QUERY COMPLEXITY LOWER BOUNDS FOR LOCAL LIST-DECODING AND HARD-CORE PREDICATES

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Talk outline

- List decoding and local list decoding

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- Our results and applications to blackbox proofs for hardcore predicates

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- List decoding and local list decoding
- Our results and applications to blackbox proofs for hardcore predicates
- Lower bound argument based on a coin problem
- Open problems

List decodable and local list decodable codes

- Binary code $Enc: \{0,1\}^k \to \{0,1\}^n$ is $\left(\frac{1}{2} - \varepsilon, L\right)$-list decodable if [Elias, Wozencraft 60's]

$w \in \{0,1\}^n$

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$dist(\mathrm{x}, \mathrm{y})$: Hamming distance between x and y

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$w \in \{0,1\}^n$

$m^{(1)} \in \{0,1\}^k$

$m^{(2)} \in \{0,1\}^k$

$m^{(L)} \in \{0,1\}^k$

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List of $\leq L$ messages whose encodings differ from w in at most $\left(\frac{1}{2} - \varepsilon\right) n$ locations

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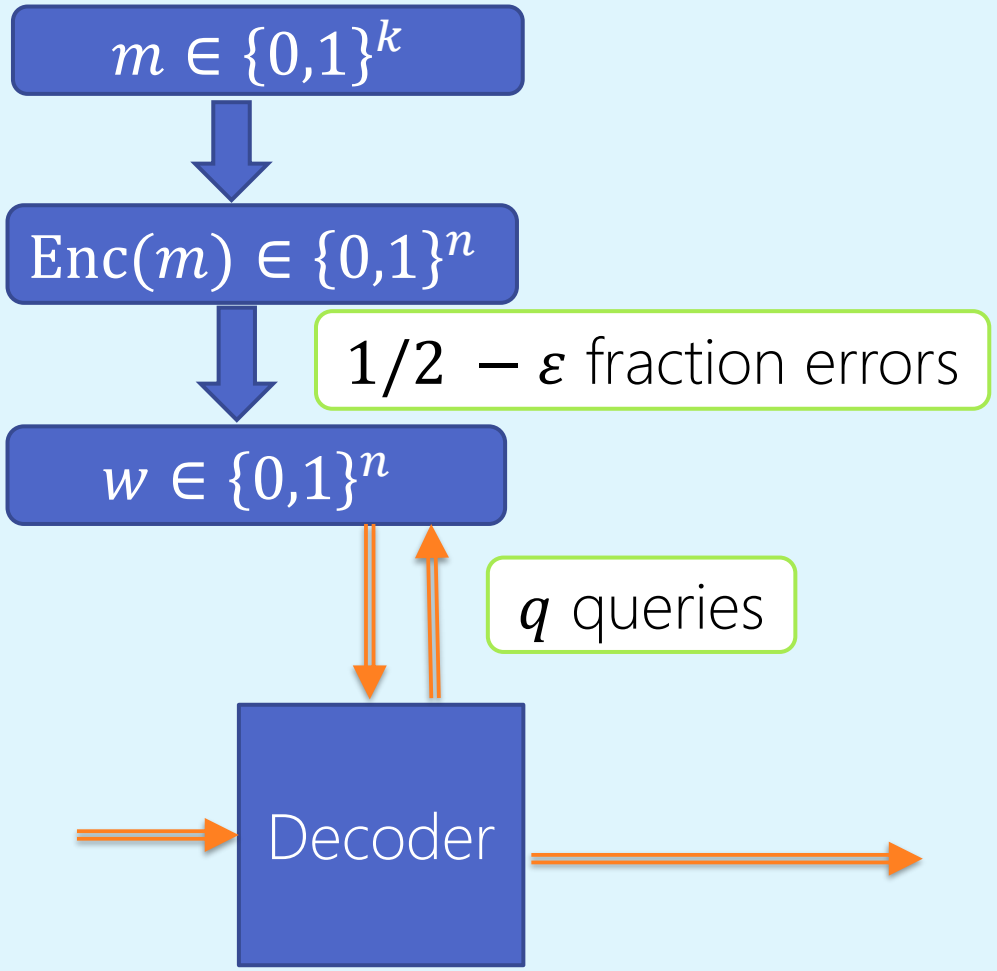
$(\frac{1}{2} - \varepsilon, q, L)$-Locally List Decodable Code

[Sudan, Trevisan, Vadhan 99]



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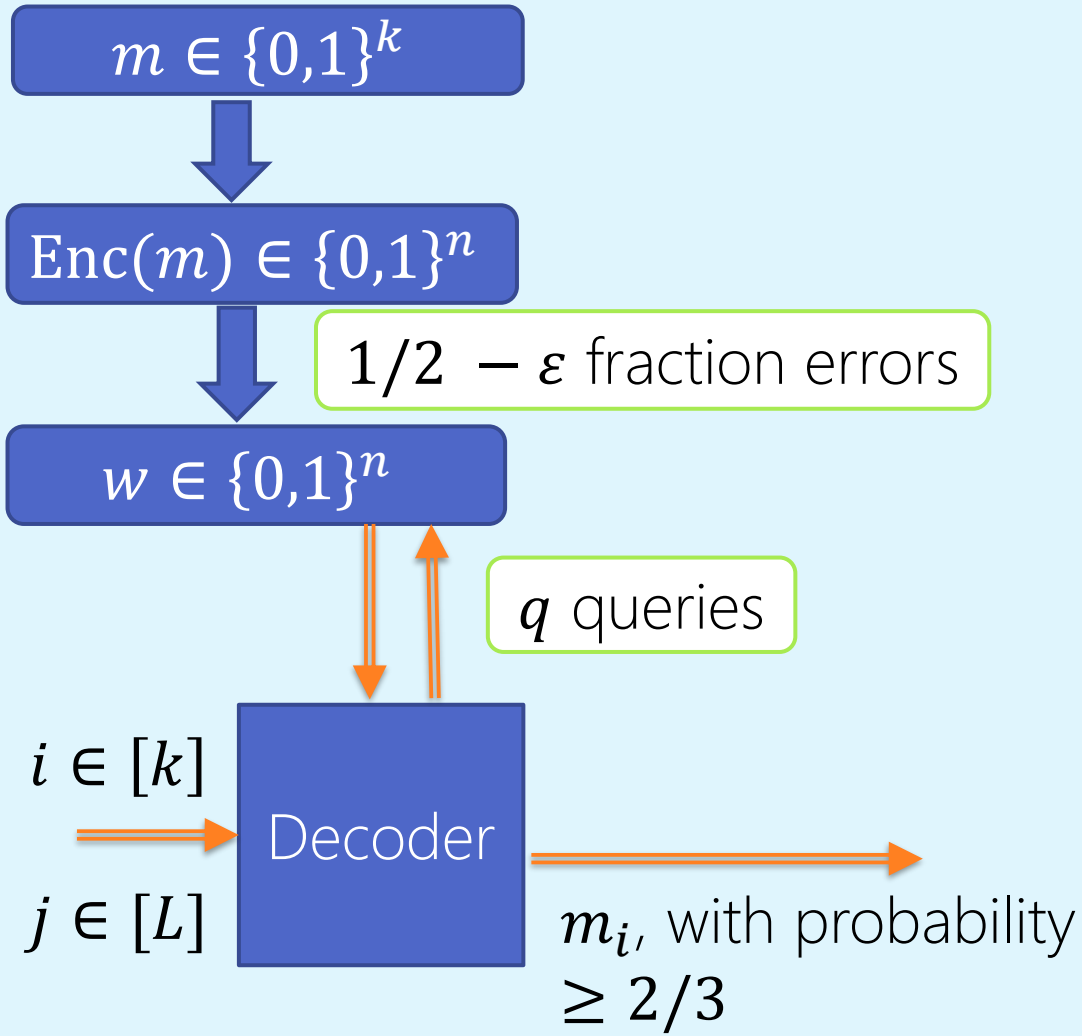

m_i: ith bit of message m

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- For every word w, and every message m, such that $\mathbf{dist}(Enc(m), w) \leq \left(\frac{1}{2} - \varepsilon\right) \cdot n$, with probability at least $2/3$,
  $\exists\ j \in [L]$ such that for all $i \in [k]$
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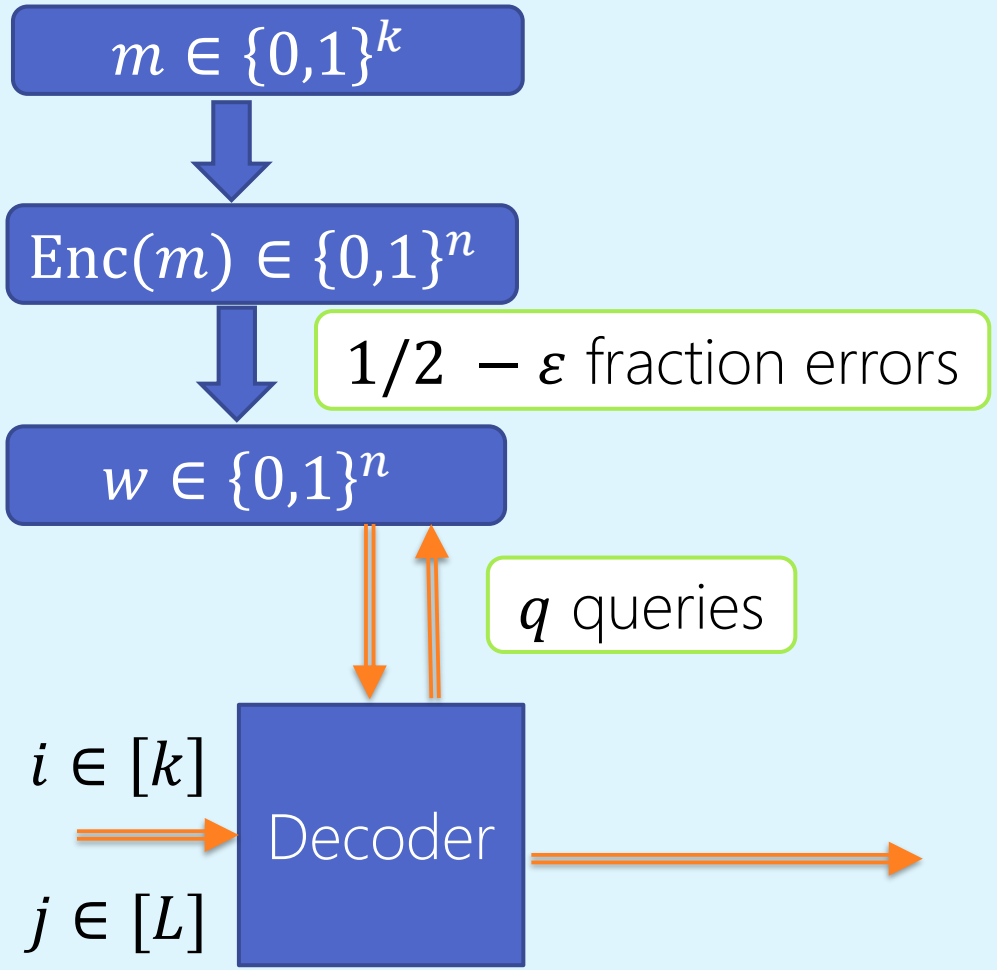

