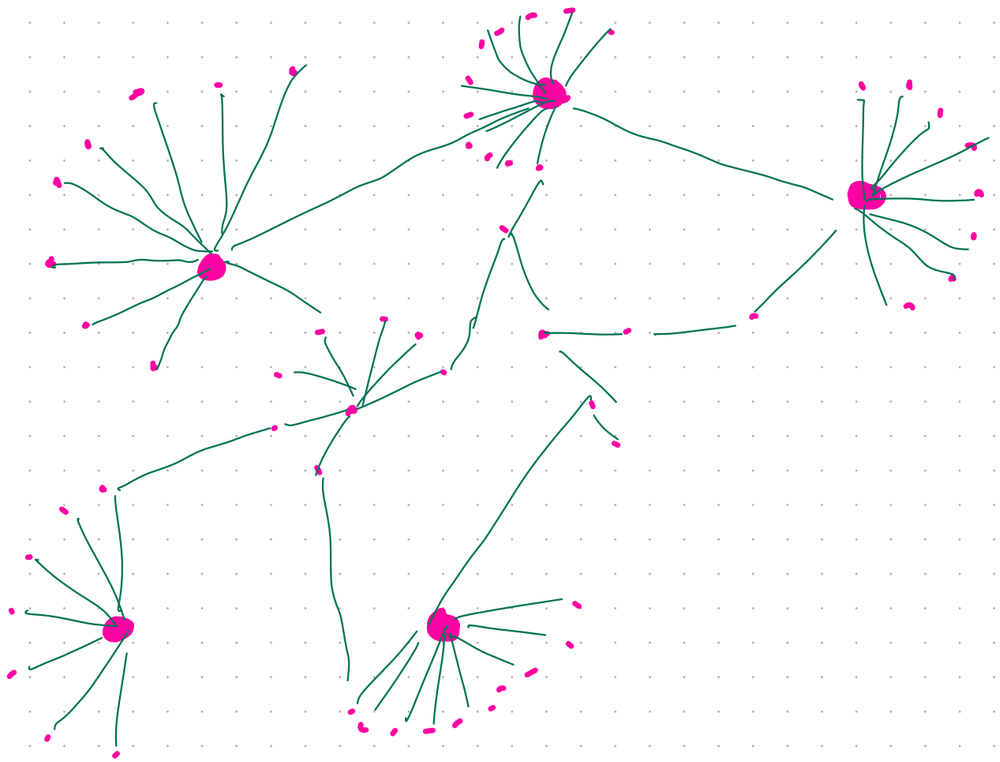
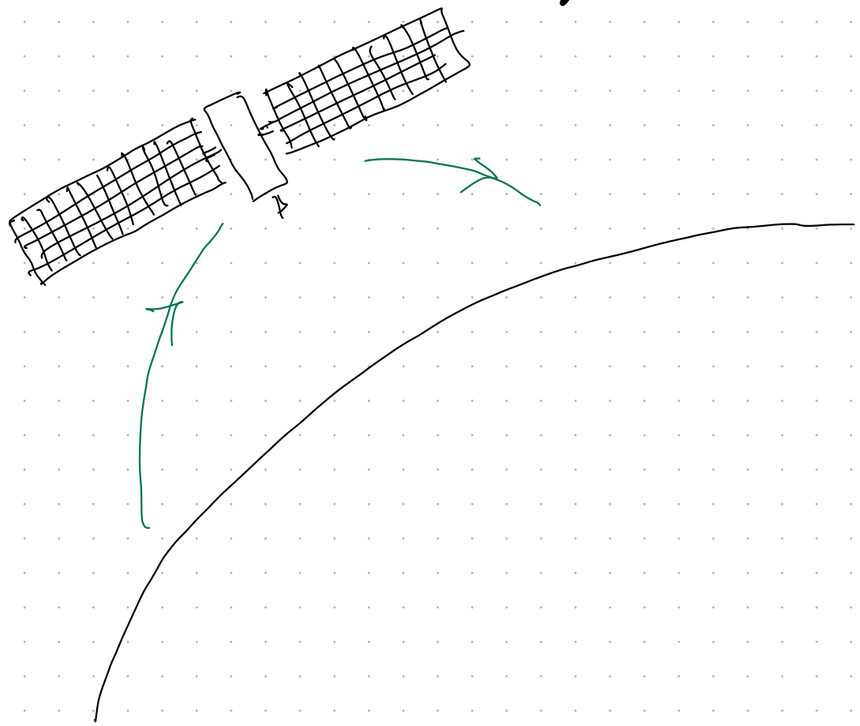
Something For Almost Nothing:

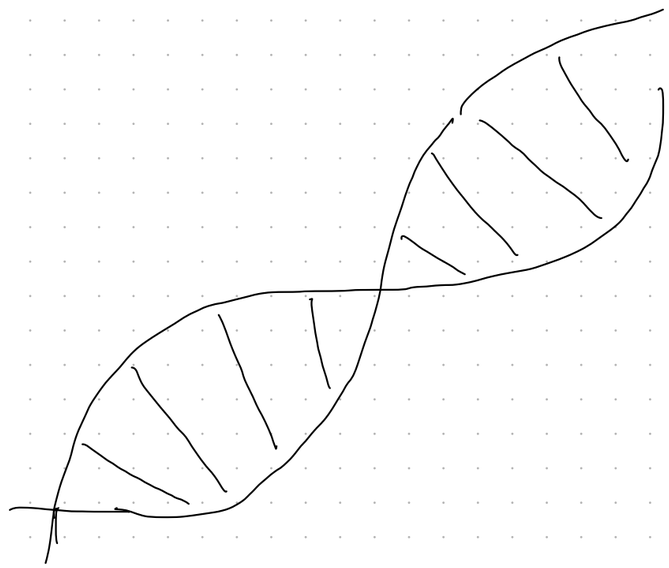
A gentle Introduction to Sublinear Algorithms

Nithin Varma

Massive Datasets are Everywhere!







