How to multiply numbers fast?

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Multiplication

- One of the most fundamental and oldest mathematical problems
- Different methods used by different civilizations

Sumerian Tablet 2600 BCE, Iraq
Multiplication Table

Rhind Papyrus 1550 BCE, Egypt
Describes a method to multiply

400 BCE, China
Multiplication Table

Modern Multiplication

- Often called the "long multiplication method", or the "grade school multiplication method"
- Can be attributed to the writings of
  - Apollonius of Perga (3rd century BCE, Anatolia)
  - Sunzi (5th century CE, China),
  - Eutocius (6th century CE, Palestine & Greece),
  - Brahmagupta (7th century CE, India),
  - Al Khwarizmi (8th century CE, Persia),
  - Leonardo Fibonacci of Pisa (13th century CE, Italy)

…and others whose names were forgotten by history

Review of Long Multiplication

$$\begin{array}{r} \mathbf{1234} \times \\ \mathbf{4321} \\ \hline \end{array}$$

Review of Long Multiplication

$$\begin{array}{r} 1234 \times \\ 4321 \\ \hline \end{array}$$

= 5332114

Review of Long Multiplication

$$\begin{array}{r} 1234 \times \\ 4321 \\ \hline 1234 \end{array}$$

$= 5332114$

Review of Long Multiplication

$$
\begin{array}{r}
1234 \times \\
4321 \\
\hline
1234 \\
2468
\end{array}
$$

$= 5332114$

Review of Long Multiplication

$$
\begin{array}{r}
1234 \times \\
4321 \\
\hline
1234 \\
2468 \\
3702
\end{array}
$$

$= 5332114$

Review of Long Multiplication

$$
\begin{array}{r}
1234 \times \\
4321 \\
\hline
1234 \\
2468 \\
3702 \\
4936 \\
\hline
\end{array}
$$

$= 5332114$

Review of Long Multiplication

$$
\begin{array}{r}
1234 \\
4321 \\
\hline
1234 \\
2468 \\
3702 \\
4936 \\
\hline
5332114
\end{array}
$$

$\times$

$= 5332114$

Review of Long Multiplication

```
   1234 ×
   4321
 -------
   1234
  2468
 3702
4936
 -------
5332114
```

= 5332114

How many digit multiplications did we perform?

Review of Long Multiplication

$$
\begin{array}{r}
1234 \times \\
4321 \\
\hline
1234 \\
2468 \\
3702 \\
4936 \\
\hline
5332114
\end{array}
$$

= 5332114

Multiplying two digits

How many <u>digit multiplications</u> did we perform?

Review of Long Multiplication

$$
\begin{array}{r}
1234 \;\; \times \\
4321 \;\; \\
\hline
1234 \\
2468 \\
3702 \\
4936 \\
\hline
5332114
\end{array}
$$

= 5332114

How many <u>digit multiplications</u> did we perform?

= 16

Number of Digit Multiplications

- How many digit multiplications are performed to multiply two
  - 5-digit numbers

Number of Digit Multiplications

- How many digit multiplications are performed to multiply two
  - 5-digit numbers 25 digit multiplications

Number of Digit Multiplications

- How many digit multiplications are performed to multiply two
  - 5-digit numbers 25 digit multiplications
  - 6-digit numbers

Number of Digit Multiplications

- How many digit multiplications are performed to multiply two
  - 5-digit numbers 25 digit multiplications
  - 6-digit numbers 36 digit multiplications

Number of Digit Multiplications

- How many digit multiplications are performed to multiply two
    - 5-digit numbers 25 digit multiplications
    - 6-digit numbers 36 digit multiplications
    - 3-digit numbers

Number of Digit Multiplications

- How many digit multiplications are performed to multiply two
  - 5-digit numbers 25 digit multiplications
  - 6-digit numbers 36 digit multiplications
  - 3-digit numbers 9 digit multiplications

Number of Digit Multiplications

- How many digit multiplications are performed to multiply two
  - 5-digit numbers — 25 digit multiplications
  - 6-digit numbers — 36 digit multiplications
  - 3-digit numbers — 9 digit multiplications
  - 18-digit numbers

