Erasure-Resilience vs. Tolerance to Errors

Sofya Raskhodnikova, **Nithin Varma**

Boston University

BOSTON UNIVERSITY

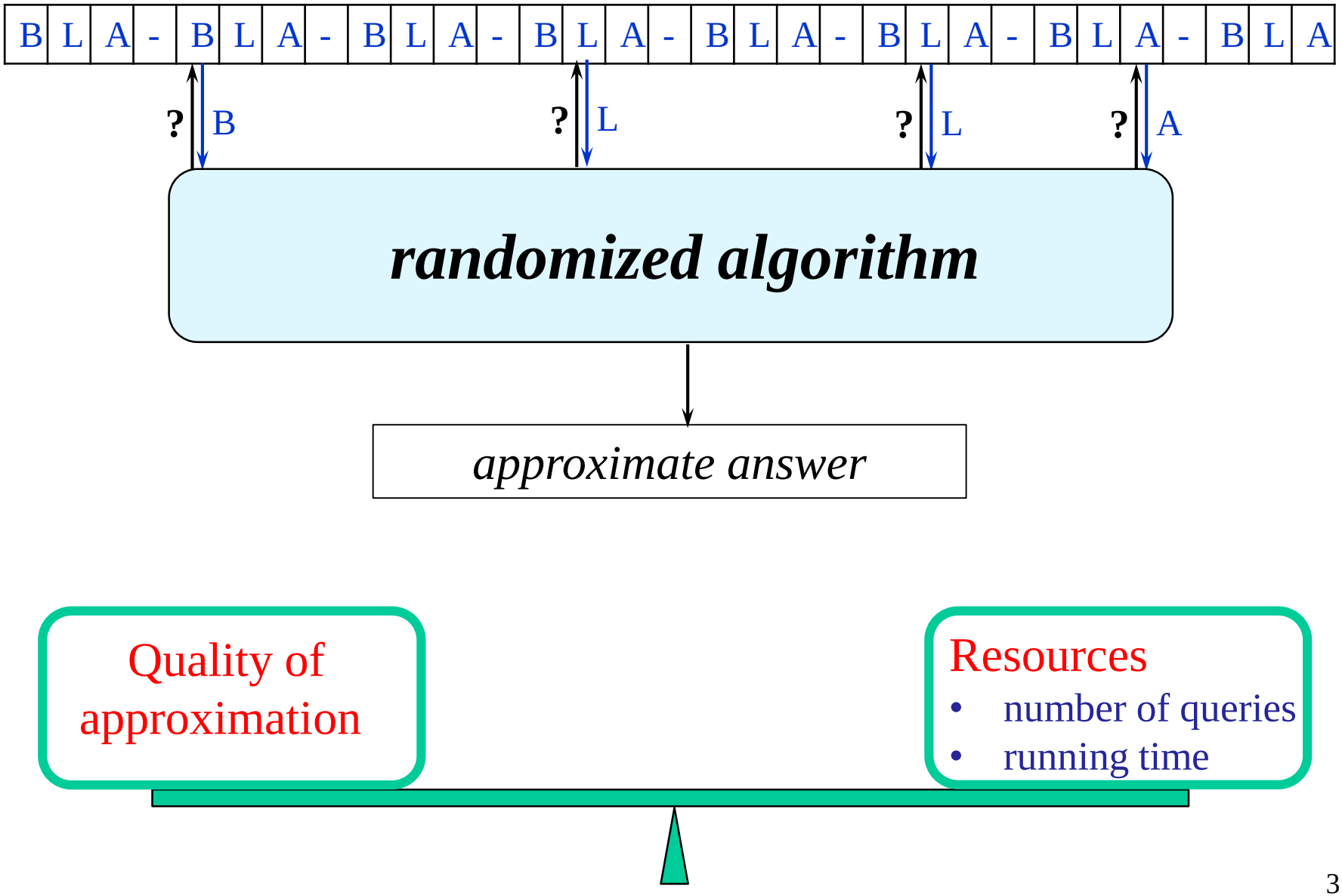
Goal: study of sublinear algorithms resilient to adversarial corruptions in the input