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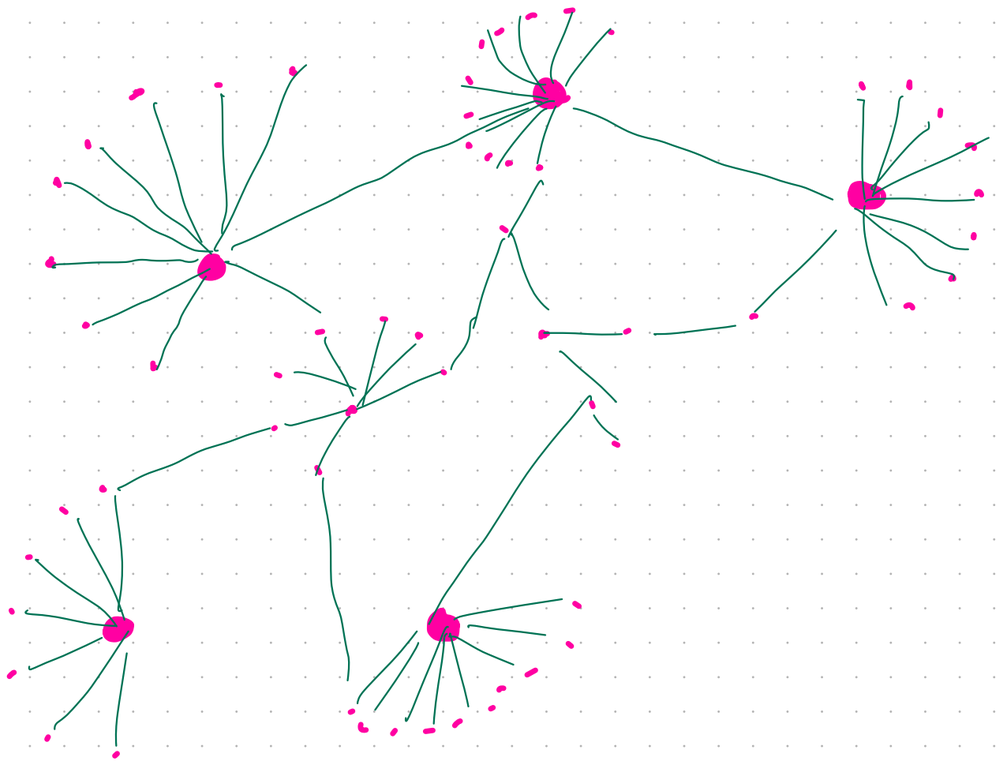
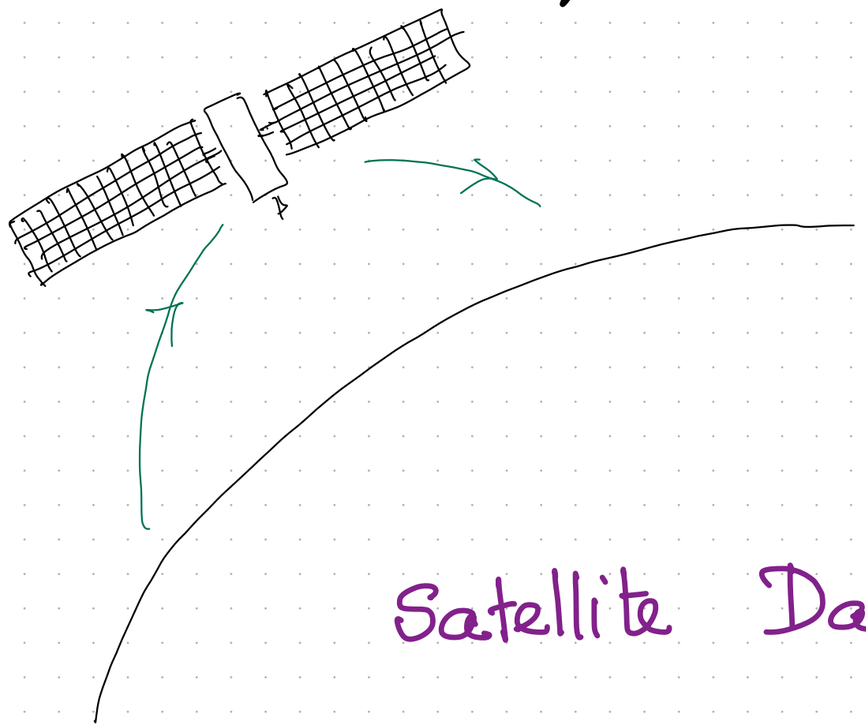


Large Networks



Satellite Data

Genetic Data

- Can draw useful conclusions by analyzing such datasets

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- Main Question: How to efficiently analyze large datasets ?

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Even reading a large dataset can take a lot of time

SUBLINEAR-TIME COMPUTATION

- Cannot read the entire input

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- Running time of algorithm is a sublinear function of input size n

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Do you already know any sublinear algorithms?

- Can we solve more challenging computational problems in sublinear-time ?

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- YES !

