Conjuctive Normal Forms

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Boolean operations:

A variable x is said to be a boolean variable if it takes only two values $0, 1$ and if some certain boolean operations can be performed on it.

There are mainly three boolean operation namely negation ($\neg$), And ($\wedge$) , Or ($\vee$). There are also some other boolean operations based on these three.

Now

$$\neg x = \begin{cases} 1 & \text{if} x = 0 \\ 0 & \text{if} x = 1 \end{cases}$$

Now to understand the operation for $x \wedge y$ and $x \vee y$ you can check the truth table for those operations

Disjunctive clause:

Now c is said to be a disjunctive clause of $x_1, \ldots, x_n$ variables if c is of the form $l_1 \vee l_2 \vee l_3 \cdots \vee l_m$ where each $l_i = x_j$ or $\neg x_j$ for some x_j
For example $(x_1 \vee \neg x_4), (x_3 \vee x_6 \vee x_5), (\neg x_1 \vee \neg x_2)$ are disjunctive clauses. It is basically an expression where some boolean variables or there negations are attached with "or"

CNF formula:

A boolean expression φ is said to be in CNF if $\phi = c_1 \wedge c_2 \wedge \cdots \wedge c_n$ where each c_i is a disjunctive clause
For example
i. $(x_1 \vee \neg x_4) \wedge (x_3 \vee x_6 \vee x_5)$
ii. $(x_3 \vee x_6 \vee x_5) \wedge (\neg x_1 \vee \neg x_2)$
iii. $(\neg x_1 \vee \neg x_2) \wedge (x_1 \vee \neg x_4)$
iv. $(x_1 \vee \neg x_4) \wedge (x_3 \vee x_6 \vee x_5) \wedge (\neg x_1 \vee \neg x_2)$
are examples of CNF. It is basically an expression where some disjunctive clauses are attached with "and". You can write some CNF formulas by yourselves to understand it better

Relation with $\mathbb{F}_2$:

Now since there are only two possible boolean values which are $0, 1$ we can extend the boolean world to the field $\mathbb{F}_2$ by finding some equivalent expression for the operations negation, and(conjction) , or (disjunction)

Now it will be a simple exercise to check
i. $x \wedge y$ in boolean $\equiv xy$ in $\mathbb{F}_2$
ii. $\neg x$ in boolean $\equiv (1 - x)$ in $\mathbb{F}_2$
iii. x xor $y \equiv x + y$
Equivalent means for each input (x, y) the output in left side and the right side will be same

Now can you find an equivalent expression for "or" in $\mathbb{F}_2$?
(Think of demorgan's law $x \vee y = \neg(\neg x \wedge \neg y)$ as a hint)

The extension from boolean world to $\mathbb{F}_2$ will help us in many ways. You will understand this later. This modification is called **Arithmetization**.

Now a possible hint for the problem 2, can you write an equivalent statement of the boolean statement $p(x, y) = (1 - y)p(x, 0) + yp(x, 1)$
If you can then for a n variable boolean expression $p(x_1, \ldots, x_n)$ can be written in terms of $(n - 1)$ variable boolean expression and then you can apply your induction hypothesis (also remember $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$)