QUANTUM COMPUTATION

Lecture 1

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1 Basic notions

Though we will assume some familiarity with linear algebra, we will review the basic concepts in this section.
Let V be a vector space over a field $\mathbb{K}$.

Definition 1.1. For a set $S \subseteq V$ define its *span* $\langle S \rangle$ to be the set of all linear combinations of the elements of S.

Proposition 1.2. *For $S \subseteq V$, $\langle S \rangle$ is the smallest subspace of V that contains S.*

Definition 1.3. A set $\mathcal{S} \subseteq V$ is said to be *spanning* iff every element in V can be written as a linear combination of finitely many elements in $\mathcal{S}$. We then write $\langle \mathcal{S} \rangle = V$ and say $\mathcal{S}$ spans V.

Example. $\mathcal{S} = V$ is a spanning set of V.

Example. $\mathcal{S} = \mathbb{Z}$ is a spanning set of $V = \mathbb{R}$ with $\mathbb{K} = \mathbb{R}$.

Definition 1.4. We say V is *finite dimensional* over $\mathbb{K}$ iff it contains a finite spanning set.

Example. $V = \mathbb{R}^2$ is a finite dimensional vector space over $\mathbb{K} = \mathbb{R}$ with $\mathcal{S} = \{(0,1),(1,0),(1,1)\}$ so that if $(a,b) \in V$ then $(a,b) = (b-a)(0,1) + a(1,1)$

Definition 1.5. A finite set $\mathcal{L} \subseteq V$ is said to be *linearly independent* in V iff there is no non-trivial solution to $\sum_{i=1}^{k} \lambda_i \boldsymbol{v_i} = 0 \quad (\lambda_i \in \mathbb{K}, \boldsymbol{v_i} \in \mathcal{L})$.

Exercise 1
Any subset of a finite linearly independent set is linearly independent.

Definition 1.6. A set $\mathcal{L} \subseteq V$ is said to be *linearly independent* in V iff every finite subset of $\mathcal{L}$ is linearly independent.

Definition 1.7. A set $X \subseteq V$ is said to be *linearly dependent* if it is not linearly independent, that is, there is a finite subset $Y \subseteq X$ and $\lambda_{\boldsymbol{y}} \in \mathbb{K} \ \ (\boldsymbol{y} \in Y)$, not all 0 such that $\sum_{y \in Y} \lambda_{\boldsymbol{y}} \boldsymbol{y} = 0$.

Example. In $V = \mathbb{R}^2$ the set $L = \{(0,1)\}$ is linearly independent but $X = \{(1,0),(0,1),(1,1)\}$ is linearly dependent.

Definition 1.8. A set $\mathcal{B} \subseteq V$ is said to be a *basis* of V iff $\mathcal{B}$ is linearly independent and $\langle \mathcal{B} \rangle = V$.

EXAMPLE. For $V = \mathbb{R}^2$, the set $\mathcal{B}_1 = \{(1,0),(0,1)\}$ is a basis. A different basis would be $\mathcal{B}_2 = \{(1,1),(0,1)\}$.

EXAMPLE. For $V = \mathbb{R}[x]$, $\mathcal{L} = \{1, x^5\}$ is linearly independent and $\mathcal{B} = \{x^k : k \in \mathbb{Z}_{\geq 0}\}$ is a basis.

Theorem 1.9. *Let V be a $\mathbb{K}-$vector space and $\mathcal{B} \subseteq V$. The following are equivalent:*

1. *$\mathcal{B}$ is a maximal linearly independent set in V*
2. *$\mathcal{B}$ is a minimal spanning set in V*
3. *$\mathcal{B}$ is linearly independent and spanning in V*
4. *Every $\boldsymbol{v} \in V$ is uniquely expressible as $\boldsymbol{v} = \sum_{i=1}^{n} \lambda_i v_i$ for some $n \in \mathbb{Z}_{\geq 0}, \boldsymbol{v}_i \in \mathcal{B}, \lambda_i \in \mathbb{K}$*

Lemma 1.10. *Let V be a finite dimensional $\mathbb{K}-$vector space and let $\mathcal{L} \subseteq \mathcal{S} \subseteq V$ be linearly independent and spanning. There is a basis $\mathcal{B} \subseteq V$ such that $\mathcal{L} \subseteq \mathcal{B} \subseteq \mathcal{S}$.*

Proposition 1.11. *Every basis of a finite dimensional vector space V has the same cardinality and denote it by* $\dim V$.