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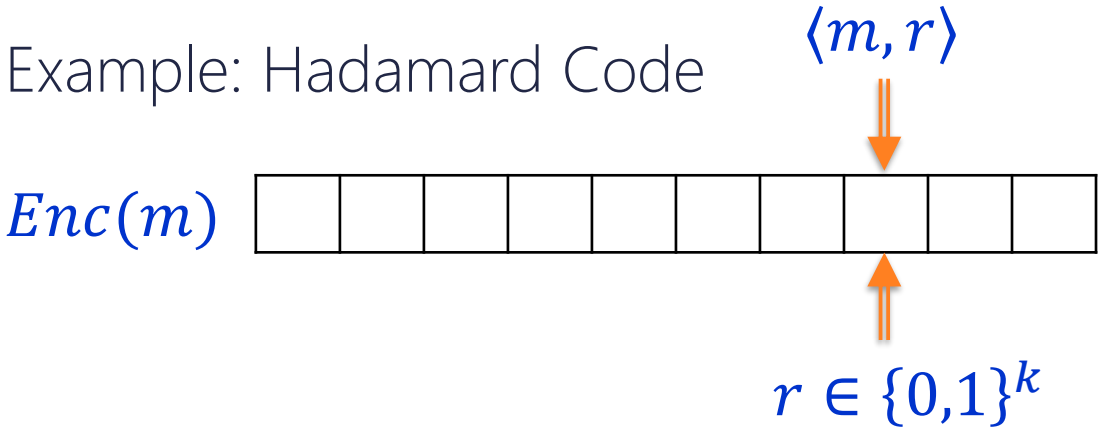
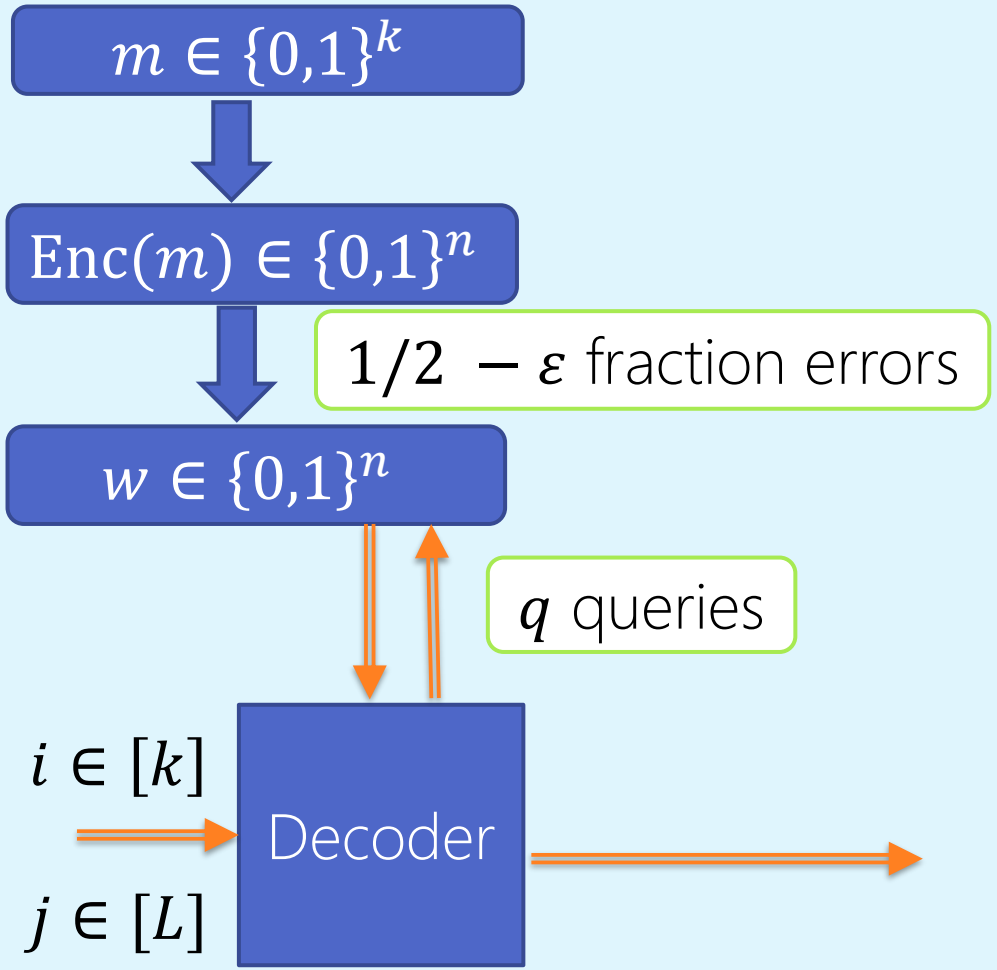
What we study

$Enc: \{0,1\}^k \to \{0,1\}^n;\ n = 2^k$ (low rate)

- Example: Hadamard Code





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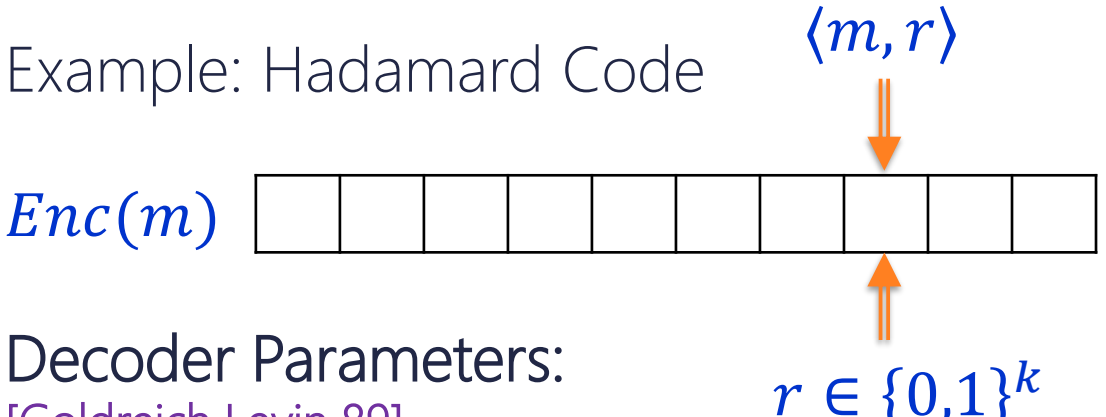

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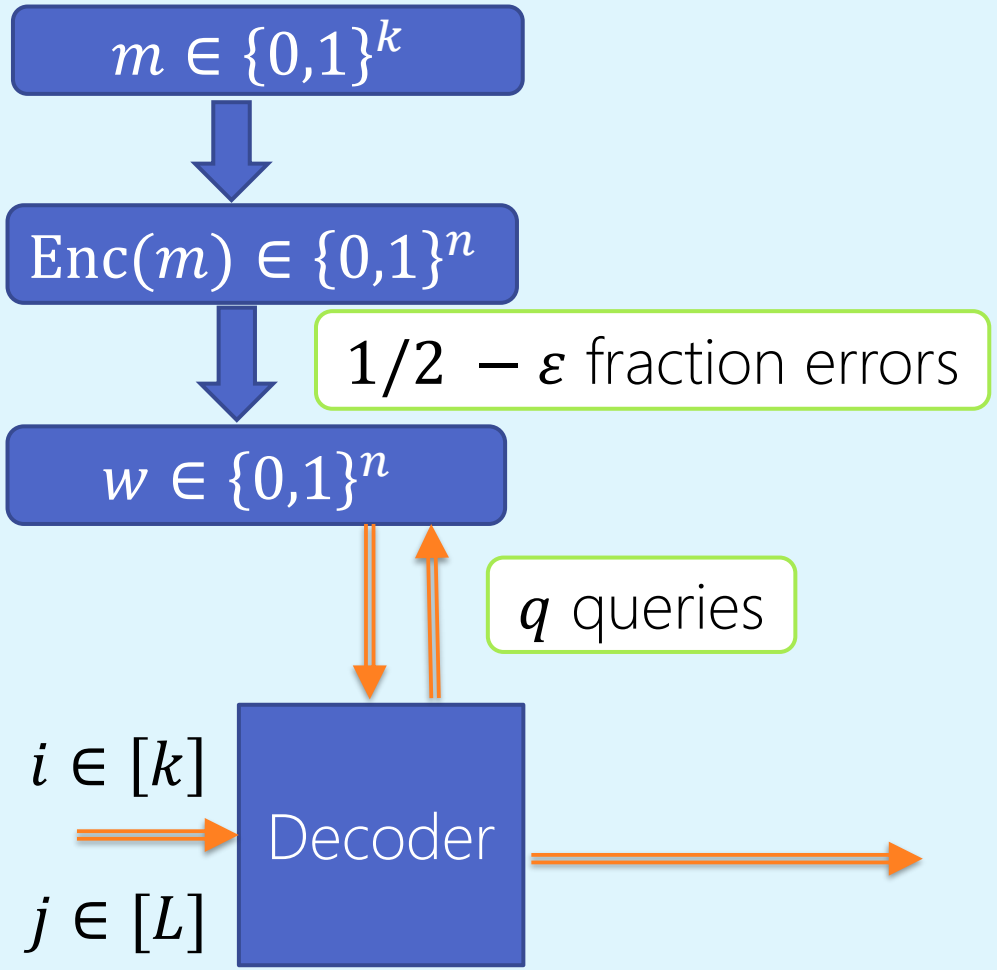