---

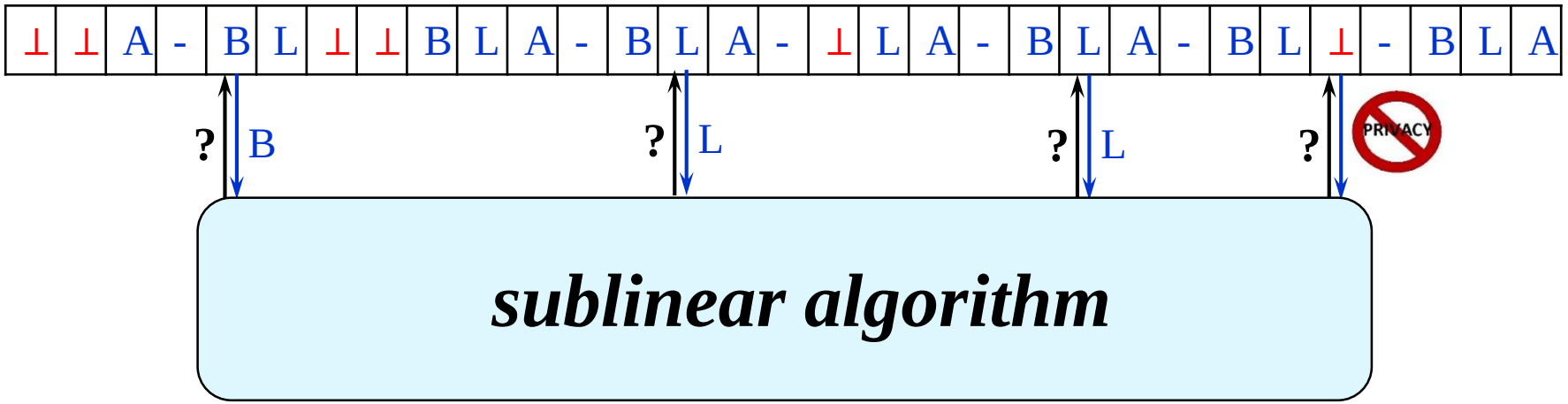
Focus: Property Testing Model

[Rubinfeld Sudan 96, Goldreich Goldwasser Ron 98]

A Sublinear-Time Algorithm



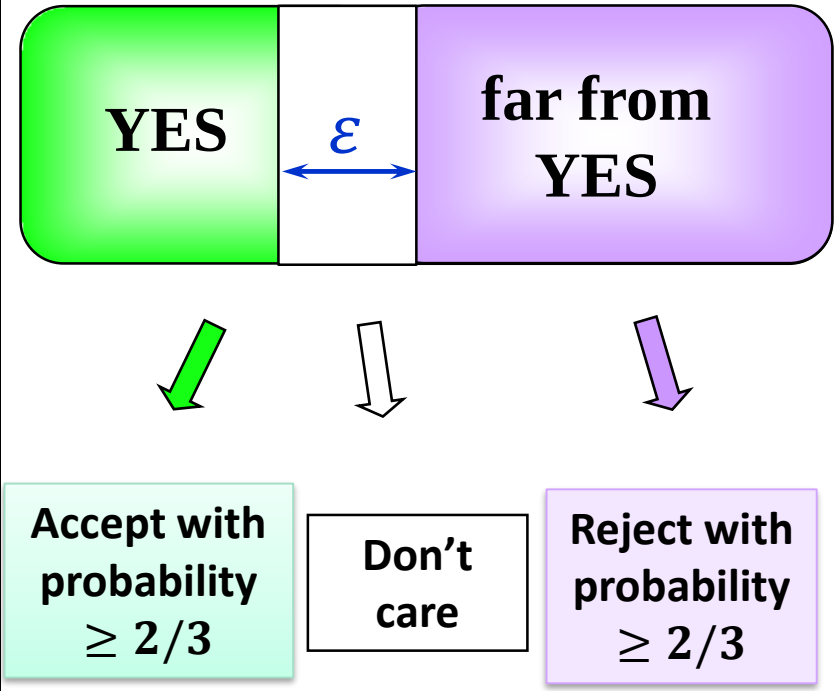
Algorithms Resilient to Erasures (or Errors)



- $\leq \alpha$ fraction of the input is erased (or modified) adversarially before algorithm runs
- Algorithm does not know in advance what's erased (or modified)