Definition 1.12. The cardinality of a basis is called the *dimension* of the vector space.

EXAMPLE. The dimension of $V = \mathbb{R}^2$ is 2.

Definition 1.13. For $\mathbb{K}-$vector spaces V, W, a map $T : V \to W$ is said to be *linear* iff $T(\boldsymbol{v} + \lambda \boldsymbol{v}) = T(\boldsymbol{v}) + \lambda T(\boldsymbol{v})$ for each $\boldsymbol{u}, \boldsymbol{v} \in V, \lambda \in \mathbb{K}$.
In case $\mathbb{K} = \mathbb{C}$, we say $T : U \to V$ is *conjugate-linear* iff $T(\boldsymbol{v} + \lambda \boldsymbol{v}) = T(\boldsymbol{v}) + \overline{\lambda} T(\boldsymbol{v})$ for each $\boldsymbol{u}, \boldsymbol{v} \in V, \lambda \in \mathbb{K}$.

Definition 1.14. For $\mathbb{K}-$vector spaces V, W define $\mathrm{Hom}_{\mathbb{K}}(V, W) := \{T : V \to W \mid T \text{ is linear}\}$ and $\mathrm{End}_{\mathbb{K}}(V) := \mathrm{Hom}_{\mathbb{K}}(V, V)$.

Definition 1.15. $T \in \mathrm{End}_{\mathbb{K}}(V)$ is said to be diagonalizable if there exists a basis $\mathcal{B}$ of V such that each $\boldsymbol{v} \in \mathcal{B}$ is an eigenvector of T.

Definition 1.16. Two $\mathbb{K}-$vector spaces U, V are said to be *isomorphic* iff $\exists\ T \in \mathrm{Hom}(U, V)$ (or $\exists$ conjugate-linear $T : U \to V$, in case $\mathbb{K} = \mathbb{C}$) such that T is a bijection and write $U \cong V$.

Definition 1.17. Let V, W be finite dimensional $\mathbb{K}-$vector spaces and let $T \in \mathrm{Hom}_{\mathbb{K}}(V, W)$. Fix ordered bases $\mathcal{B}_1 = (\boldsymbol{u}_1, \ldots, \boldsymbol{u}_n), \mathcal{B}_2 = (\boldsymbol{v}_1, \ldots, \boldsymbol{v}_m)$ of V, W respectively. Then we will have unique $\lambda_{ij} \in \mathbb{K}$ satisfying $T(\boldsymbol{u}_i) = \sum_{j=1}^{m} \lambda_{ij} \boldsymbol{v}_j$ for each $i \in [n], j \in [m]$. The matrix $[T]_{\mathcal{B}_1, \mathcal{B}_2} := [(\lambda_{ij})]$ is called the matrix of T with respect to $\mathcal{B}_1, \mathcal{B}_2$.

If $V = W$ and $\mathcal{B}_1 = \mathcal{B}_2 = \mathcal{B}$ then we write $[T]_{\mathcal{B}} := [T]_{\mathcal{B},\mathcal{B}}$ and call it the matrix of T with respect to $\mathcal{B}$.

EXERCISE 2
For each pair of bases $\mathcal{B}_1, \mathcal{B}_2$ of finite dimensional $\mathbb{K}-$vector space V, W (respectively) there is an isomorphism $\mathrm{Hom}_{\mathbb{K}}(V, W) \cong M_{m \times n}(\mathbb{K})$ where $n = \dim V, m = \dim W$, given by the above correspondence.

EXERCISE 3
Let $S : U \to V, T : V \to W$ be linear operators on finite dimensional $\mathbb{K}-$vector spaces U, V, W and fix (ordered) bases $\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3$ respectively. Then

$$[TS]_{\mathcal{B}_1,\mathcal{B}_3} = [T]_{\mathcal{B}_2,\mathcal{B}_3}[S]_{\mathcal{B}_1,\mathcal{B}_2}$$

2 Hilbert spaces

For the sake of this seminar, we will mostly look at complex vector spaces with inner products. Another common notation we will use is $x^* = \overline{x}$ for $x \in \mathbb{C}$.