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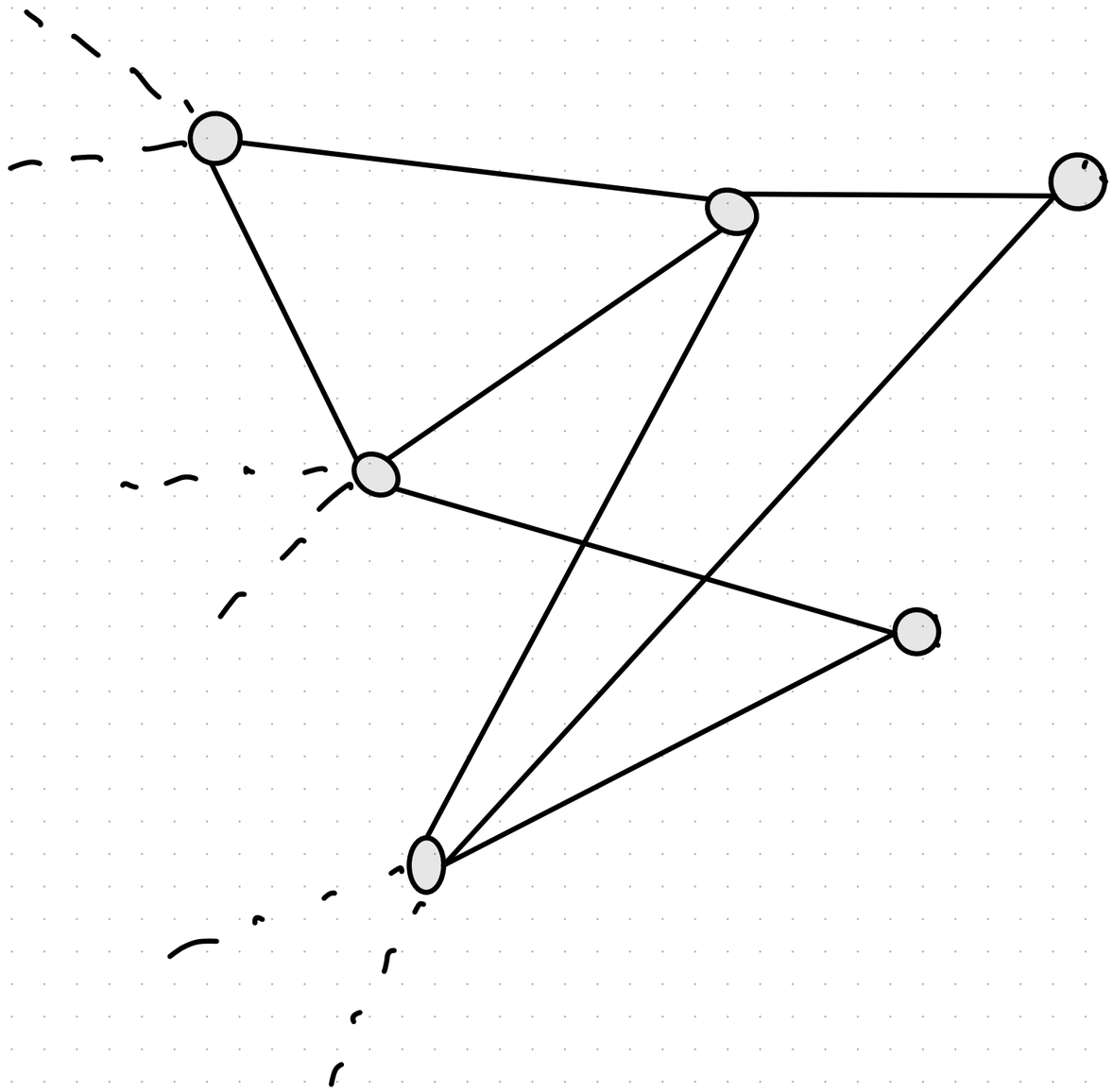
  * Output approximate answers

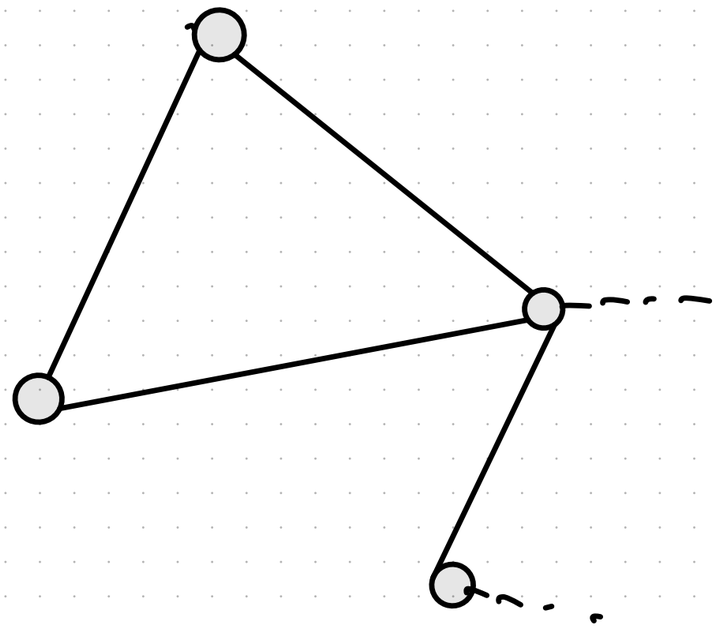
- Can we solve more challenging computational problems in sublinear-time ?

- YES ! IF WE ARE ALLOWED TO
  - * Output approximate answers
  - * Fail 1% of the time