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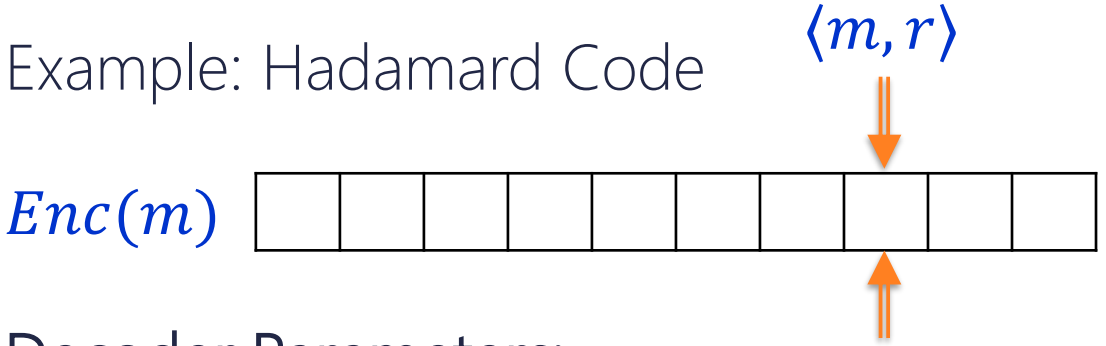

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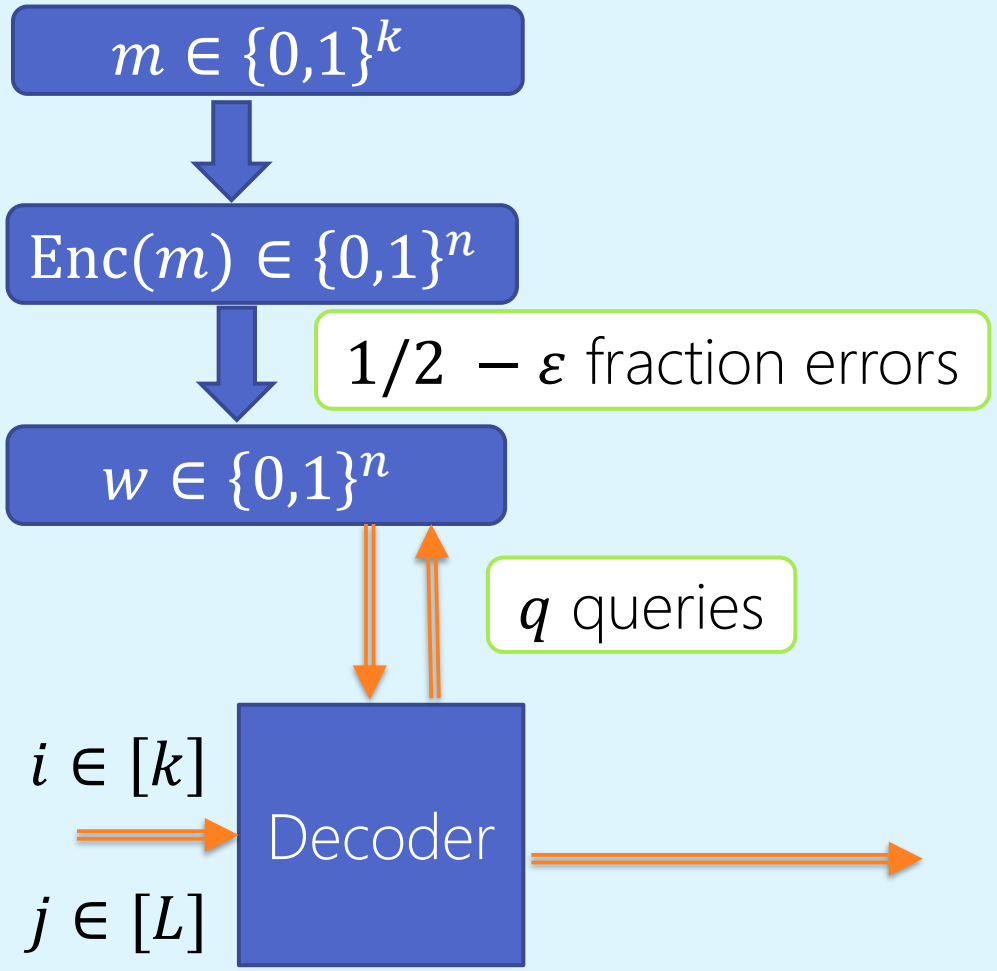

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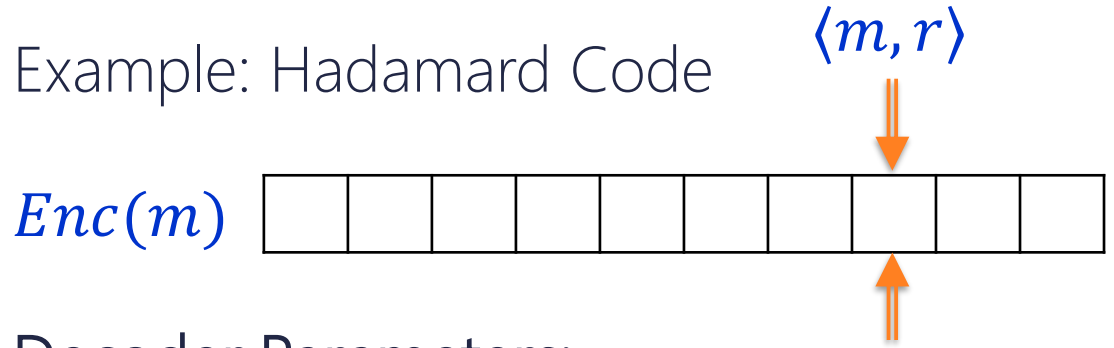

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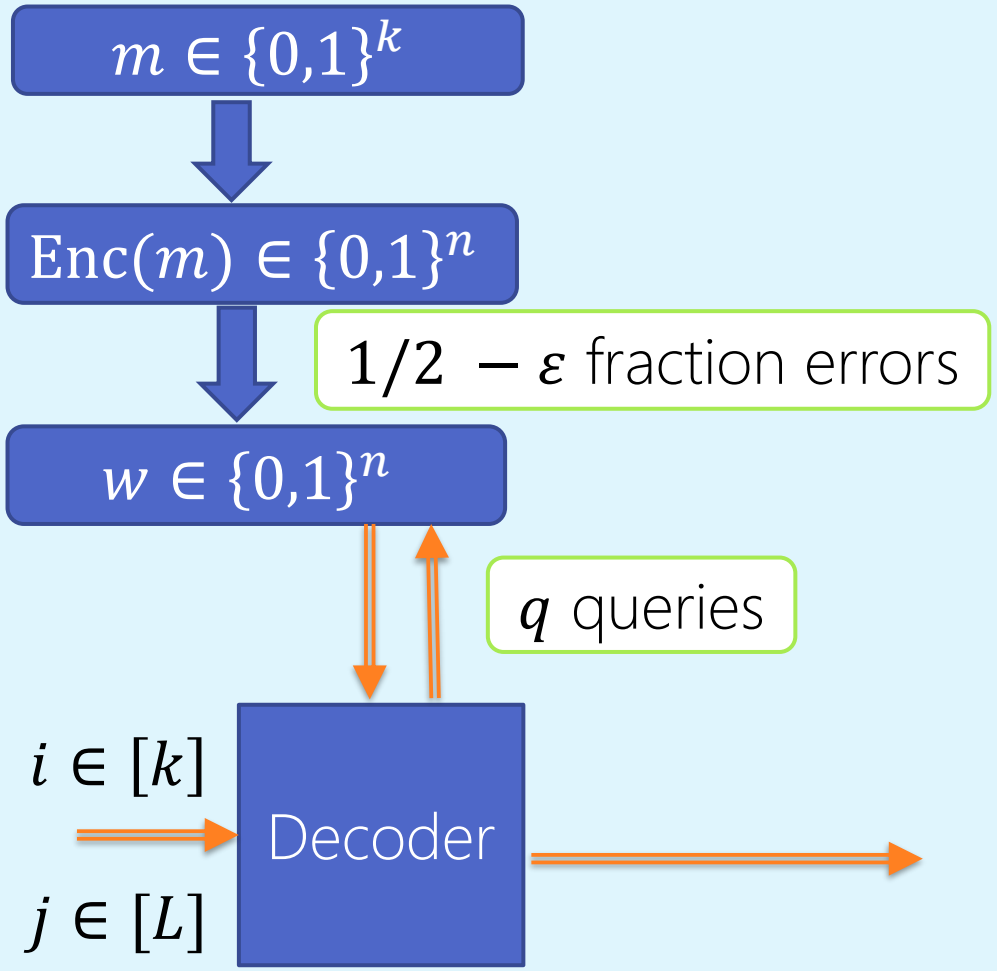

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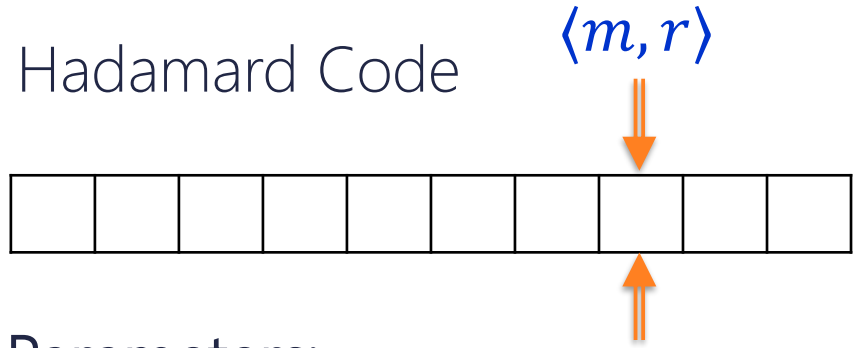

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Question: Can the number of queries be smaller, say $\left(\frac{1}{\varepsilon}\right)^{o(1)}$ if we allow for large lists, say $L = 2^{\frac{k}{100}}$?