Number of Digit Multiplications

- How many digit multiplications are performed to multiply two
  - 5-digit numbers 25 digit multiplications
  - 6-digit numbers 36 digit multiplications
  - 3-digit numbers 9 digit multiplications
  - 18-digit numbers 18^2 digit multiplications

Number of Digit Multiplications

- How many digit multiplications are performed to multiply two
  - 5-digit numbers 25 digit multiplications
  - 6-digit numbers 36 digit multiplications
  - 3-digit numbers 9 digit multiplications
  - 18-digit numbers 18^2 digit multiplications
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Number of Digit Multiplications

- How many digit multiplications are performed to multiply two
  - 5-digit numbers 25 digit multiplications
  - 6-digit numbers 36 digit multiplications
  - 3-digit numbers 9 digit multiplications
  - 18-digit numbers 18^2 digit multiplications
  - n-digit numbers n^2 digit multiplications

Number of Digit Multiplications

We perform n^2 digit multiplications in order to multiply two n-digit numbers

Number of Digit Multiplications

We perform n^2 digit multiplications in order to multiply two n-digit numbers

Is there a method to multiply two n-digit numbers by performing fewer than n^2 digit multiplications?

Can we multiply with fewer digit multiplications?

- Question pursued by many mathematicians in the first half of 20th century

- Andrey Kolomogorov (1903-1987) *conjectured* that:

  There is no general method to multiply two n-digit numbers with *much* fewer than n^2 digit multiplications

- Kolomogorov organised a seminar series in Moscow University in 1960 to discuss this conjecture and other related conjectures

- He publicly stated this conjecture in the first seminar in the series

Can we multiply with fewer digit multiplications?

- Andrey Kolomogorov publicly *conjectured* in 1960 that:

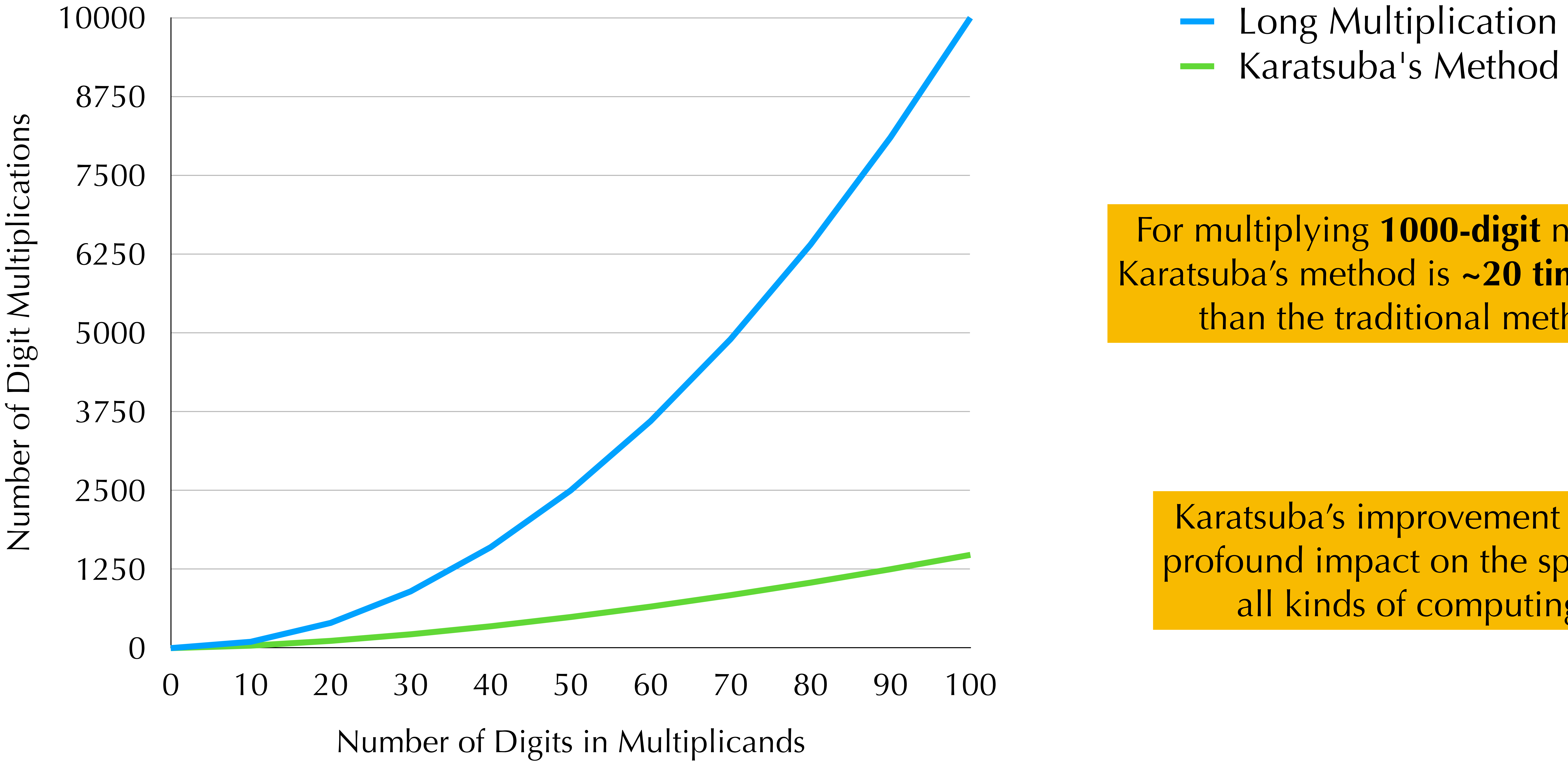
  There is no general method to multiply two n-digit numbers with *much* fewer than n^2 digit multiplications