Today: Sublinear-Time Algorithm for a Fundamental Graph Problem

Massive Graphs: Social Network

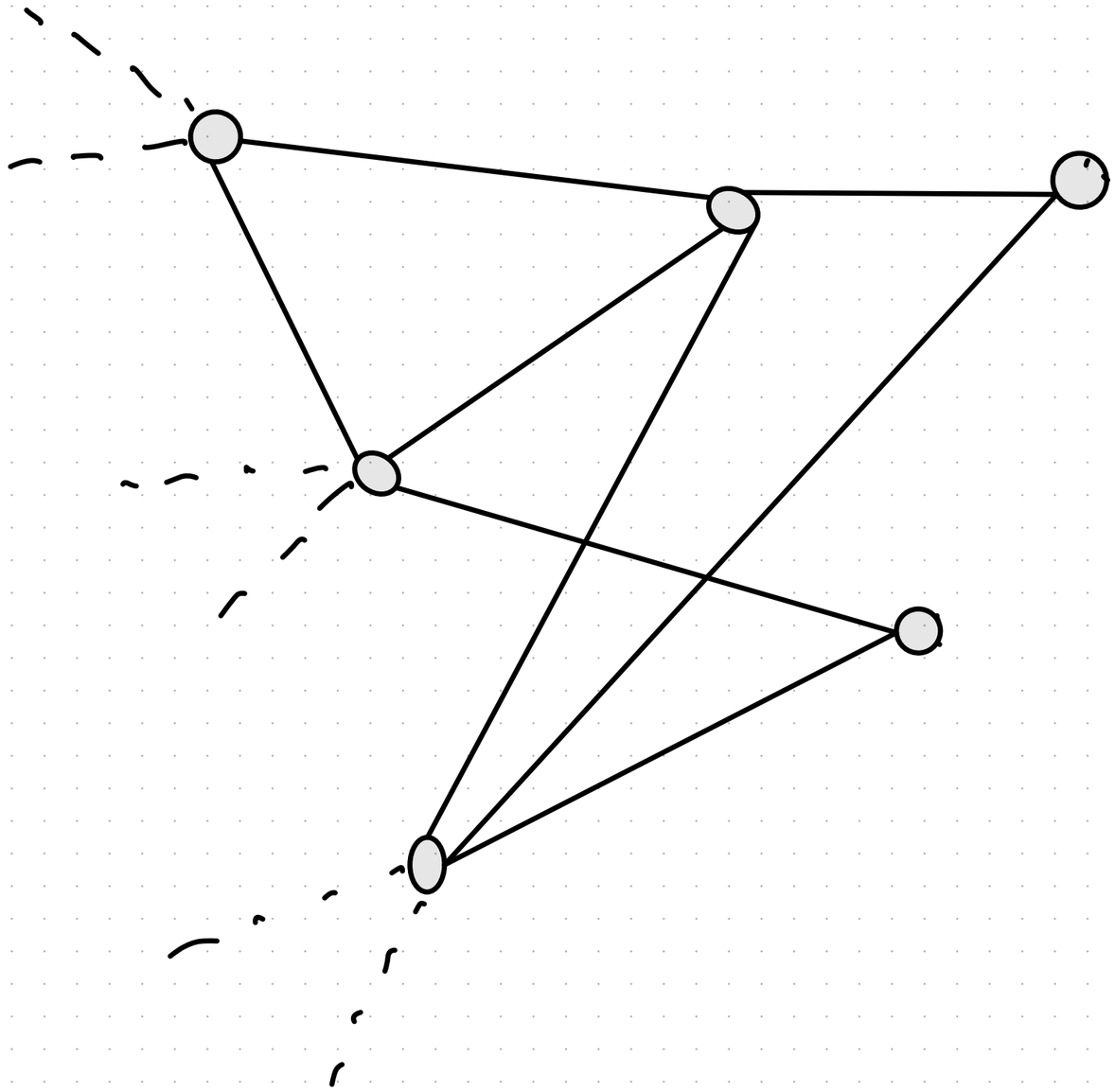


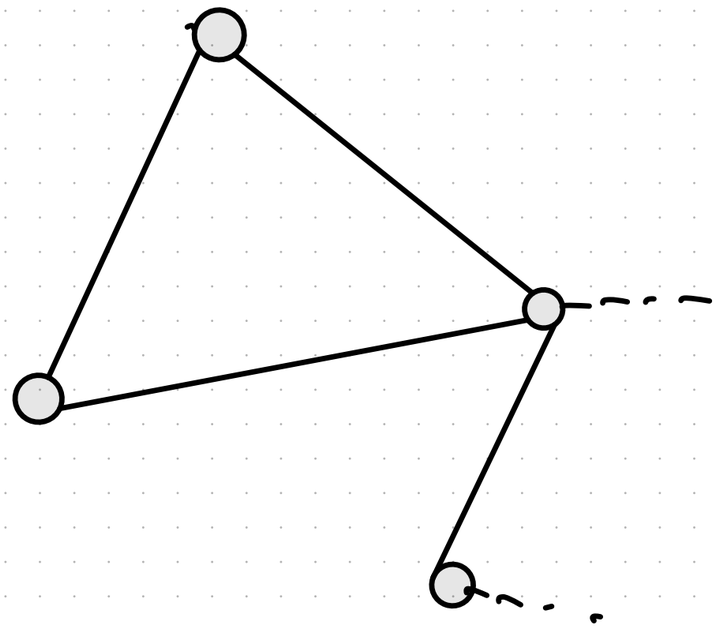


Massive Graphs : Social Network

Vertices : Users

Edges : Friendship





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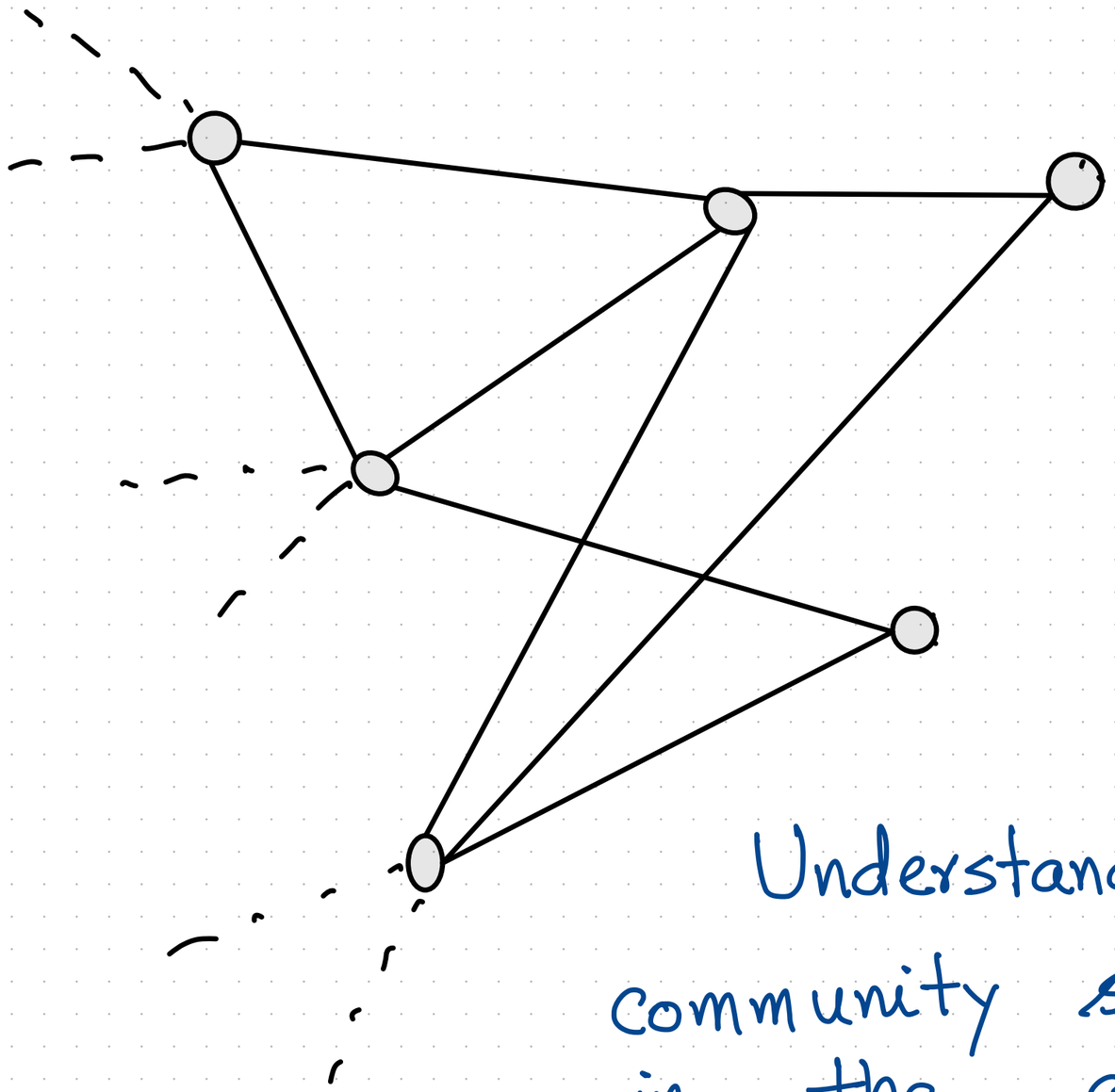

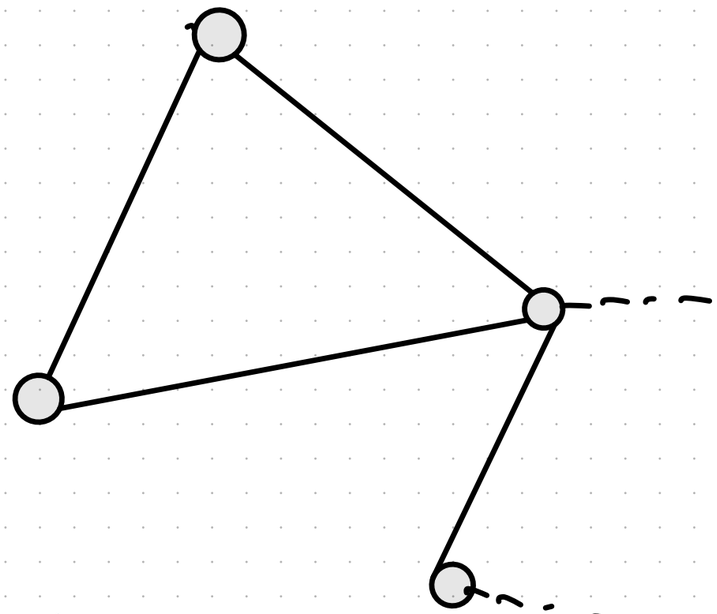

Understand community structure in the graph

Massive Graphs: Social Network

Big Goal: Understand community structure in the social network.