Motivation

- Improvement in query complexity has the potential to provide "better" hardcore predicates from one-way functions (OWF) that are hard to invert

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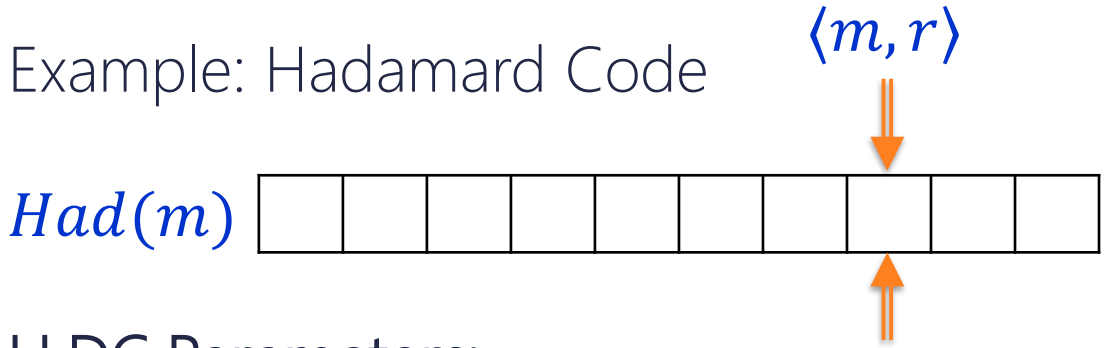

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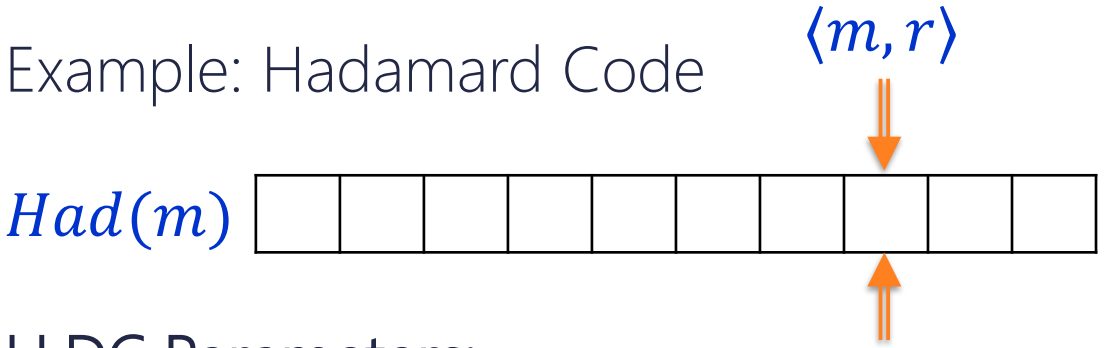

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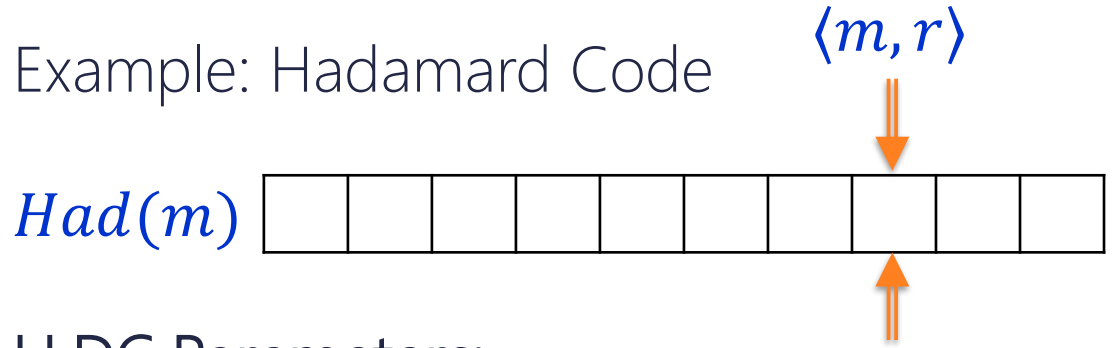

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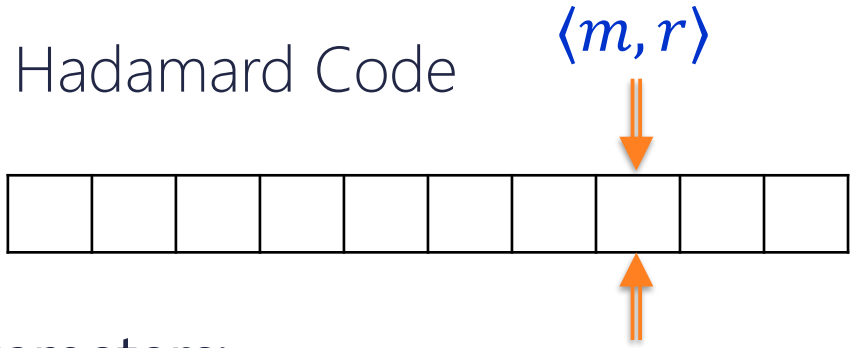

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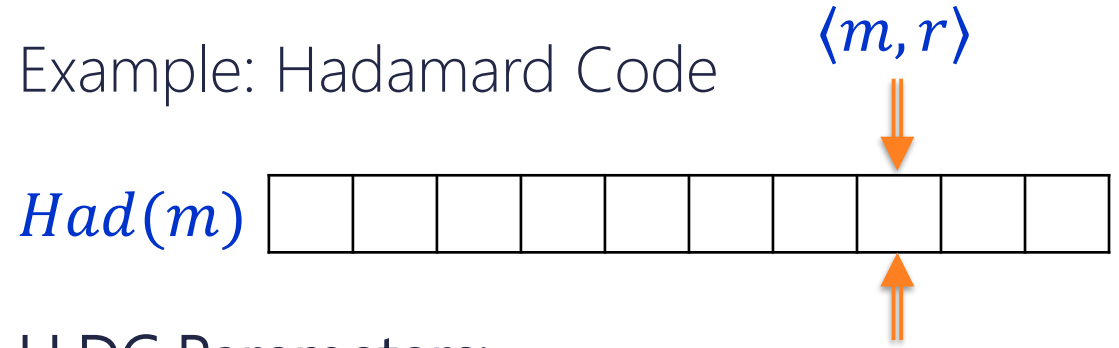

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Restatement of [Goldreich Levin 89]

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1/poly(k)

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Even if we start with a harder function, possible to obtain only $\varepsilon = 1/poly(k)$ via this approach

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Is it possible obtain a hardcore predicate with $\varepsilon = \frac{1}{k^{\omega(1)}}$ via a different "black-box method" ?

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What we will show

- Local list decoding of $Enc: \{0,1\}^k \to \{0,1\}^n$ from $\frac{1}{2} - \varepsilon$ fraction errors needs $\Omega\left(\frac{1}{\sqrt{\varepsilon}}\right)$ queries for small $\varepsilon < \frac{1}{k^{0.01'}}$

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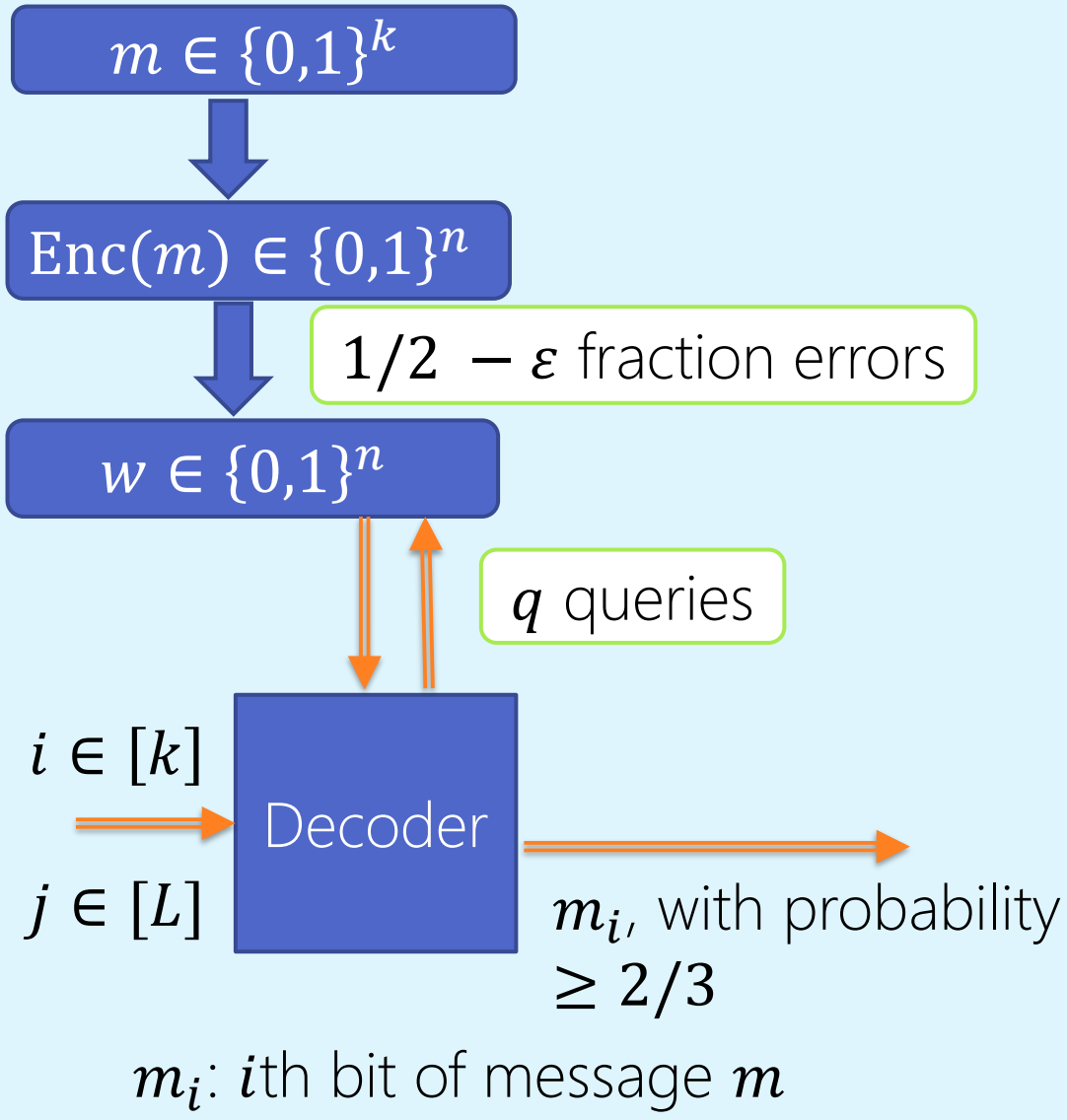
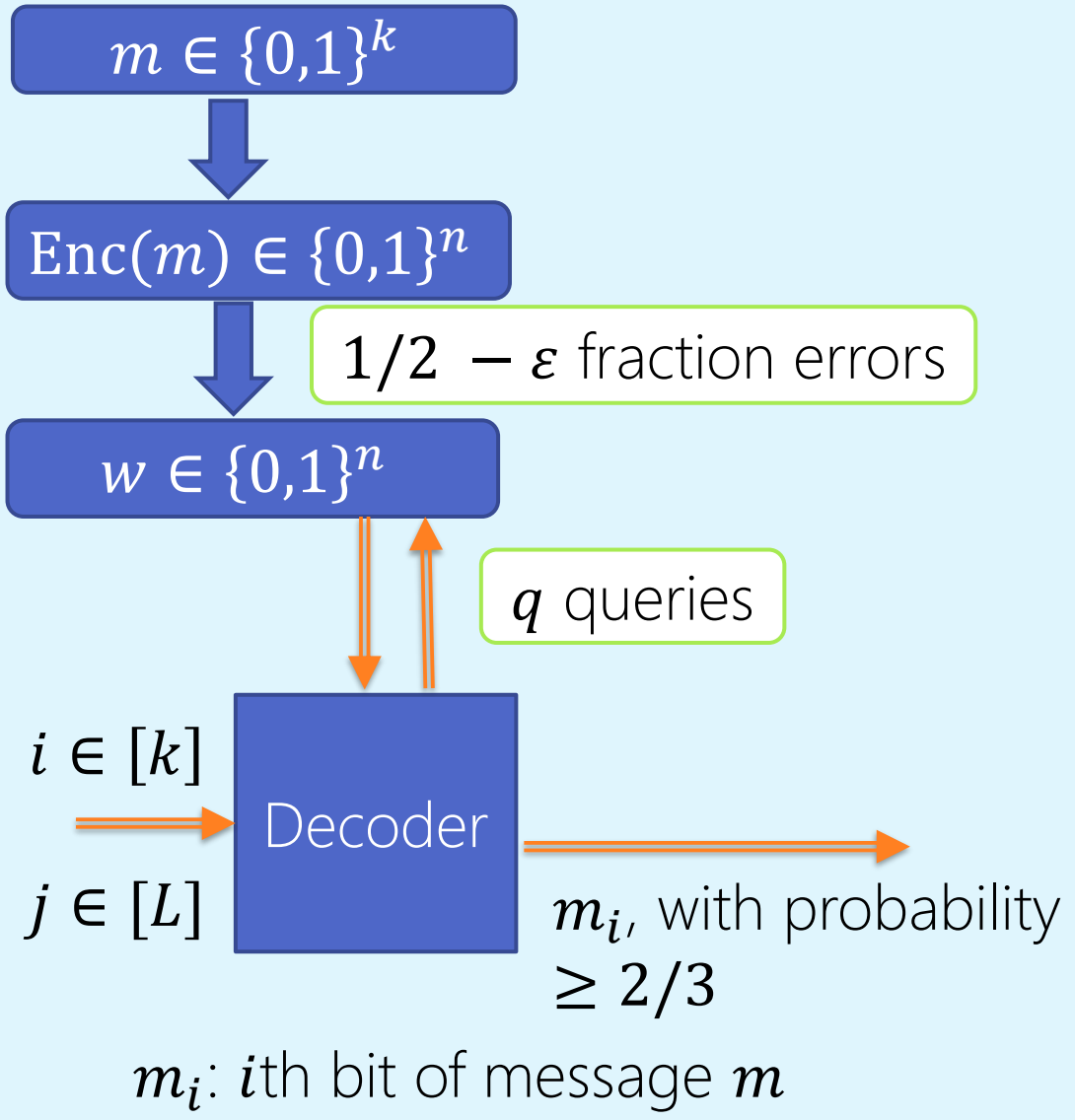
Proof idea inspired from
Applebaum, Artemenko, Shaltiel, Yang 16