- Almost exactly a week after the conjecture, a 23 year old student named **Anatoly Karatsuba** disproved the conjecture!

What did Karatsuba do?

- **Karatsuba gave a novel method to multiply n-digit numbers by performing at most $n^{\log_2 3}$ digit multiplications**
- $\log_2 3 = 1.58496.. < 2$
- This was a huge improvement!
    - How huge?

Multiplying large numbers became so much faster!



For multiplying **1000-digit** numbers, Karatsuba's method is **~20 times faster** than the traditional method!

Karatsuba's improvement had a profound impact on the speed of all kinds of computing!

What is this super-duper method?

- Multiplying two 2-digit numbers with fewer than 4 digit multiplications
- Let a, b, c, d be digits
- Let the multiplicands be $10 \times a + b$ and $10 \times c + d$
- What is the product of these numbers?

$$(10 \times a + b) \times (10 \times c + d) = 10^2 \times ac + 10 \times (ad + bc) + bd$$

- How many digit multiplications?

What is this super-duper method?

$$\begin{array}{r} 12 \times \\ 34 \\ \hline \end{array}$$

$a = 1$, $b = 2$

$c = 3$, $d = 4$

$12 = 1 \times 10 + 2$

$34 = 3 \times 10 + 4$

Karatsuba's method

- Multiplying two 2-digit numbers with fewer than 4 digit multiplications
- Let a, b, c, d be digits. Let the multiplicands be $10 \times a + b$ and $10 \times c + d$
- $(10 \times a + b) \times (10 \times c + d) = 10^2 \times ac + 10 \times (ad + bc) + bd$
- **Karatsuba's Observation:**
  - One does not need all of ac, bc, ad, bd to compute the above product.
  - We can compute above product if we have $ac, bd, ad + bc$

Karatsuba's method

- Multiplying two 2-digit numbers with fewer than 4 digit multiplications
- Let a, b, c, d be digits. Let the multiplicands be $10 \times a + b$ and $10 \times c + d$
- $(10 \times a + b) \times (10 \times c + d) = 10^2 \times ac + 10 \times (ad + bc) + bd$
- **Karatsuba's Observation:** $ad + bc = ac + bd - (a - b)(c - d)$
- Overall, we need only 3 digit multiplications!!