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* Several definitions of "community" possible

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Simpler Goal: Is the graph connected or not?

Problem: Given a graph, is it connected or not?

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Goal: Design a sublinear-time algorithm for the problem

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Connectedness in Sublinear-Time ?

Graph $G = (V, E)$ n vertices m edges

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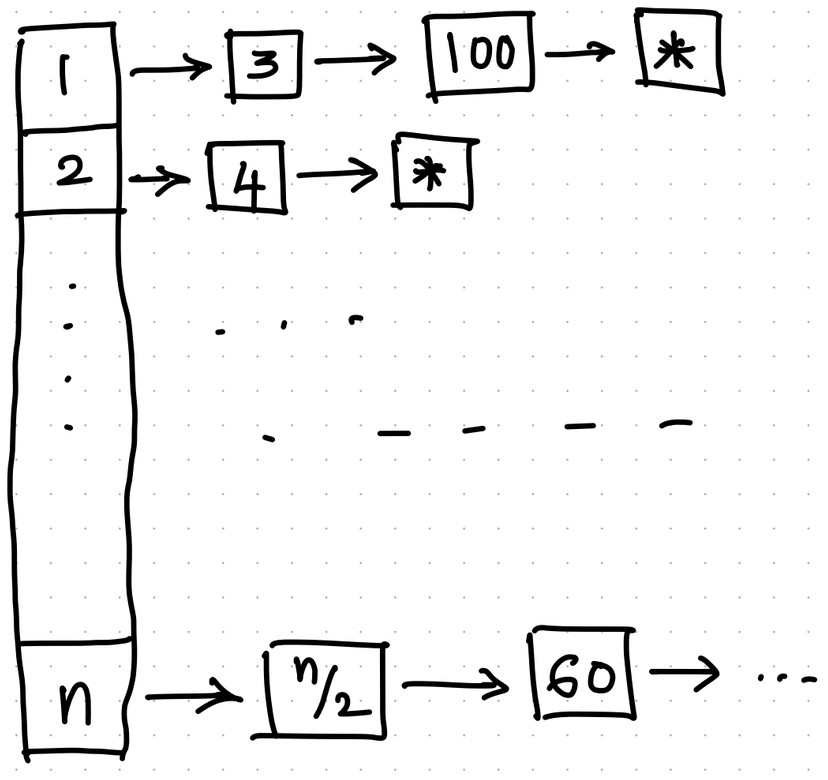
★ Represented as adjacency lists