Reducing to a different problem





m_i: ith bit of message m

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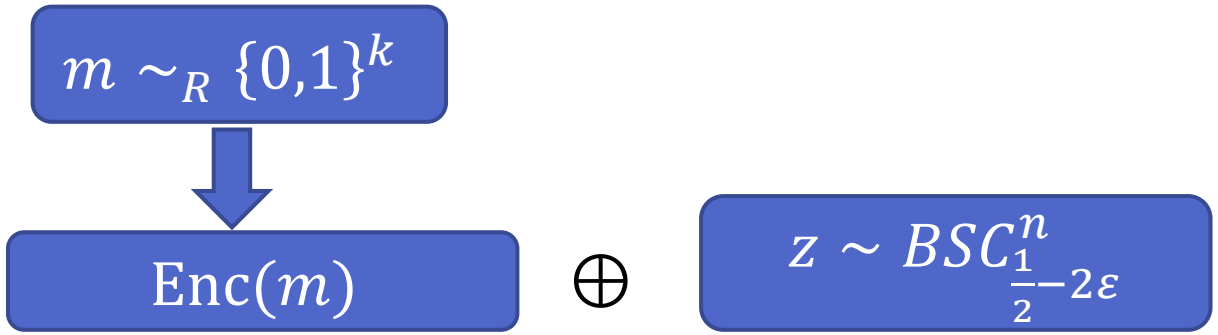
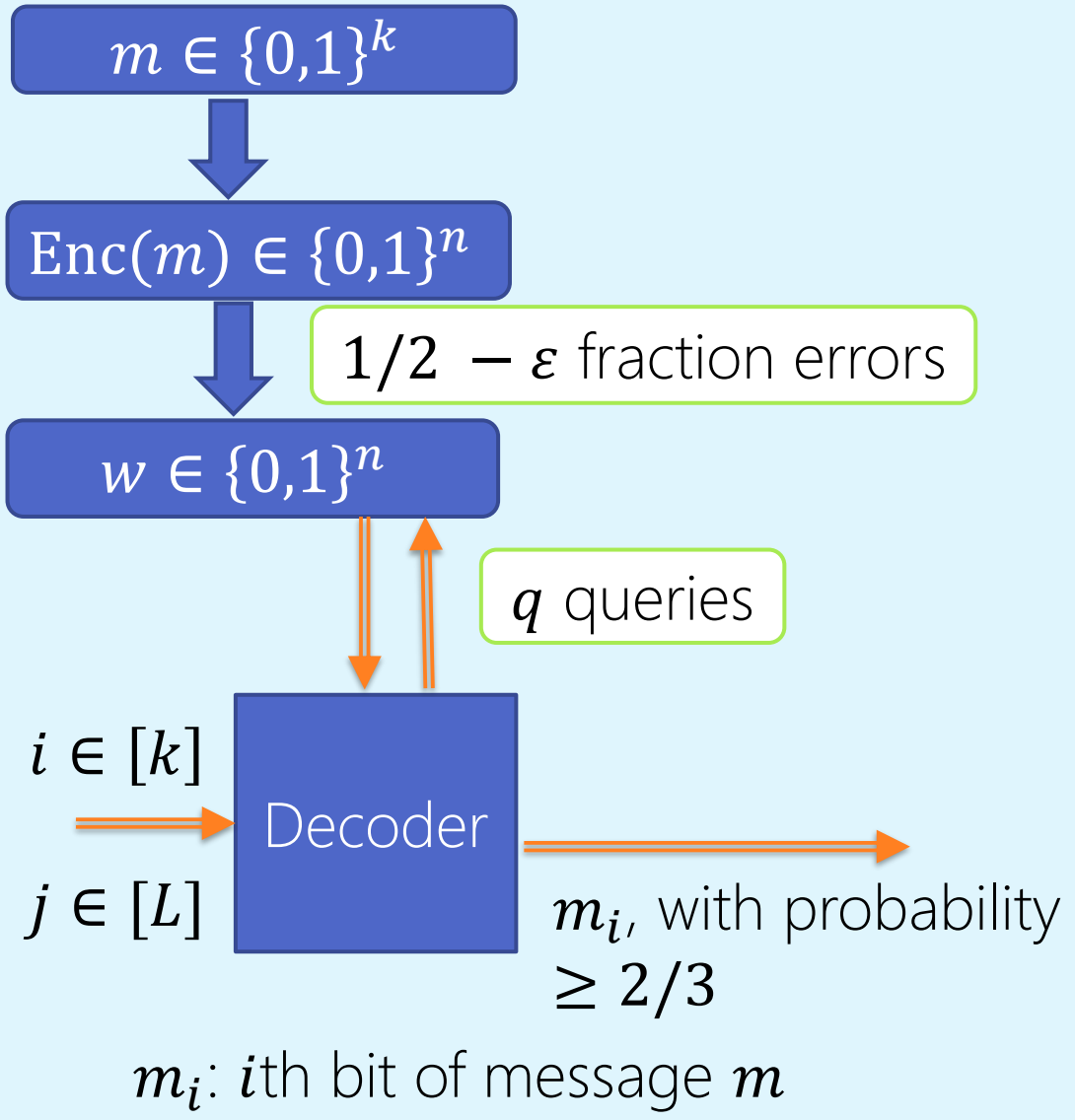
- Decode from word $w \in \{0,1\}^n$





