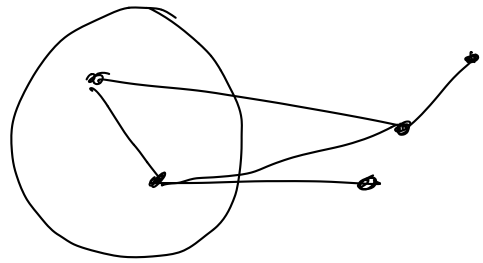
Let $G = (V, E)$ be a graph. For $S \subseteq V$ define

$$K(S) = \{ e \in E \mid \text{one endpt of } e \text{ is in } S \text{ \& the other in } V \setminus S \}$$



Prove : $\exists \; S_0 \subseteq V$ s.t. $|K(S_0)| \geq \frac{1}{2} |E|$.

Motivation : If $\mu = (x_1 + \cdots + x_n)/n$ then $\exists \; i, j$ s.t. $x_i \geq \mu, x_j \leq \mu$.

I will show that $\underbrace{\text{"the average of"}}_{\text{Expectation}}$ $|K(S)|$'s is $\geq \frac{|E|}{2}$.

So we compute $\mathbb{E}(|K(S)|)$:

Let $I_{ij} = \begin{cases} 1 & \text{if edge } \{i,j\} \text{ is in } K(S) \\ 0 & \text{o/w} \end{cases}$

$(i, j \in V)$

$$\mathbb{E}(|K(S)|) = \mathbb{E}\Big(\sum_{\{i,j\} \in E} I_{ij}\Big) = \sum_{ij \in E} \mathbb{E}(I_{ij})$$

$$= \sum_{ij \in E} 1 \cdot \mathbb{P}(I_{ij} = 1)$$

$$\mathbb{P}(I_{ij} = 1) = \mathbb{P}\big((i \in S, j \in V \setminus S) \sqcup (i \in V \setminus S, j \in S)\big)$$

$$= \mathbb{P}(i \in S, j \in V \setminus S) + \mathbb{P}(i \in V \setminus S, j \in S)$$

$$= \mathbb{P}(i \in S) \cdot \mathbb{P}(j \in V \setminus S) + \mathbb{P}(i \in V \setminus S)\, \mathbb{P}(j \in S)$$

$$= \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{2}.$$

$$= \sum_{ij \in E} \frac{1}{2} = \frac{1}{2} |E|$$

We conclude that $\exists \; S_0$ s.t. $|K(S_0)| \geq \frac{|E|}{2}$.

Defn: A poset L is called a lattice if any two elements $x, y \in L$ have a minimum common upper bd a & a maximum common lower bd b.

If $S, T \subseteq X$, then the minimum common u.b. of S & T is $S \cup T$, and maximum common lower bd is $S \cap T$.

In case of a general lattice, min common u.b. of x, y is written as $a = x \vee y$ & max common l.b of x, y is written as $b = x \wedge y$.

$\wedge$ - meet
$\vee$ - join

The problem I gave yesterday:

Suppose 7 boys and 13 girls line up in a row. Let S be the number of places in the row where a boy and a girl are standing next to each other. For example, in GBBGGGBGBGGGBGBGGBGG we have S=12. All such arrangements are enlisted and the S value written next to them. What is the sum of all these S?

Fix a naming: $B_1, \dots, B_7$, $G_1, \dots, G_{13}$

$$I_{ij} = \begin{cases} 1 & \text{if } B_i \text{ \& } G_j \text{ are next to each other} \\ 0 & \text{o/w} \end{cases}$$

Easy to see $S = \sum_{\substack{1 \le i \le 7 \\ 1 \le j \le 13}} I_{ij}$

What is $\mathbb{E}(I_{ij})$?

$\mathbb{E}(I_{ij}) = 1 \cdot \mathbb{P}(I_{ij} = 1)$

$= \dfrac{19! \times 2}{20!}$

$$\therefore \mathbb{E}(S) = \mathbb{E}\left(\sum_{i,j} I_{ij}\right) = \sum_{i=1}^{7} \sum_{j=1}^{13} \mathbb{E}(I_{ij})$$

$$= \sum_{i=1}^{7} \sum_{j=1}^{13} \frac{19! \times 2}{20!} = \frac{19! \times 2}{20!} \underbrace{91}_{13 \times 7}$$

The no. we are looking for $= \mathbb{E}(S) \times (\underbrace{\text{total no. of arrangements}}_{20!})$

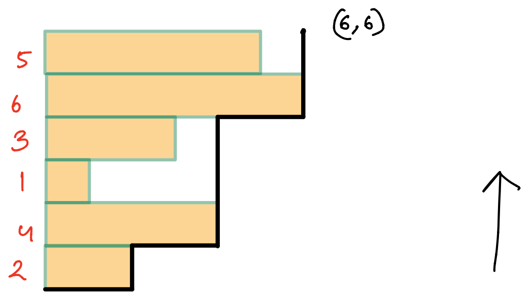
$= 19! \times 2 \times 91$

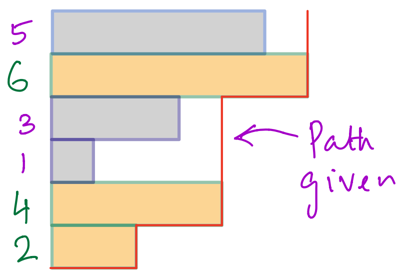
Expected no. of fixed points (in a permutation of n letters) is 1.

$\therefore$ If you look at all permutations on n letters & count the total no. of fixed points, it comes out to be $n!$.

P3 of Quiz 2

2 4 1 3 6 5 ← Seq given





Remaining are 1, 3, 5.

Place them in increasing order in the gaps.