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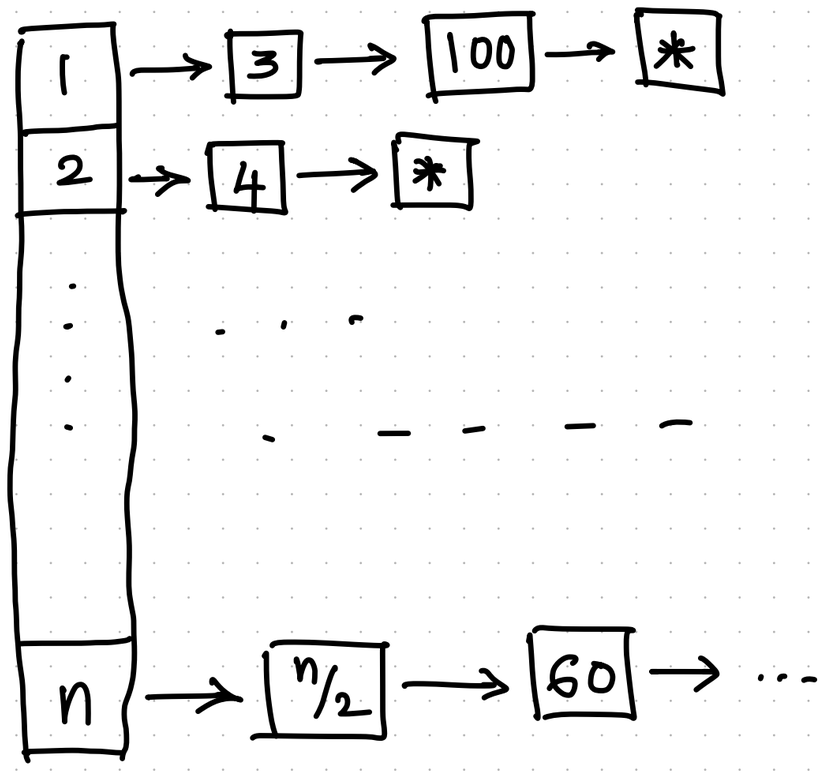


★ Access via queries

Connectedness in Sublinear-Time ?

Graph $G = (V, E)$ n vertices m edges

★ Represented as adjacency lists

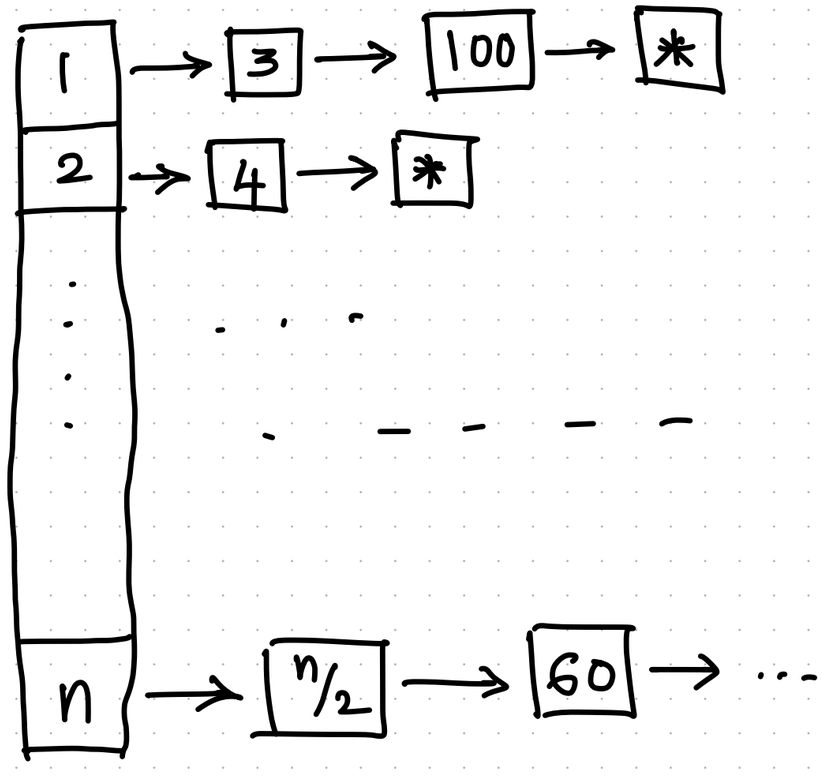
★ Access via queries

- Degree query:

  What is degree of vertex v?

- Neighbor query:

  Who is i^{th} nbr of v?



Can we check if a graph is connected without examining the entire adjacency lists ?

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NO !