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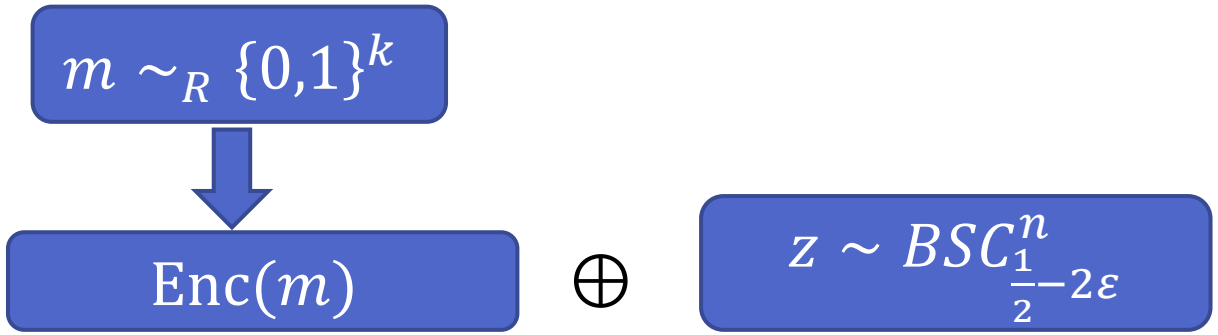
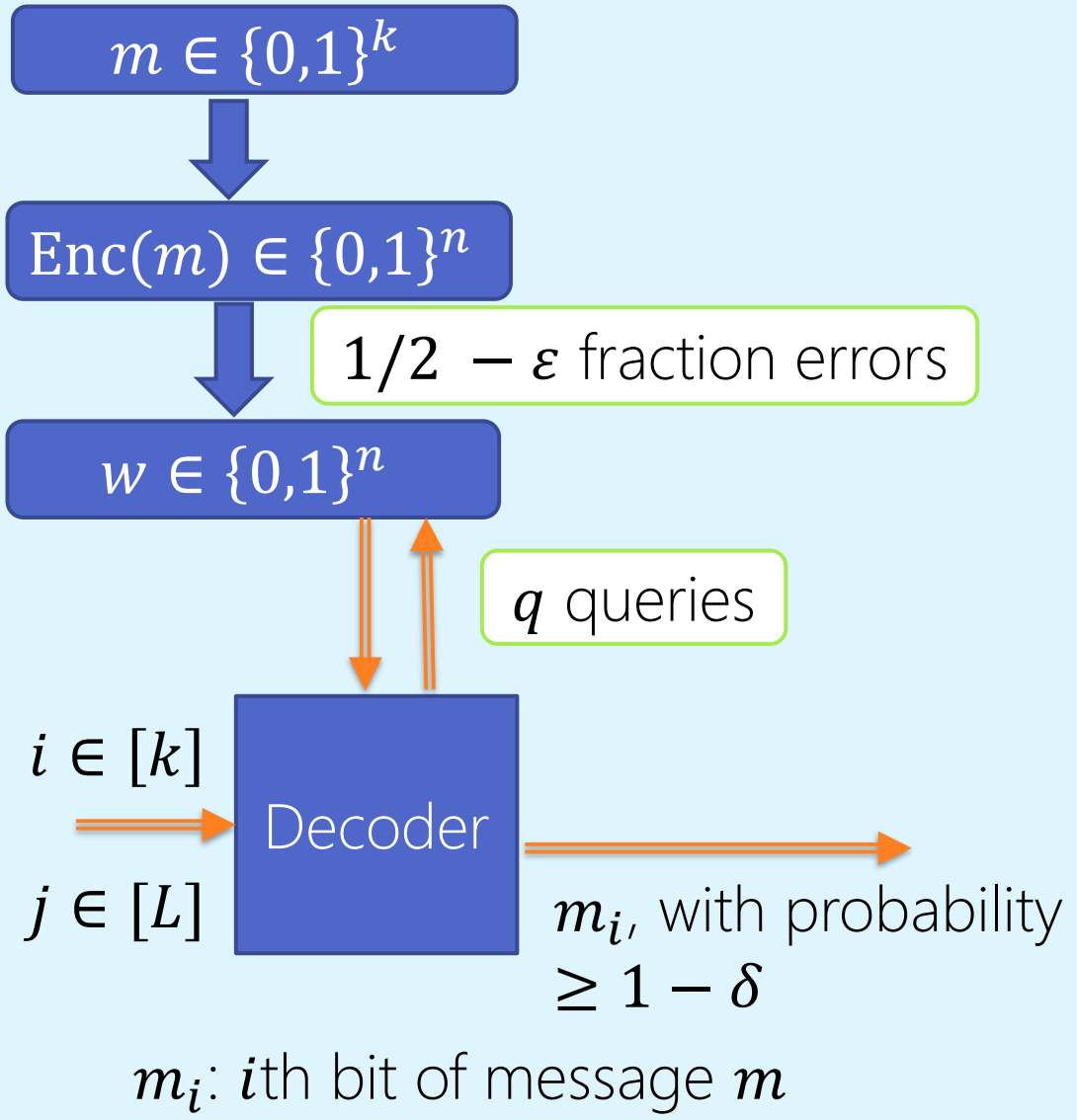


- Decode at least $1 - 1/2k$ fraction of message bits correctly



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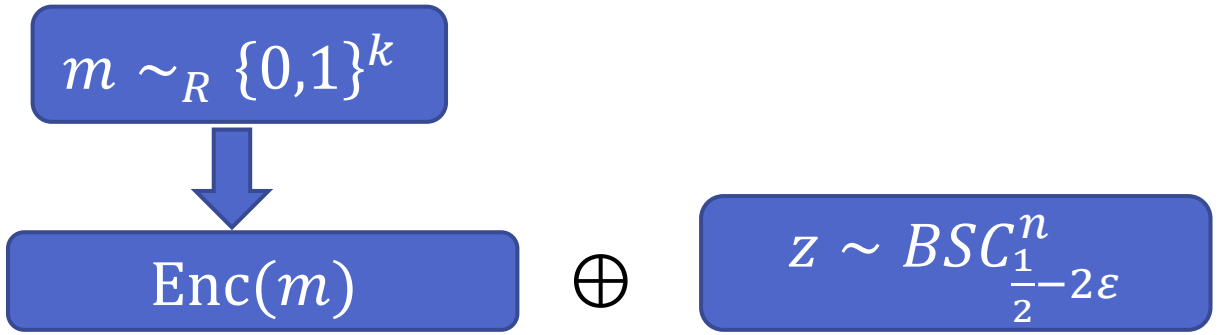

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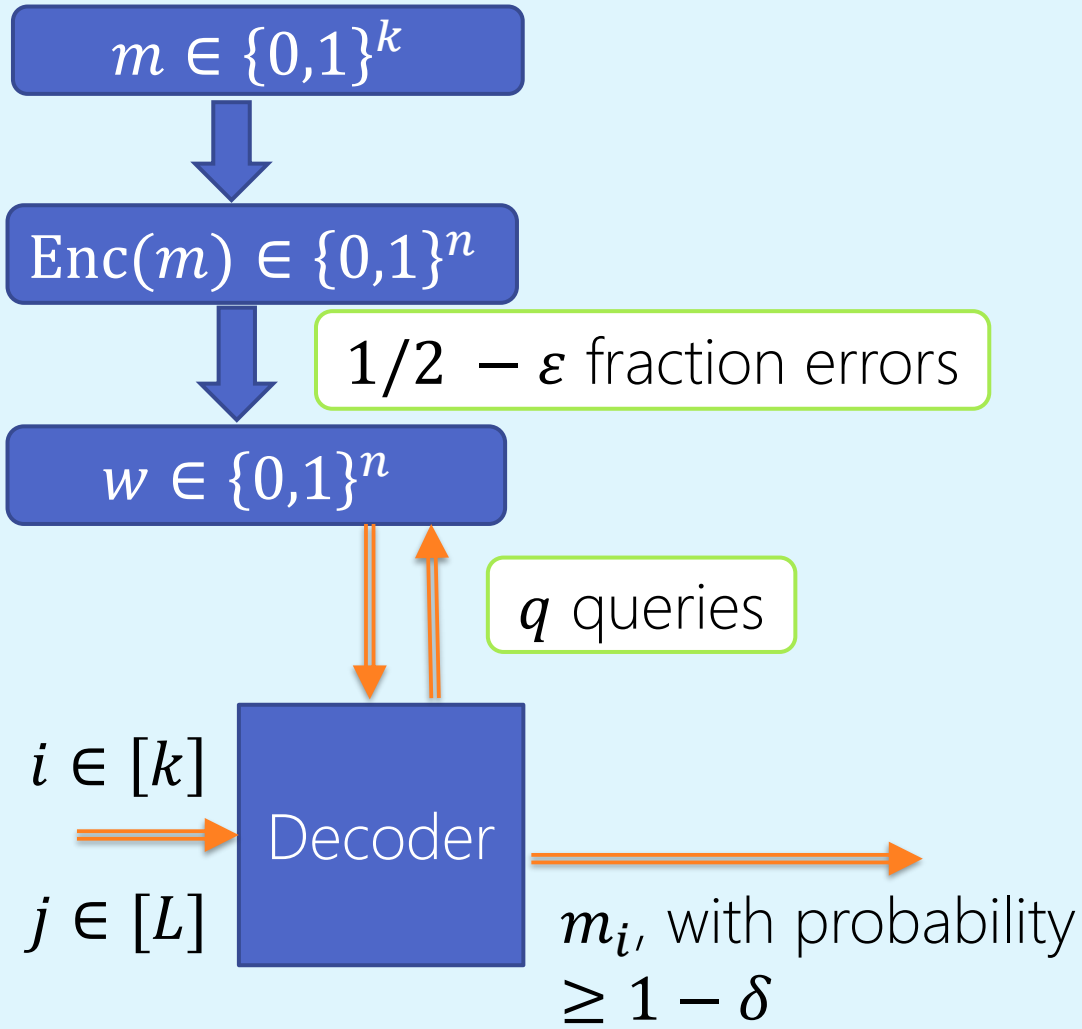

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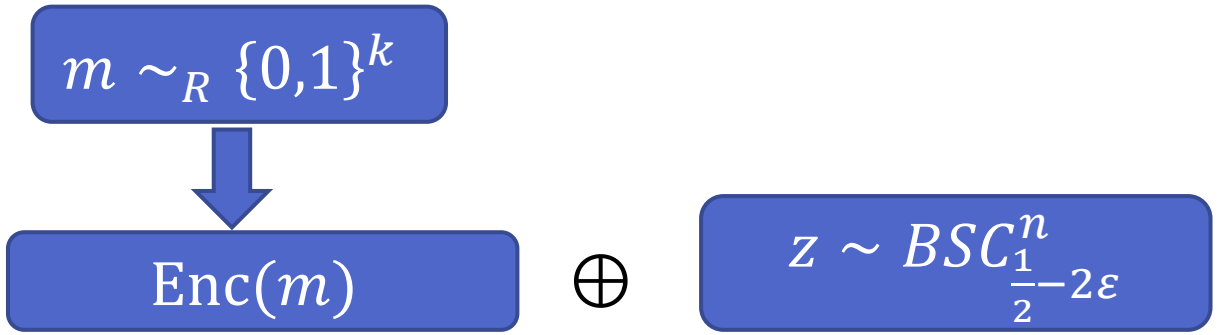

