

and let basis be $\{\alpha_i|1 \leq i \leq n\}$
now $f(T)(\alpha) = c(\alpha)$
$\alpha = \sum_{i=1}^{n} x_i \alpha_i$
compare the coeff of $\alpha_n$
we get the answer