# Tutorial

June 18, 2021

**Definition 0.1.** A Lattice is a Poset where every $x, y$ in it have a LUB and a GLB. The GLB of $x, y$ is denoted by $x \vee y$ (join) and the LUB by $x \wedge y$ (meet). (We will deal with locally finite posets.)

**Example:** Consider the poset $L = \{S | S \subseteq [n]\}$. Check it is a lattice.
(For $S, T \in L$ we have $S \wedge T = S \cap T$ and $S \vee T = S \cup T$. )

A obvious observation is $x \vee y \geq x, y \geq x \wedge y$

Clearly we can extent the relation $\wedge, \vee$ to multiple variables and which will essentially follow all ”boolean relations”.

Basically if $x, y$ lies in different chain then $x \wedge y$ is the place where the chains get separated and $x \vee y$ is the place where the chains reunite again.
Using this can you construct a poset $L$ which have one of these following properties:

1. there exist $x, y$ s.t. $x \wedge y$ or $x \vee y$ does not exists
2. there exist $x, y$ s.t. there are multiple $x \wedge y$ or $x \vee y$.

Clearly in any of these above cases it won't be a lattice.

Now can we say any relation between the existence of $x \wedge y$ and $x \vee y$?

**Claim.** If for a finite poset $L$ every element $x, y$ has a GLB (or LUB respectively) and bounded above then it is a lattice

Fix $x, y$ in $L$. Let $B$ be the set of all upper bounds of $x, y$
Now $B$ is finite. you can find a GLB of $B$ which will lie inside $B$ and that will be LUB of $x, y$

Now we will relate the mobious inversion with lattice in the following theorem:

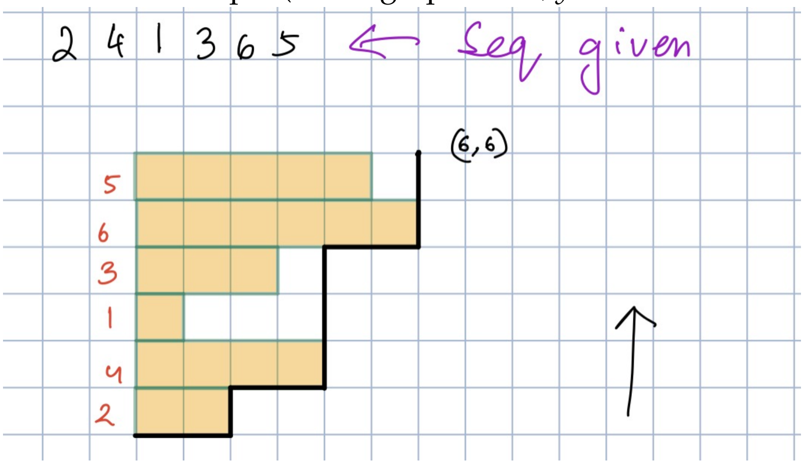
**Proposition 0.1.** For a Lattice $L$ if all elements of $L$ is bounded above by $\hat{1}$ and bounded below by $\hat{0}$ then for any $a$ in $L$ then

$$\mu(\hat{0},\hat{1}) = -\sum_{x, x \wedge a = \hat{0}} \mu(x,\hat{1})$$

We will not going to prove this theorem but will discuss the importance.
This basically shows that we can reduce the problem $\mu(\hat{0},\hat{1})$ by computing all $\mu(x,\hat{1})$ where $x$ and $a$ lies in different chains separated from the origin 0. (Clearly choosing $a = \hat{1}$ is meaningless) So the interval length reduces but number of sub problems increases. So if we choose $a$ close to 1 then there will be lesser number pf such $x$s and we can efficiently compute $\mu(\hat{0},\hat{1})$

**Solution of quiz 2 last problem:**
Set a bijection to paths which don't cross the $x = y$ line on the plane.
Draw a $n \times n$ grid. And put dots at positions $(\pi_i, i)$.
Pick out $pi_1 = i_1$, the next number bigger than $pi_1$ is $i_2$, the number bigger than that $i_3$ and so on.
So we get $i_1 < i_2 < i_3, \ldots$
In between $i_1$ and $i_2$ all numbers you see must be less than $i_1$.
Between $i_2$ and $i_3$ they are all less than $i_2$. Start at $(i_1, 1)$. As you go from $(i_1, 1)$ to $(i_2, j)$, go up on seeing the next number only if all numbers less than that have already occured before it. This ensures you don't cross $y = x$

Here is an example (in the graph the $x, y$ co ordinates have been swapped).




Now here is a problem related to prbaility. This is from Nilava's note, you can find the solution there or you can ask me for that.

Suppose 7 boys and 13 girls line up in a row. Let S be the number of places in the row where a boy and a girl are standing next to each other. For example, in GBBGGGBGBGGGBGBGGBGG we have S=12. All such arrangements are enlisted and the S value written next to them. What is the sum of all these S