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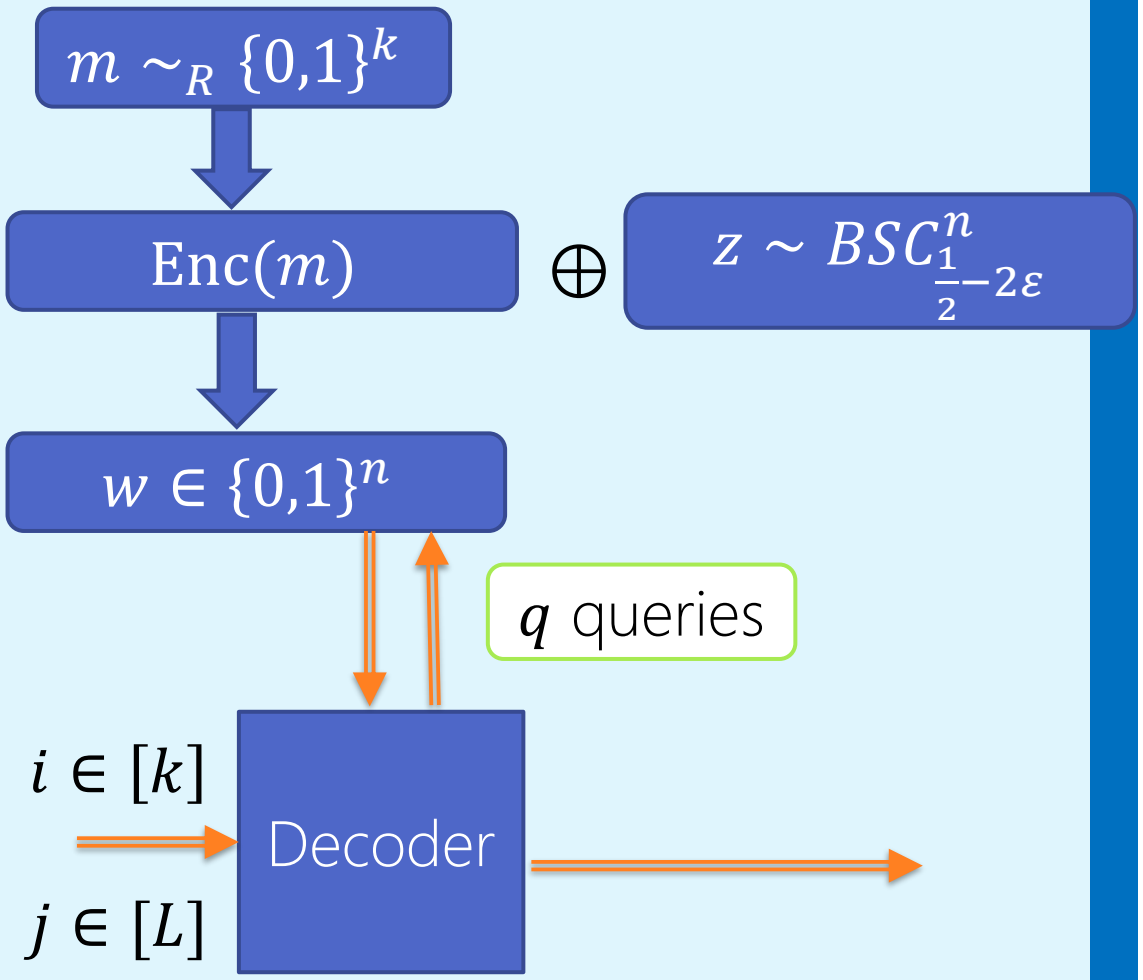

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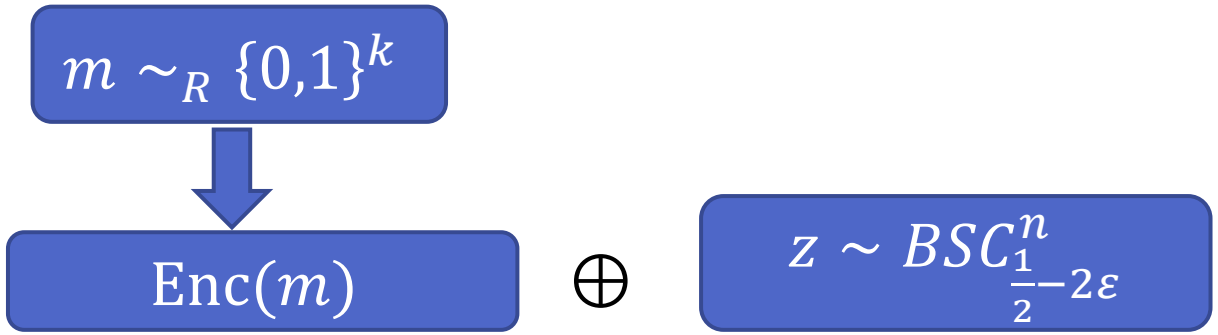

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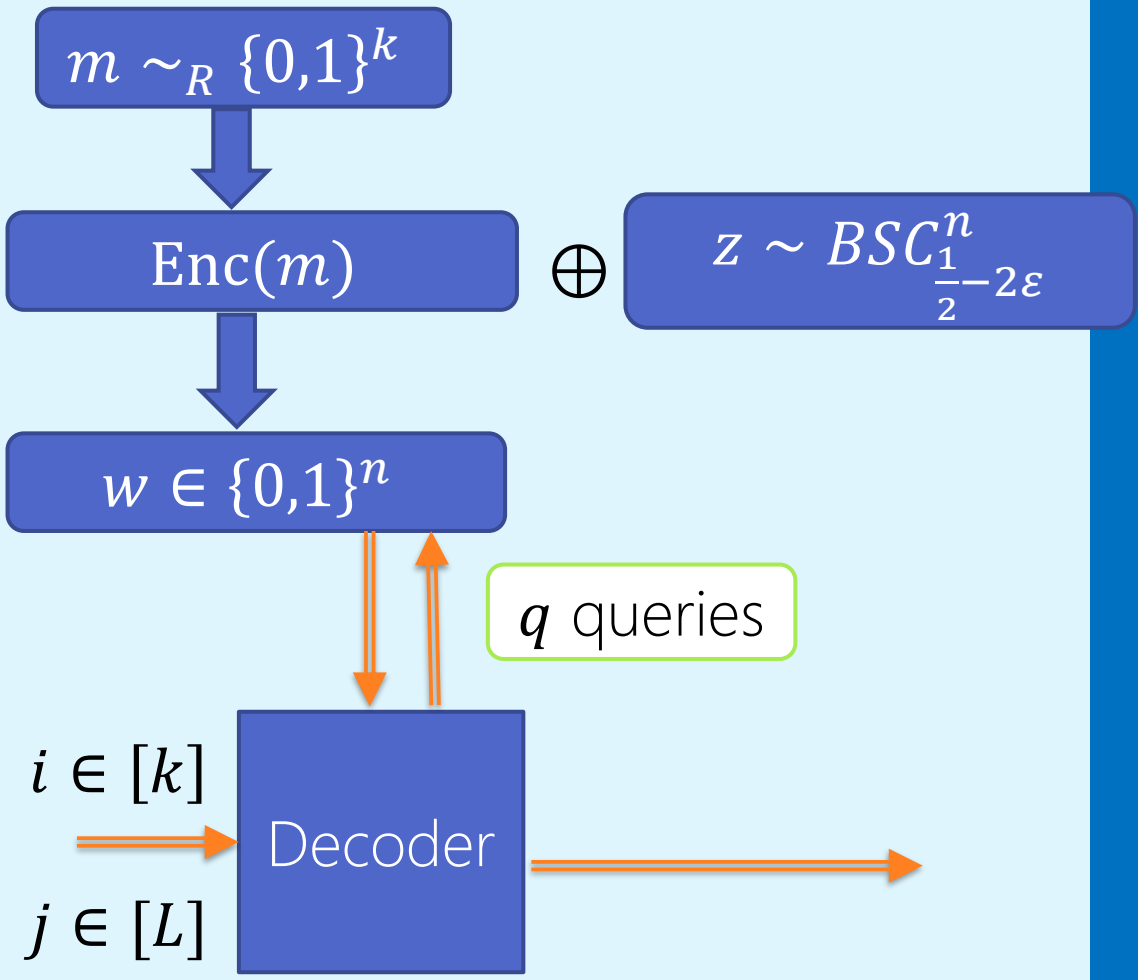

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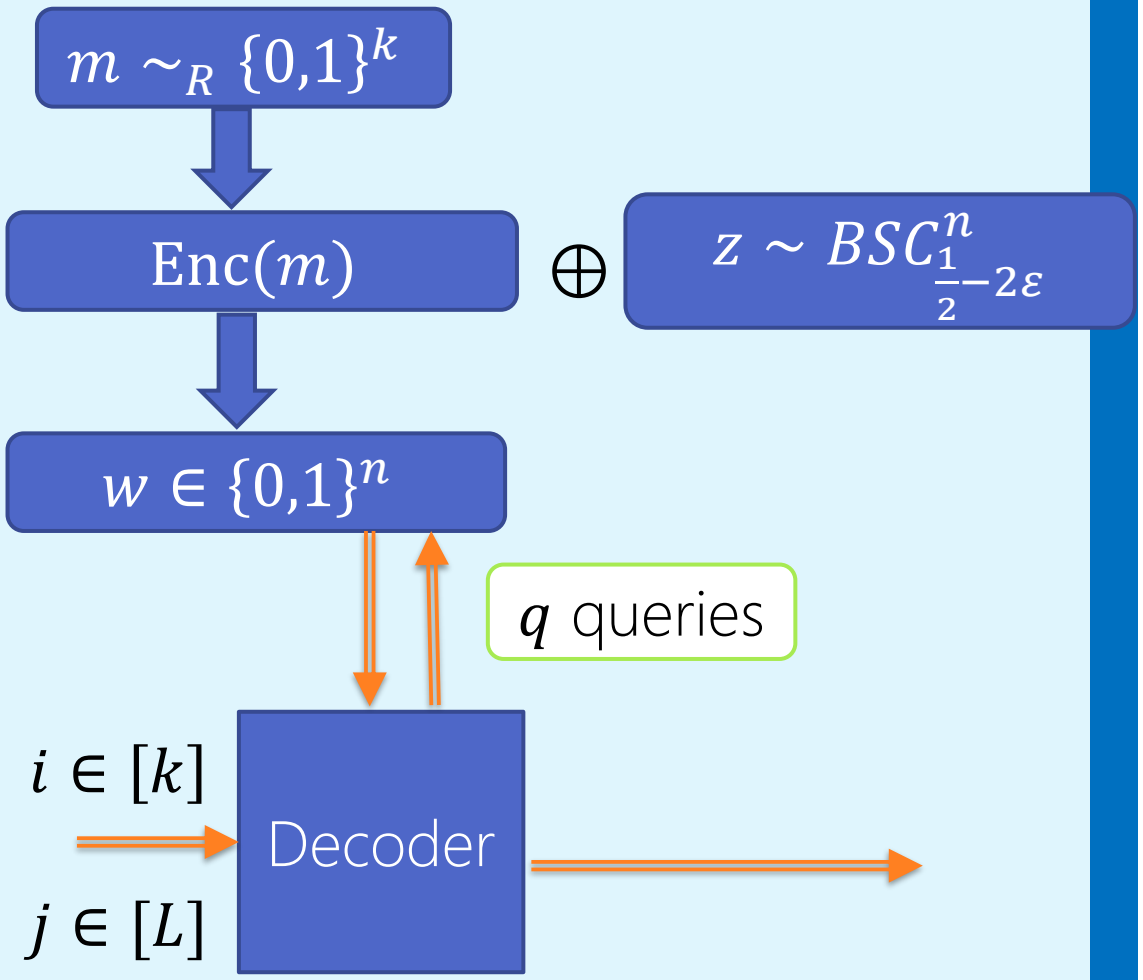
Lower bounds on circuit sizes for coin problem