Karatsuba's method in action

$$\begin{array}{r} 12 \times \\ 34 \\ \hline \end{array}$$

$a = 1$, $b = 2$

$c = 3$, $d = 4$

$$ac = 1 \times 3 = 3$$

$$bd = 2 \times 4 = 8$$

$$(a-b)(c-d) = -1 \times -1 = 1$$

$$ad + bc = 3 + 8 - 1 = 10$$

Karatsuba's method in action

$$\begin{array}{r} 12\ \times \\ 34 \\ \hline \end{array}$$

$a = 1$, $b = 2$

$c = 3$, $d = 4$

$$ac = 1 \times 3 = 3$$

$$bd = 2 \times 4 = 8$$

$$ad + bc = 3 + 8 - 1 = 10$$

$$12 \times 34 = 3 \times 10^2 + 10 \times 10 + 8 = 408$$

Karatsuba's method in action

$$1234 \times 4321$$

How do we perform Karatsuba's algorithm here?

Need one more idea!

(1)

$$1234 = \underbrace{12}_{a} \times 10^2 + \underbrace{34}_{b}$$

$$4321 = \underbrace{43}_{c} \times 10^2 + \underbrace{21}_{d}$$

Karatsuba's method in action

1234×4321

(2) $= 12 \times 43 \times 10^4 + (12 \times 21 + 34 \times 43) \times 10^2$

a c ad + bc

$+ 34 \times 21$

b d

Karatsuba's method in action

$ac = 12 \times 43$

$bd = 34 \times 21$

(3)

$ad + bc = ac + bd - (a-b)(c-d)$

$a - b = -22$

$c - d = 22$

$(a-b)(c-d) = -22 \times 22$

Karatsuba's method in action

Three multiplications :

$ac = 12 \times 43$

$bd = 34 \times 21$

$(a-b)(c-d) = -22 \times 22$

How to do these ?

Karatsuba again !

Karatsuba's method in action

Three multiplications :

$ac = 12 \times 43$

$bd = 34 \times 21$

$(a-b)(c-d) = -22 \times 22$

Each involves 3 digit mults.

How to do these?

Karatsuba again!

Karatsuba's Method for Larger Numbers

◉ **Problem**: Want to compute $x \times y$, where they are n-digit numbers

1. $x = 10^{n/2} \times a + b, y = 10^{n/2} \times c + d,$ where a, b, c, d are $\frac{n}{2}$-digit numbers