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Need to "approximately" check connectedness

Relaxed Problem

Design an algorithm that

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- Outputs YES if graph connected

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Design an algorithm that

- Outputs YES if graph connected
- Outputs NO if graph contains "many" connected components

Problem

Design an algorithm that gets query access to a graph G, input $\varepsilon \in (0,1)$

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* Trivial if $\varepsilon > \frac{n}{m}$

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→ output YES without querying

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* Smaller ε $\Rightarrow$ More challenging problem

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G connected

$\varepsilon m \rightarrow$ # edges in G

} Sufficient to be correct with probability $\geq \frac{2}{3}$

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$$\left.\begin{array}{l}\#\text{components with} \\ > \frac{2n}{\varepsilon m} \text{ vertices}\end{array}\right\} < \frac{n}{2n/\varepsilon m} = \frac{\varepsilon m}{2}$$

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Remaining $\geq \varepsilon m - \frac{\varepsilon m}{2}$ components are "small"

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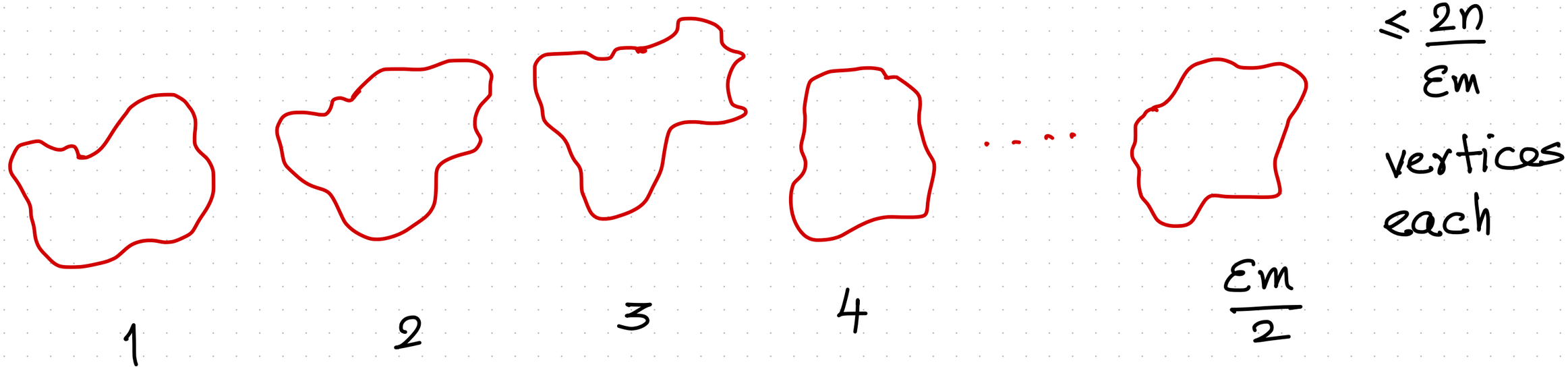
A NO instance G



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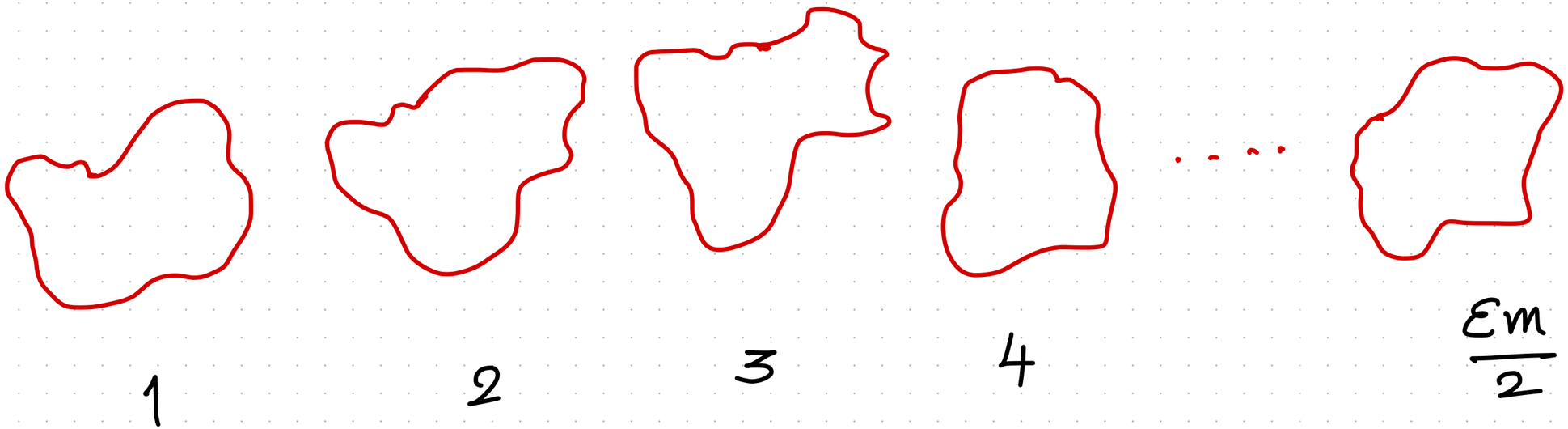