- Circuits of AND, OR, NOT gates of unbounded fan-in



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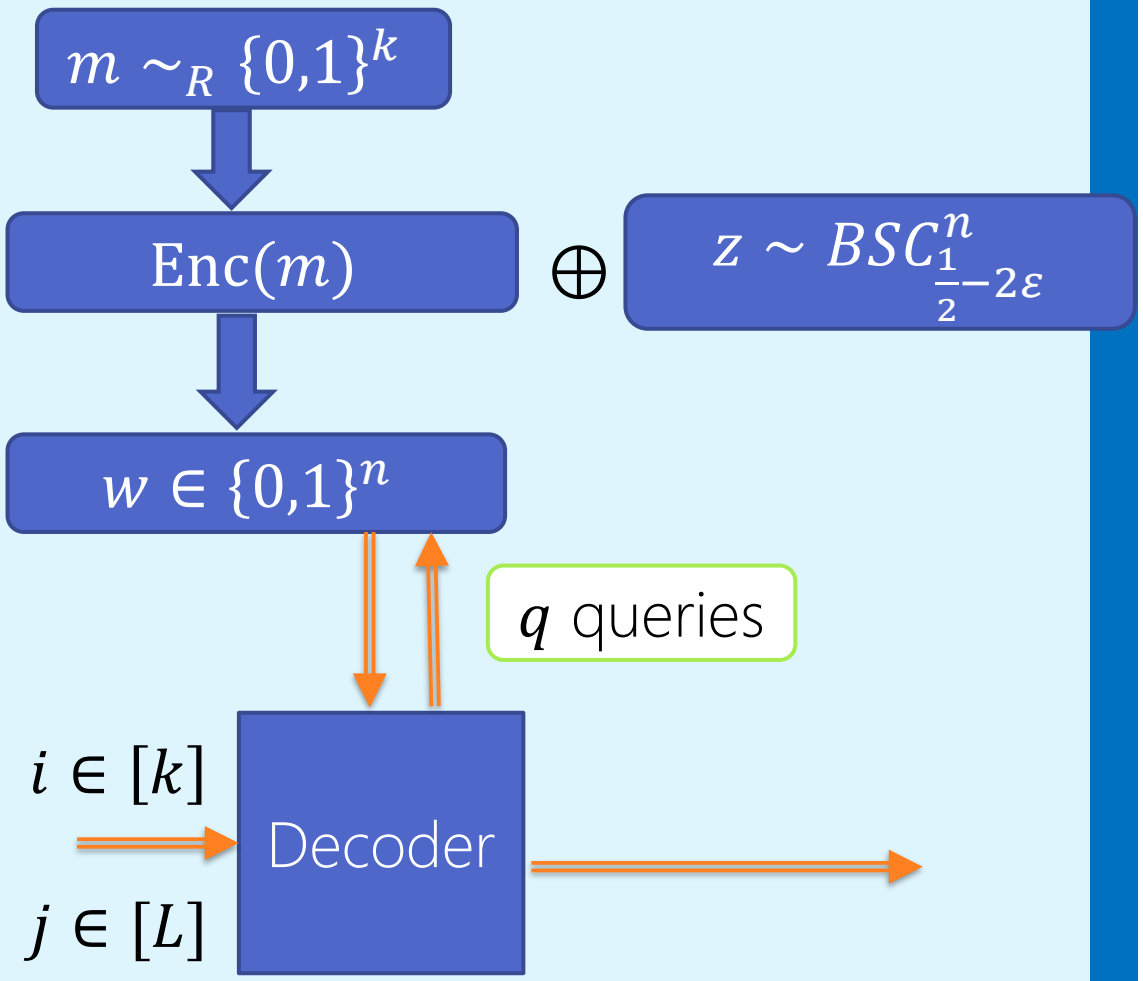

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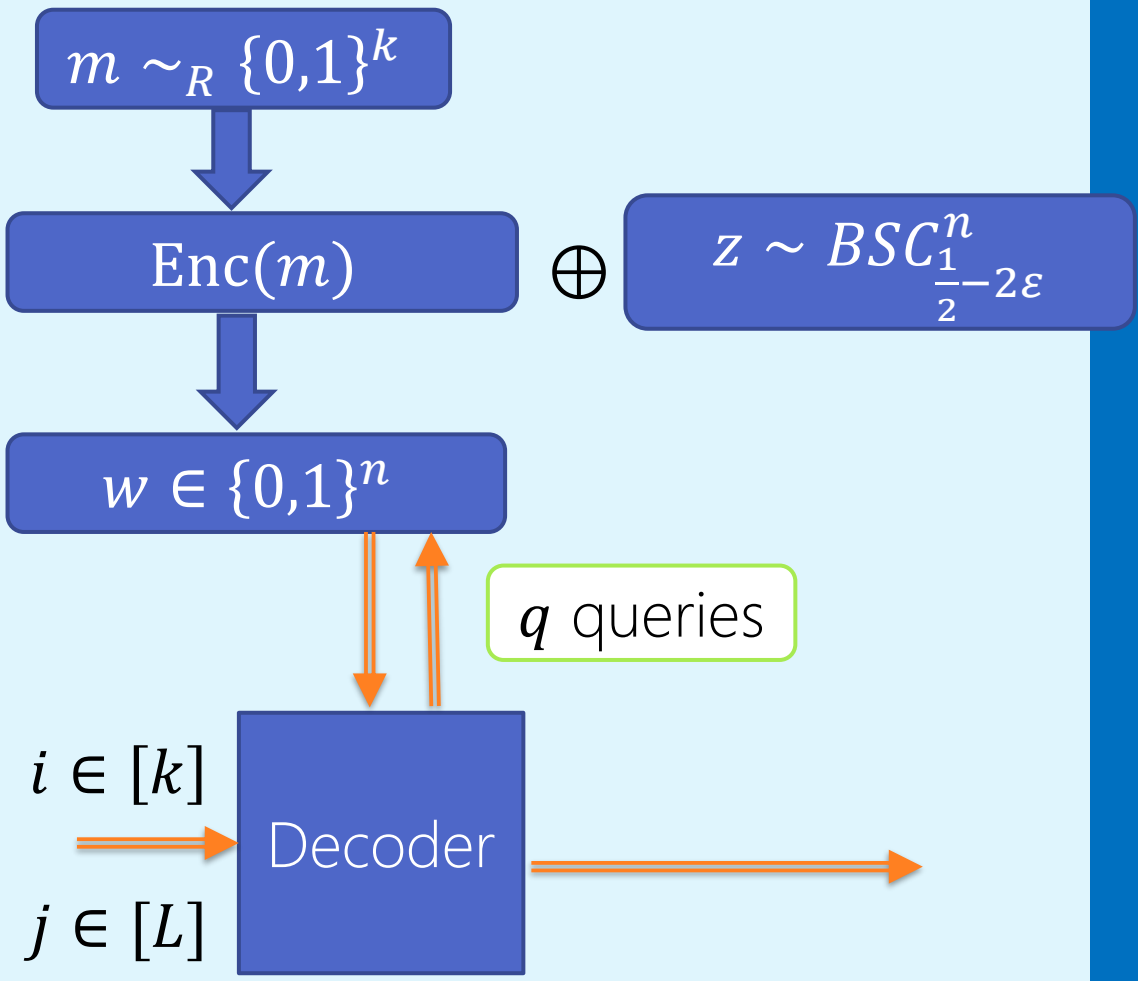

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Shaltiel Viola 10, Aaronson 10, Cohen Ganor Raz 14, Limaye Sreenivasaiah Srinivasan Tripathi Venkitesh 19

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- Can do this with a circuit of depth 3 and size $O(k \cdot L \cdot q2^q)$

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  1. $\Pr_{z\sim BSC^n_{\frac{1}{2}-2\varepsilon}}[C(z) = 1] \geq 0.99$
  2. $\Pr_{z\sim BSC^n_{\frac{1}{2}}}[C(z) = 1] \leq 0.01$

  Then C has size at least

$$\exp\left(\Omega\left(d \cdot \left(\frac{1}{\varepsilon}\right)^{d-1}\right)\right)$$

Circuit based on decoder

- For $m \in \{0,1\}^k$ define a circuit C_m that on input $z \in \{0,1\}^n$:
  - Evaluate $w = Enc(m) \oplus z$
  - For $i \in [k], j \in [L]$, evaluate a bit $b_{i,j}$ for whether $Dec^w(i,j) = m_i$
  - Output $\vee_{j\in[L]} \wedge_{i\in[k]} b_{i,j}$
- Can do this with a circuit of depth 3 and size $O(k \cdot L \cdot q2^q)$
- Main Idea: q-query deterministic decoder can be represented with $O(q \cdot 2^q)$ size and only $O(k \cdot L \cdot q2^q)$ "useful bits " in $Enc(m)$

Lower bounds on circuit sizes for coin problem

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$$kL2^{2q} \geq \exp\left(\Omega\left(\frac{1}{\sqrt{\varepsilon}}\right)\right)$$

$$\Rightarrow q \geq \frac{1}{\log k\sqrt{\varepsilon}} - \log L = \Omega\left(\frac{1}{\varepsilon^{0.5-o(1)}}\right)$$

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What we saw

- Local list decoding of $Enc: \{0,1\}^k \to \{0,1\}^n$ from $\frac{1}{2} - \varepsilon$ fraction errors needs $\Omega\left(\frac{1}{\sqrt{\varepsilon}}\right)$ queries for small $\varepsilon < \frac{1}{k^{0.01'}}$ even for codes with $n \geq 2^k$ and even by allowing large list sizes $L \leq \beta 2^k$ for some constant $\beta > 0$

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  - Is it possible to do a similar extension for the small ε case?
- Non-black-box techniques that help obtain better hardcore predicates from hard functions?

Thank You!