2. Compute $ac, bd, (a-b)(c-d)$ using three applications of Karatsuba's method

3. Combine the results to obtain $x \times y$

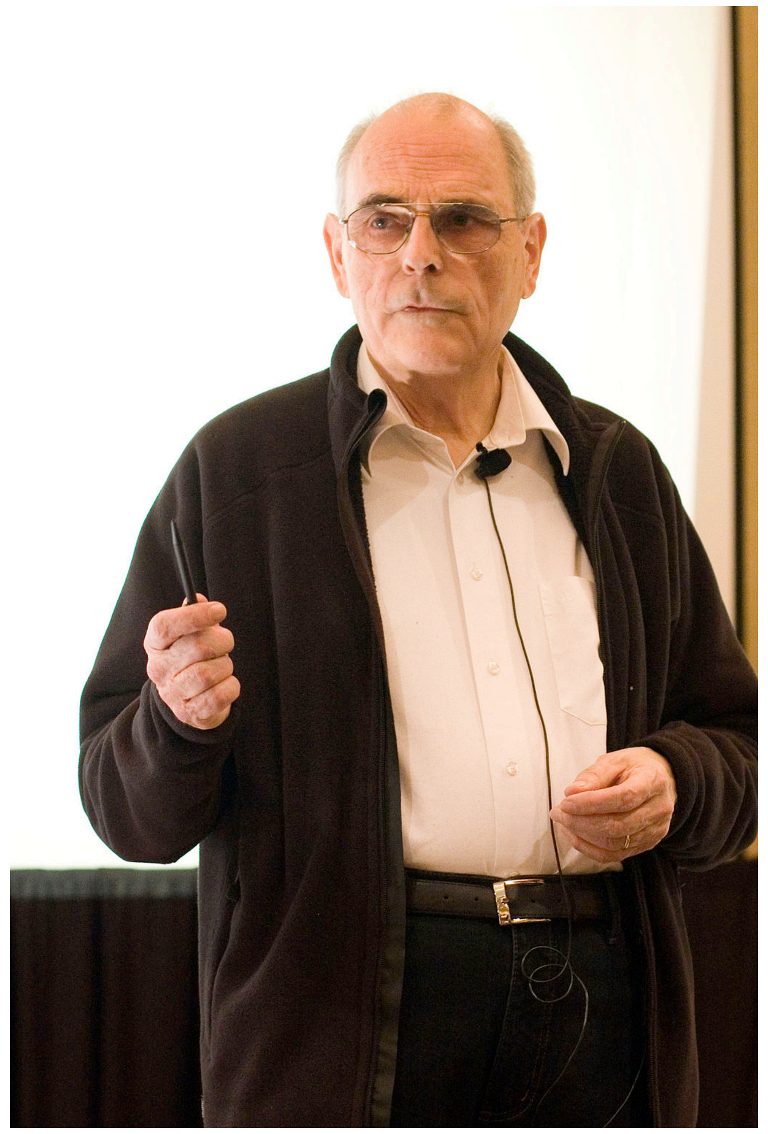
After Karatsuba…

- Multiplication of numbers has been made even faster in the decades following Karatsuba's discovery

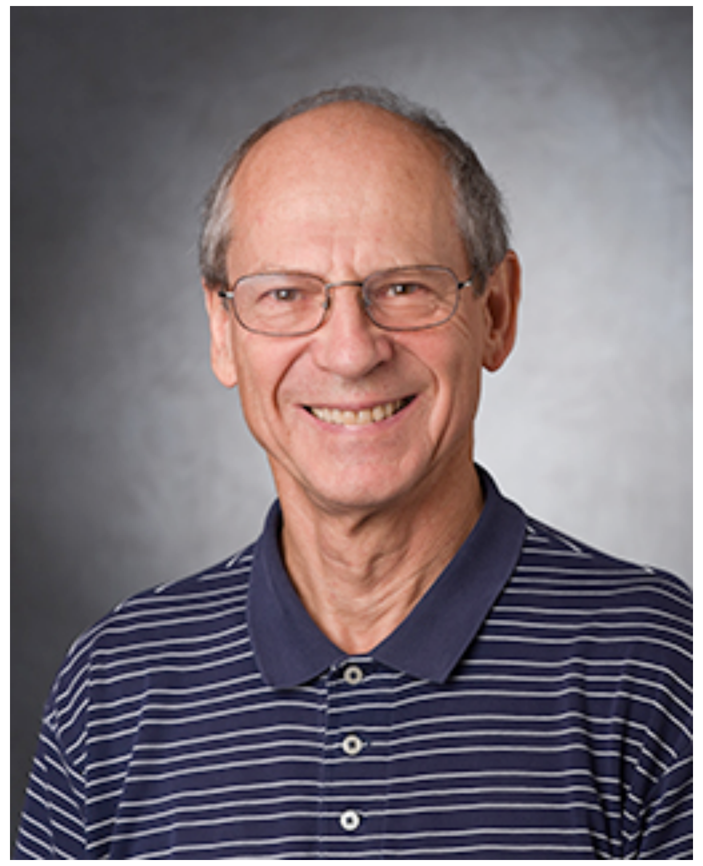
Andrei Toom

Stephen Cook

Arnold Schönhage



Volker Strassen



Martin Fürer

David Harvey

Joris van der Hoeven

and several others….

What we saw today…

- Multiplying numbers is a fundamental problem that has been studied for long
- Until 1960s, the fastest known multiplication method was the "Long Multiplication"
- Karatsuba, in 1960, gave a novel and surprising method to multiply numbers much faster!
- Karatsuba's work gave rise to a lot of interesting further work and left a lasting impact on computer science and mathematics

Thank you! Questions?