In quantum mechanics, a Hilbert space is the same as a complex inner product space. We will restrict ourselves to finite dimensional Hilbert spaces at least for the next few talks, if not all.

2.1 Inner products

Definition 2.1. An inner product on a $\mathbb{C}-$vector space V is a function $\langle\cdot|\cdot\rangle : V \times V \to \mathbb{C}$ satisfying the following:

1. $\langle \boldsymbol{x}|\boldsymbol{x}\rangle \geq 0 \ \forall\ \boldsymbol{x} \in V$ with equality iff $\boldsymbol{x} = 0$
2. $\langle \boldsymbol{x}|\boldsymbol{u}+\boldsymbol{v}\rangle = \langle \boldsymbol{x}|\boldsymbol{u}\rangle + \langle \boldsymbol{x}|\boldsymbol{u}\rangle \ \ \forall\ \boldsymbol{x}, \boldsymbol{u}, \boldsymbol{v} \in V$
3. $\langle \boldsymbol{x}|\boldsymbol{y}\rangle = \langle \boldsymbol{y}|\boldsymbol{x}\rangle^*$

Definition 2.2. Let H be a Hilbert space. The *norm* of $\boldsymbol{u} \in H$ is defined as $\|\boldsymbol{u}\| := \sqrt{\langle \boldsymbol{u}|\boldsymbol{u}\rangle}$.

Theorem 2.3 (Cauchy Schwarz inequality)**.** *Let H be a Hilbert space. For any vectors $\boldsymbol{u}, \boldsymbol{v} \in H$ we have $|\langle \boldsymbol{u}|\boldsymbol{v}\rangle|^2 \leq \langle \boldsymbol{u}|\boldsymbol{u}\rangle \langle \boldsymbol{v}|\boldsymbol{v}\rangle$*

Definition 2.4. $\boldsymbol{u}, \boldsymbol{v}$ in Hilbert space H are said to be *orthogonal* iff $\langle \boldsymbol{u}|\boldsymbol{v}\rangle = 0$. We write $\boldsymbol{u} \perp \boldsymbol{v}$.

Definition 2.5. Let H be a finite dimensional Hilbert space and $\mathcal{B} = (\xi_i)_{i=1}^n$ be a basis of H. $\mathcal{B}$ is said to be an *orthogonal* basis (of H) iff $\langle \xi_i|\xi_j\rangle = \delta_{ij} \ \forall\ i, j \in [n]$. $\mathcal{B}$ is said to be an *orthonormal* basis (of H) iff $\mathcal{B}$ is an orthogonal basis and $\|\xi_k\| = 1 \ \forall\ k \in [n]$.

Theorem 2.6 (Gram-Schmidt)**.** *Every finite dimensional Hilbert space has an orthonormal basis.*

Definition 2.7. Let H be a Hilbert space and V be a subspace of H. The orthogonal complement of V is defined as $V^\perp := \{\boldsymbol{u} \in H \mid \boldsymbol{u} \perp \boldsymbol{v} \ \forall\ \boldsymbol{v} \in V\}$.

Lemma 2.8. *Let V be a subspace of a finite dimensional Hilbert space H. Then $H = V \oplus V^\perp$.*

Definition 2.9. For an inner product space V, an operator $T \in \mathrm{End}(V)$ is said to be *non-negative definite* if $\langle T\boldsymbol{u}|\boldsymbol{u}\rangle \geq 0 \ \forall\ \boldsymbol{u} \in V$ and write $T \geq 0$. We sometimes call such operators simply *non-negative.*

2.2 Dual spaces and the adjoint operator

Definition 2.10. For a $\mathbb{K}-$vector space V define its *dual space* as $V^* := \{\varphi : V \to \mathbb{K} \mid \varphi \text{ is linear}\}$.