Property Testing

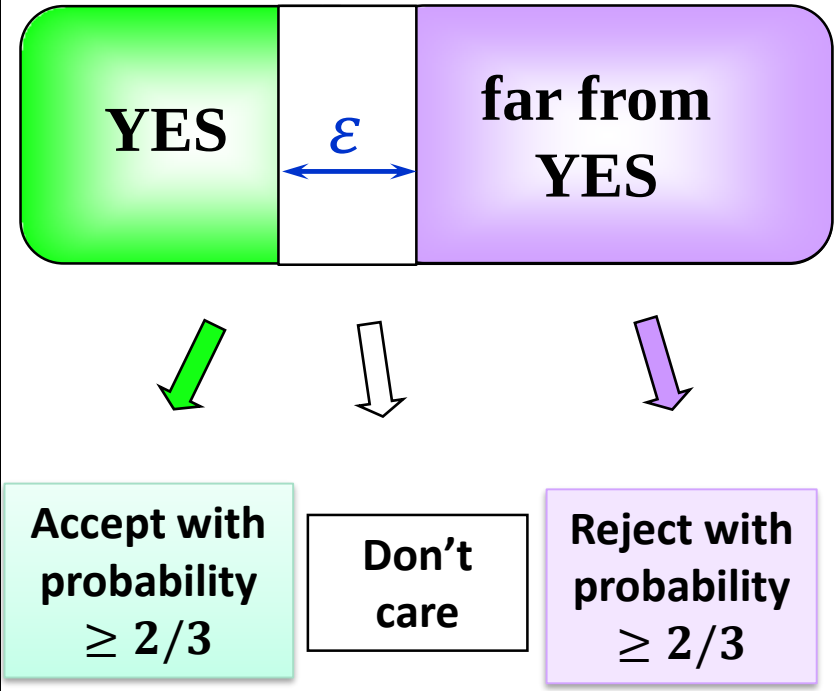
Property Tester [Rubinfeld Sudan 96, Goldreich Goldwasser Ron 98]



Two objects are at distance ε = they differ in an ε fraction of places

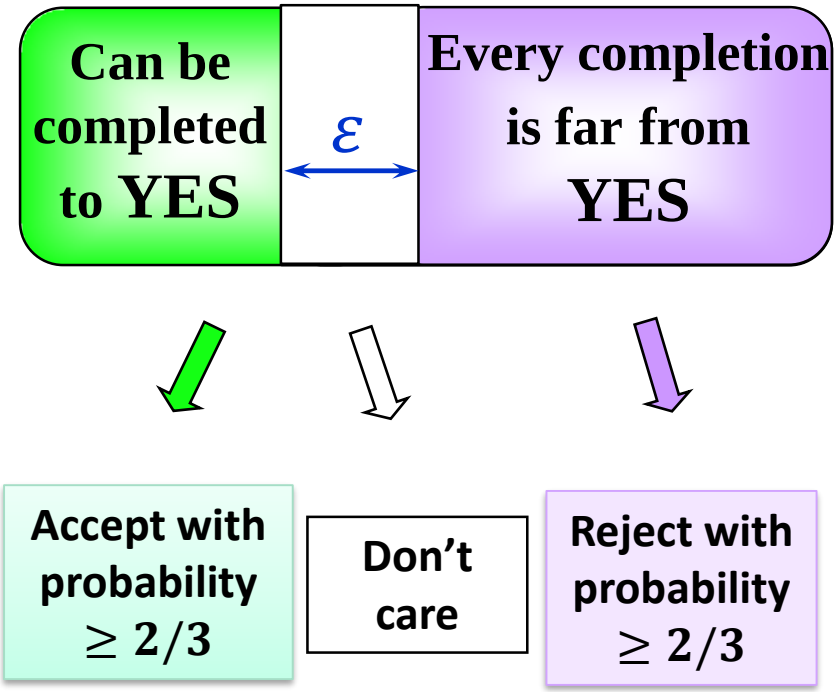
Property Testing with Erasures

Property Tester [Rubinfeld Sudan 96, Goldreich Goldwasser Ron 98]



Erasure-Resilient Property Tester [Dixit Raskhodnikova Thakurta Varma 16]

