Tutorial

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Example of presets which are not poset:

Consider the set $A := \{x \in \mathbb{N} | x \text{ divides } 6\}$ i.e., A is the set of all positive dividors of 6
Define the relation $\leq$ such way that $a \leq b \iff a|b$
Now we can say A is a poset on the relation $\leq$

Now if we modify the set A to A' by including the negative divisors of 6 as well then A will be a preset but not poset under the same relation as $-3 \leq 3$ and $3 \leq -3$ but $-3 \neq 3$

So the main problem is $\forall a \in A$ we have $a \leq -a$ and $-a \leq a$ but $a \neq -a$
So if we declare that a and $-a$ are the same elements (i.e. equality property only depends on the absolute value) then the case will be similar to the set A and hence it will be a poset
So this gives a motivation for the conversion a preset to a poset

Conversion of a preset into a poset:

The idea is to create a partition on the preset A where in each partition if we declare the elements are same it will be a poset
Then continue with the set of partitions

Let A is a preset but not poset under the operation $\leq$
Define a relation $\tilde{=}$ such way that $a \tilde{=} b \iff a \leq b \wedge b \leq a$
One can easily check that $\tilde{=}$ is an equivalence relation. Hence it creates some partition on A
Let A' be the set of partitions of A
Define the relation $\tilde{\leq}$ on A' such way that $[a] \tilde{\leq} [b] \iff a \leq b$
Check the relation $\tilde{\leq}$ is well defined on A'
Check A' is a poset under the relation $\tilde{\leq}$

Tutorial problems:

Question 1.

So for the first position there are 3 choices and for each choice of the first position there are 6! many choices for the rest 6 positions. Hence $3 \times 6!$

Question 2.

total $\frac{7!}{2}$ many permutations.
Out of which 5! many have same elements at the beginning and at the end.(clearly first and the last position will be 1)

Question 3.

Take any k consecutive natural number amongst them the largest one in n
Clearly their product will be $(n)_k$
Now an easy observation is $\binom{n}{k} = \frac{(n)_k}{k!}$

Question 4.

Observe $\binom{n}{2} = \sum_{i=1}^{n} i$
Hence

$$\begin{aligned}
\binom{k}{2} + k(n-k) + \binom{(n-k)}{2} &= \sum_{i=1}^{k} i + \sum_{i=1}^{n-k} k + \sum_{i=1}^{n-k} i \\
&= \sum_{i=1}^{k} i + \sum_{i=1}^{n-k} (i+k) \\
&= \sum_{i=1}^{k} i + \sum_{i=k+1}^{n} i \\
&= \sum_{i=1}^{n} i = \binom{n}{2}
\end{aligned}$$

Question 5.

observe

$$\binom{p}{k} = \frac{p}{k}\binom{p-1}{k-1}$$

Now if $k \Big| \binom{p-1}{k-1}$ then we are done. Otherwise

$$p \Big| k\binom{p}{k}$$
$$\implies p \Big| \binom{p}{k} \vee p \Big| k$$

Now p can not divide k (**why?**)

Question 6.

The problem is same as saying how to distribute $k-2n$ sweets into n students without any lower bound constrain as we can simply give two sweets each of them before hand.

Now we have to create n partitions on $k-2n$ sweets that is same as saying we have to put $n-1$ identical stick in between $k-2n$ identical balls

WLOG assue there are n balls and r sticks
So to put the first stick we have $n+1$ gaps. After placing the first one, for the second stick we have $n+2$ gaps, $(n+3)$ for the third and so on. So $(n+r)_r$ possibilities.
But during that calculation we are considering the order of the sticks as well.
Hence total number of permutations are $\dfrac{(n+r)_r}{r!} = \dbinom{(n+r)}{r}$

Question 7.

For $f(i) = i$:
Fix an i from $[n]$ which will be the fixed point. So for each the rest $j \neq i \in [n]$ there are $(n-1)$ choices, there are $(n-1)$ such js and choices of i are n
Hence $n \times (n-1)^{(n-1)}$

For $f(i) = i^2$:
The idea is more or less same: fix an i and then check for other js
But the possibilities of i here are $\sqrt{n}$[1] many. And for a fixed i, for each $j \neq i \in [\sqrt{n}]$ each of them has $n-1$ choices.(**Note: during the tutorial I made a mistake here, I wrote $\sqrt{n}-1$ many choices**)
And for $j \in [\sqrt{n}+1 \dots n]$ each j has n many choices.
Hence total number of possibilities $\sqrt{n} \times (n-1)^{\sqrt{n}-1} \times n^{n-\sqrt{n}}$

[1]by $\sqrt{n}$ we mean the floor of $\sqrt{n}$ here