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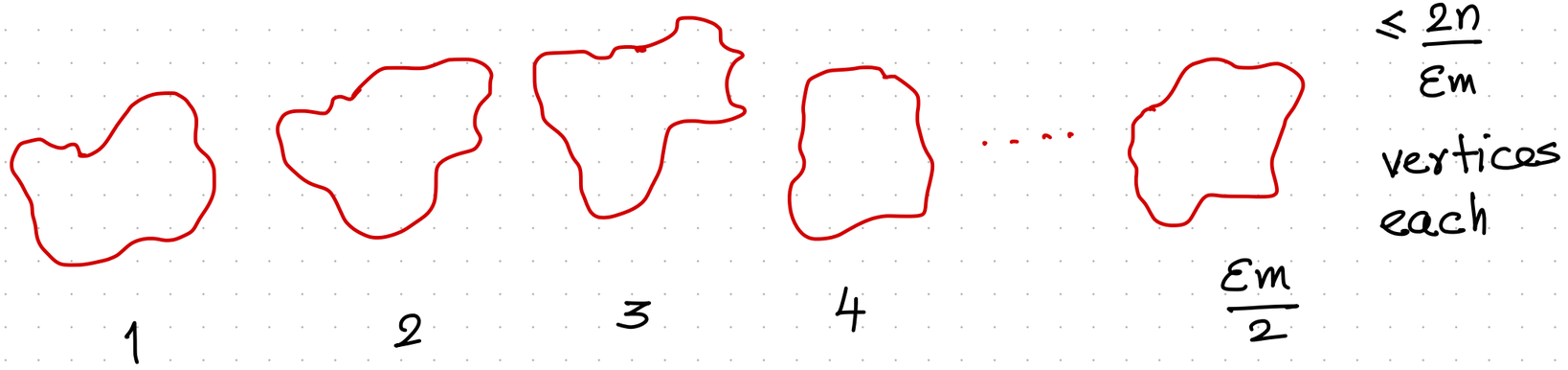

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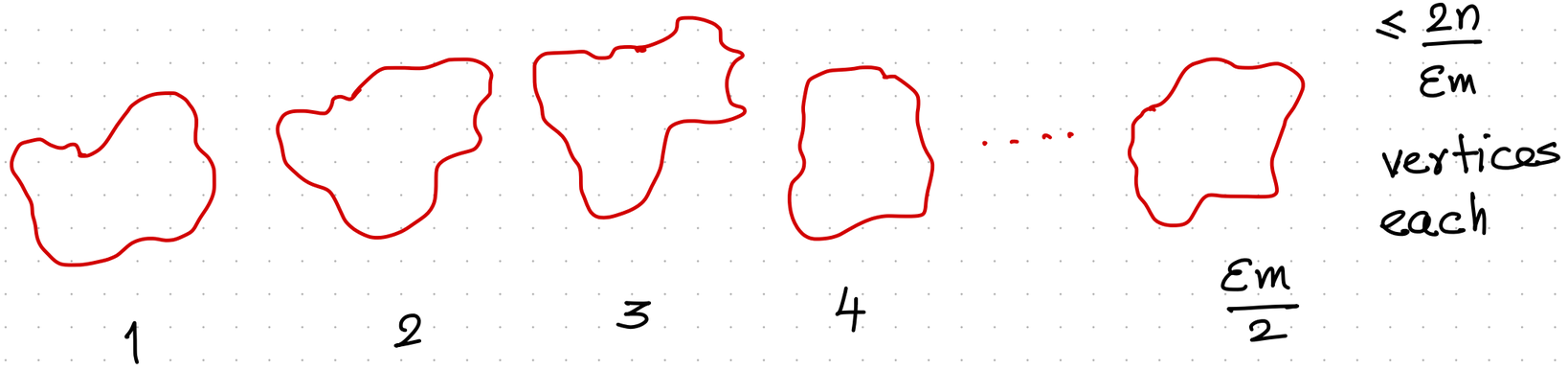

- If we can detect one such component, we have proof of disconnectedness.
- How to detect a component like this?
  - ★ A uniformly random vertex belongs to such a component with "good" probability

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If True, output NO!

Analyzing the Algorithm

- If G is connected, algo. never outputs NO

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$\geq \frac{\varepsilon m}{2}$ vertices belong to "small" components

$\therefore$ Pr [algo. outputting NO] $\geq \frac{\varepsilon m}{2} \times \frac{1}{n}$

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    can take $\leq \frac{4n^2}{\varepsilon^2 m^2}$ time

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We need a correctness probaility $\geq \frac{2}{3}$!

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Repeat algo. $\Theta\left(\frac{n}{\varepsilon m}\right)$ times independently

$\Rightarrow$ Correctness probability $\geq \frac{2}{3}$.

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$$\Big\Vert$$

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$\bar{d}$: average degree

Theorem: Algo. to ε-test connectedness of n-vertex m-edge graphs with time complexity $O\left(\frac{n^3}{\varepsilon^3 m^3}\right)$

$$= O\left(\frac{1}{\varepsilon^3 \bar{d}^3}\right)$$

Very very small compared to size of input

$\bar{d}$: average degree

CAN WE DO BETTER?

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YES!

Connectedness Testing Algorithm (Goldreich Ron '02)

Given query access to $G = (V, E)$, input ε

- Sample $v \sim V$ uniformly at random
- Check whether v belongs to a component with $\leq \frac{2n}{\varepsilon m}$ vertices by performing a BFS from v until seeing $\leq \frac{2n}{\varepsilon m}$ vertices

If True, output NO!

Connectedness Testing Algorithm

(Levi, Pallavoor, Raskhodnikova, Varma '21)

Given query access to $G = (V, E)$, input ε

- Sample $v \sim V$ uniformly at random
- Check whether v belongs to a component with $\leq \frac{2n}{\varepsilon m}$ vertices by performing a BFS from v until seeing $\leq \frac{2n}{\varepsilon m} \times d_v$ edges
  - d_v → degree of v

If True, output NO!

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$$= O\left(\frac{1}{\varepsilon}\right)$$

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For $\geq \frac{2}{3}$ success probability, $O\left(\frac{1}{\varepsilon^2 \bar{d}}\right)$

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- $\frac{1}{\varepsilon^3\bar{d}^3}$ → [Goldreich Ron '02]
- $\frac{1}{\varepsilon^2\bar{d}}$ → our result

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Better when $\bar{d}$ is small

$\bar{d}$: average degree

Summary

- Sublinear-time computation to tackle big data challenges
- Can solve non-trivial problems in sublinear-time
- Very efficient algorithms to test connectedness of graphs

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THANK YOU!