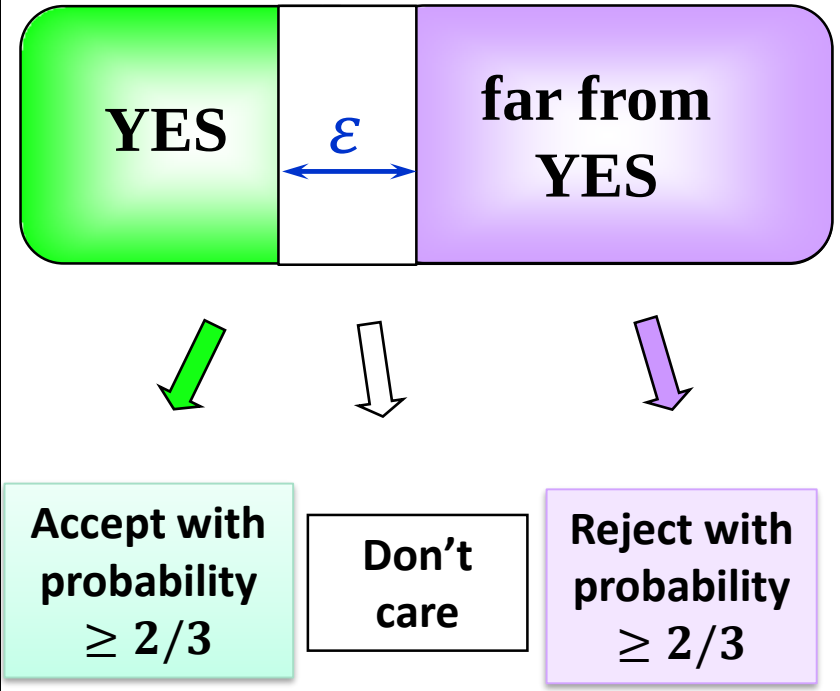
$\leq \alpha$ fraction of the input is erased adversarially



Two objects are at distance ε = they differ in an ε fraction of places

Property Testing with Errors

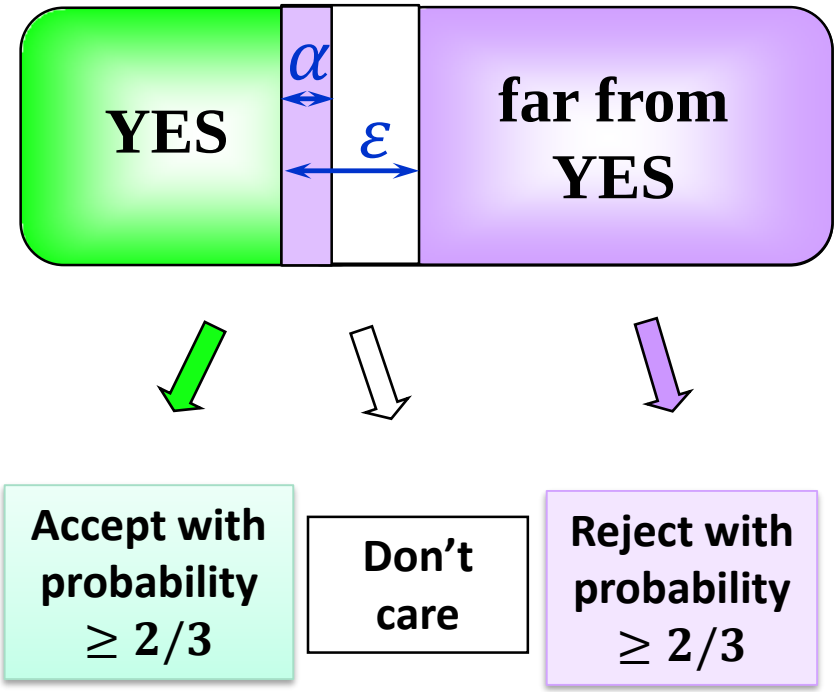
Property Tester [Rubinfeld Sudan 96, Goldreich Goldwasser Ron 98]



Tolerant Property Tester

[Parnas Ron Rubinfeld 06]

