Tutorial

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Question 1. Prove that if A is a square of length 1 then for any five points in side A there will be two points separeated by a distance at max $\dfrac{1}{\sqrt{2}}$

Divide A into four squares of side length $\frac{1}{2}$. Now apply PHP and try to figure out the solution.

Question 2. If every points of the plane $\mathbb{R}^2$ is coloured by either red black or blue then there will be a rectangle whose vertices are of same colour. Prove it.

Consider the set $S \subseteq \mathbb{Z}^2 = \{(x, y) \big| 0 \leq x \leq 3, 0 \leq y \leq 81\}$
So there are 82 columns and for each column there are $3^4 = 81$ possibilities of colouring
Hence there will two columns c_1, c_2 which will have exact pattern of colouring.(PHP)

Now apply PHP again to find two vertices from each c_i which will have exact same colour.

Question 3. Let $x_1, x_2, \ldots, x_n \in \mathbb{R}$, $|x_i| \leq 1$. Show that there exists $a_i \in \{-1, 0, 1\}$, not all zero, such that $|\sum a_i x_i| \leq \dfrac{n}{2^n - 1}$

Take $\vec{x} = (x_1, \ldots, x_n)$ s.t. $0 < x_i < 1$
Define $\langle \vec{x}, \vec{y} \rangle = \sum_{i=1}^{n} x_i y_i$
Define $A = \{-1, 0, 1\}^n, B \subseteq A = \{0, 1\}^n$
Note for $\vec{b_1}, \vec{b_2} \in B$ we have $\vec{b_1} - \vec{b_2} \in A$
Note for $\vec{b} \in B$ we have $0 \leq \langle \vec{b}, \vec{x} \rangle \leq n$ and there are 2^n possibilities for $\langle \vec{b}, \vec{x} \rangle$

Now divide the interval $[0, n]$ into $2^n - 1$ equal length intervals and apply PHP
Also remember $\langle \vec{b_1}, \vec{x} \rangle - \langle \vec{b_2}, \vec{x} \rangle = \langle \vec{b_1} - \vec{b_2}, \vec{x} \rangle$