$\leq \alpha$ fraction of the input is wrong

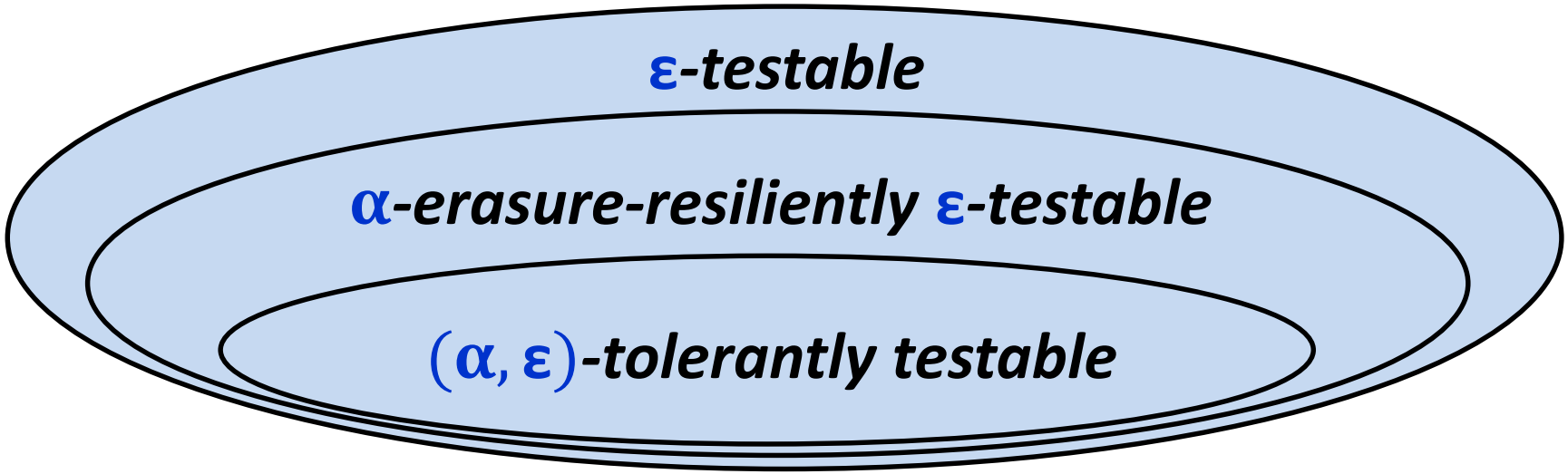


Two objects are at distance ε = they differ in an ε fraction of places

Relationships Between Models

Containments are strict:

- [Fischer Fortnow 05]: standard vs. error-tolerant
- [Dixit Raskhodnikova Thakurta Varma 16]: standard vs. erasure-resilient
- **new**: erasure-resilient vs. error-tolerant



Main Tool: Locally List Erasure-Decodable Codes

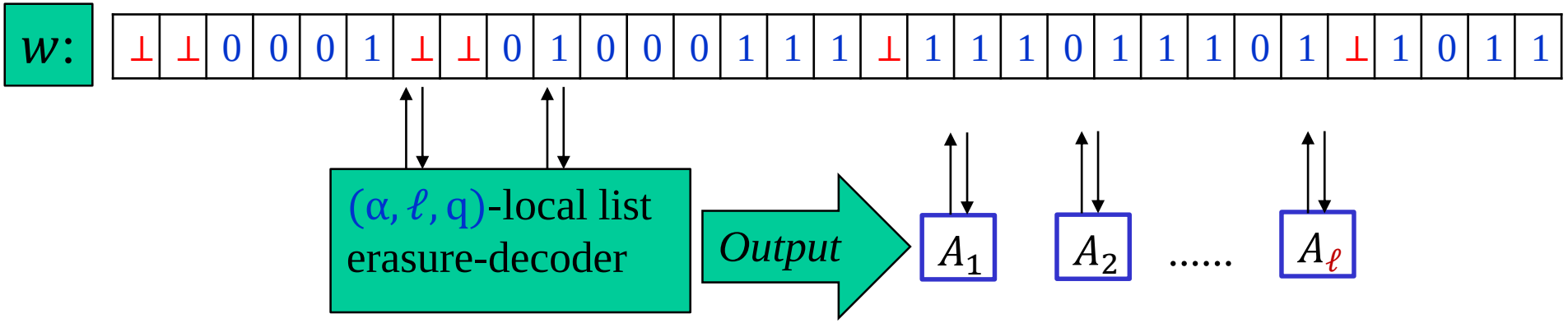
- Locally list decodable codes have been extensively studied
  [Goldreich Levin 89, Sudan Trevisan Vadhan 01, Gutfreund Rothblum 08, Gopalan Klivans Zuckerman 08, Ben-Aroya Efremenko Ta-Shma 10, Kopparty Saraf 13, Kopparty 15, Hemenway Ron-Zewi Wootters 17, Goi Kopparty Oliveira Ron-Zewi Saraf 17, Kopparty Ron-Zewi Saraf Wootters 18]

- Only errors, not erasures were previously considered
  - Not the case without the locality restriction
    [Guruswami 03, Guruswami Indyk 05]

- Can locally list decodable codes perform better with erasures than with errors?

A Locally List Erasure-Decodable Code

- An error-correcting code $\mathcal{C}_n : \Sigma^n \to \Sigma^N$
- Parameters: α fraction of erasures, list size ℓ and q queries.



  - w.p. $\geq 2/3$, for every $x \in \Sigma^n$ with encoding $\mathcal{C}_n(x)$ that agrees with w on all non-erased bits, one of the algorithms A_j, given oracle access to w, implicitly computes x (that is, $A_j(i) = x_i$);
  - each algorithm A_j makes at most q queries to w.

Hadamard Code

Hadamard: $\{0,1\}^k \to \{0,1\}^{2^k}$; $\text{Hadamard}(x) = (\langle x, y \rangle)_{y \in \{0,1\}^k}$

Type of Corruptions	Corruption Tolerance α	Number of Queries	List Size	Reference
Errors	$0 \le \alpha < 1/2$	$O\left(\frac{1}{(1/2-\alpha)^2}\right)$	$O\left(\frac{1}{(1/2-\alpha)^2}\right)$	[Goldreich Levin 89]
Erasures*	$0 \le \alpha < 1$	$O\left(\frac{1}{1-\alpha}\right)$	$O\left(\frac{1}{1-\alpha}\right)$	[new]

If fraction of errors is $\geq 1/2$, impossible to decode Hadamard codes.

*An improvement in dependence on α was suggested by Venkat Guruswami

How does separating erasures from errors in local list decoding help with separating them in property testing?

3CNF Properties: Hard to Test, Easy to Decide

- Formula ϕ_n : 3CNF formula on n variables, $\theta(n)$ clauses
- Property $P_{\phi_n} \subseteq \{0,1\}^n$: set of satisfying assignments to ϕ_n

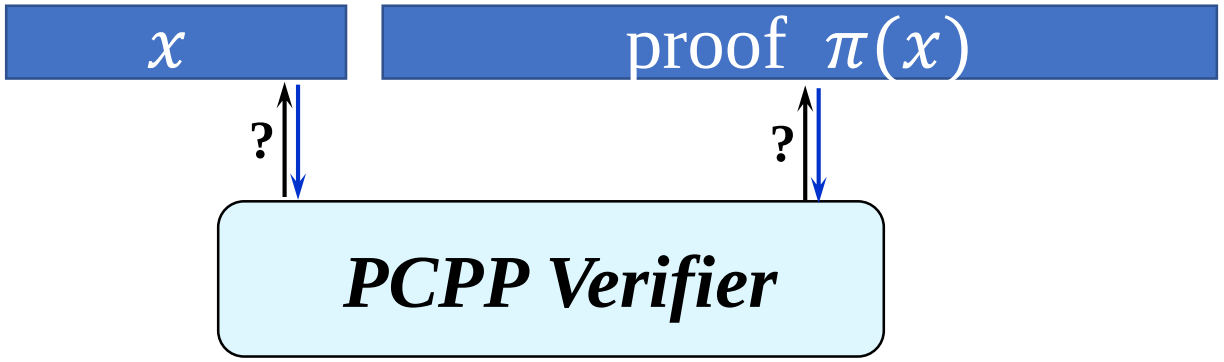
Theorem [Ben-Sasson Harsha Raskhodnikova 05]

For sufficiently small **ε**,
ε-testing P_{ϕ_n} requires $\boldsymbol{\Omega(n)}$ queries.

- P_{ϕ_n} decidable by a $\boldsymbol{O(n)}$-size circuit.

Testing with Advice: PCPs of Proximity (PCPPs)

[Ben-Sasson Goldreich Harsha Sudan Vadhan 06, Dinur Reingold 06]



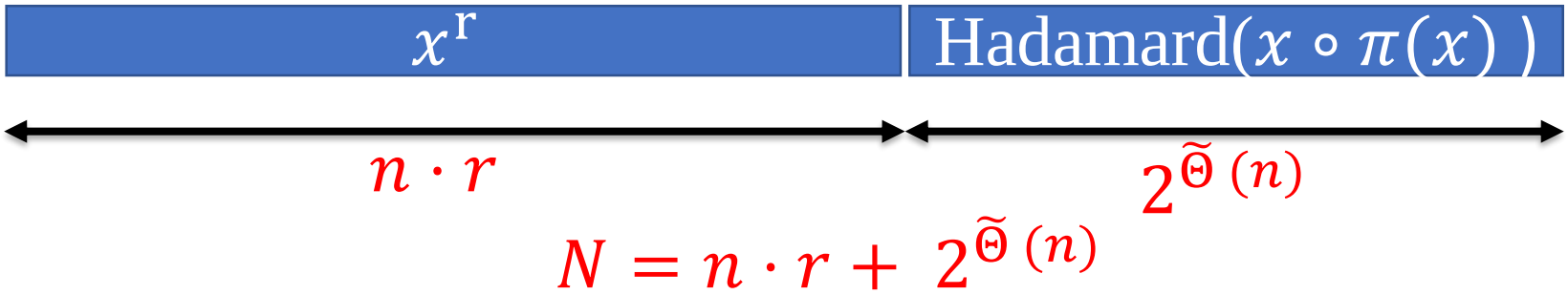
- If x has the property, then $\exists \pi(x)$ for which verifier accepts.
- If x is ε-far, then $\forall \pi(x)$ verifier rejects with probability $\geq 2/3$.

Theorem

Every property decidable with a circuit of size m
has PCPP with proof length $\tilde{O}(m)$ and constant query complexity.

3CNF properties have efficient PCPPs

Separating Property



- x satisfies the hard 3CNF property
- r is the number of repetitions (to balance the lengths of 2 parts)
- $\pi(x)$ is the proof on which the PCPP verifier accepts x
- **Idea:** Even if a 3/4 fraction of the encoding is erased, we can still locally list erasure-decode and test with constant query complexity.
  If 1/2 fraction of encoding has errors, cannot decode the proof.
  Need $\Omega(|x|) = \widetilde{\Omega}(\log N)$ queries to tolerantly test.

Bottom Line

The separating property is

- erasure-resiliently testable with a constant number of queries,
- but requires $\widetilde{\Omega}(\log N)$ queries to tolerantly test.

Error-tolerant testing is harder than erasure-resilient testing in general.

Open Questions and Directions

- Constant-query, constant list size, local list erasure-decodable codes with better rate?
  - Will imply better separation.

- Erasure-resilient testers for specific properties: linearity, dictatorship, linear threshold functions...

- Erasure-resilience for other models of sublinear algorithms.

Thank you!

Separating Property: Erasure-Resilient Testing

x^r	Hadamard$(x \circ \pi(x))$

Idea: If a constant fraction (say, 1/4) of the encoding is preserved, we can locally list erasure-decode.

Erasure-Resilient Tester

1. Locally list erasure-decode Hadamard to get a list of algorithms.
2. For each algorithm, check if:
   - the plain part is x^r by comparing u.r. bits with the corresponding bits of the decoding of x
   - PCPP verifier accepts $x \circ \pi(x)$
3. Accept if, for some algorithm on the list, both checks pass.

Constant query complexity.

Separating Property: Hardness of Tolerant Testing

x^{r}	Hadamard$(x \circ \pi(x))$

Idea: Reduce standard testing of 3CNF property to tolerant testing of the separating property.

- Given a string x, we can simulate access to

x^{r}	00000 … 00000

- All-zero string is Hadamard$(x \circ \pi(x))$ with 1/2 of the encoding bits are erroneous!
- Testing 3CNF property requires $\Omega(n)$ queries, where $n = |x|$.
  The input length for separating property is $N \approx 2^{cn}$.

$$\Omega(n) \approx \Omega(\log N) \textit{ queries are needed.}$$