Theorem 2.11 (Riesz representation theorem)**.** *Let H be a finite dimensional Hilbert space and $\varphi \in H^*$. Then there is a $\xi \in H$ such that $\varphi(\eta) = \langle \xi|\eta\rangle$ for each $\eta \in H$.*

Exercise 4

Let H be a finite dimensional Hilbert space. There is a natural (conjugate linear) isomorphism $\xi \mapsto \langle \xi|\cdot\rangle = (\eta \mapsto \langle \xi|\eta\rangle)$

Exercise 5

Let H be a finite dimensional Hilbert space and $T \in \mathrm{End}_{\mathbb{C}}(H)$. Then there is a unique $T^* \in \mathrm{End}_{\mathbb{C}}(H)$ such that $\langle \xi|T\eta\rangle = \langle T^*\xi|\eta\rangle$.

Definition 2.12. Let T and T^* be as above. Then T^* is called the *adjoint* of T.

EXERCISE 6
The adjoint operator $T \overset{*}{\mapsto} T^*$ is conjugate linear.

EXERCISE 7
$T = T^{**} := (T^*)^*$ for every $T \in \mathrm{End}_{\mathbb{C}}(H)$, where H is a finite dimensional Hilbert space.

Proposition 2.13. *Let $\mathcal{B} = (e_i)_{i=1}^n$ be an ordered basis of a finite dimensional Hilbert space H. Then $[T^*]_{\mathcal{B}} = \overline{[T]}_{\mathcal{B}}^t$.*

Definition 2.14. Let H be a Hilbert space. A linear operator $T \in \mathrm{End}_{\mathbb{C}}(H)$ is said to be *self-adjoint* or *hermitian* iff $T = T^*$.

EXERCISE 8
Let $\mathcal{B}$ be an orthonormal basis of Hilbert space H and let $T \in \mathrm{End}_{\mathbb{C}}(H)$. T is self adjoint iff $[T]_{\mathcal{B}}$ is equal to its conjugate transpose.

EXERCISE 9
All eigenvalues of a self-adjoint linear operator on a finite dimensional Hilbert space are real.

Definition 2.15. Let H be a finite dimensional Hilbert space. A linear operator $T \in \mathrm{End}(H)$ is said to be *unitary* iff $TT^* = T^*T = I$.

Theorem 2.16. *Let H be a Hilbert space and $T \in End_{\mathbb{C}}(H)$ be self-adjoint. Then T is diagonalizable (through an orthonormal basis of H).*

3 Tensor products

3.1 An initial problem to consider

To motivate tensor products, we start with a problem where one might think of something similar to tensor products in a natural way. Here is a way that leads to the notion of tensor products.

Let U, V, W be fintie dimensional $\mathbb{K}-$vector spaces. Denote by $B(U, V, W)$ the vector space of all bilinear maps $U \times V \to W$. A map $\chi : U \times V \to W$ is said to be bilinear if it is linear in both arguments, that is, $\chi(\boldsymbol{u}_1 + \lambda\boldsymbol{u}_2, \boldsymbol{v}) = \chi(\boldsymbol{u}_1, \boldsymbol{v}) + \lambda\chi(\boldsymbol{u}_2, \boldsymbol{v})$ and $\chi(\boldsymbol{u}, \boldsymbol{v}_1 + \lambda\boldsymbol{v}_2) = \chi(\boldsymbol{u}, \boldsymbol{v}_1) + \lambda\chi(\boldsymbol{u}, \boldsymbol{v}_2)$.

In case of linear maps, we know that if we choose a basis of the domain and specify the image of all the basis vectors, it uniquely determines a linear map. We would like to know something similar similar for bilinear maps: what is the 'smallest' information we need to uniquely determine a bilinear map? One might say out loud, "Math is good!" and end up with a conjecture that it is enough to specify the image of each $(\boldsymbol{u}, \boldsymbol{v}) \in U \times V$ where $\boldsymbol{u}, \boldsymbol{v}$ are basis vectors of U, V respectively. And that turns out to be the answer! (Caution: Though clear, one must still check well-definedness). In fact, we see that it is necessary and sufficient to specify $\dim U \times \dim V \times \dim W$ numbers to determine the map χ uniquely.

We might ask what minimal set $S \subseteq U \times V$ would be good enough such that specifying $\chi\left(\vec{t}\right)$, for $\vec{t} \in S$, would be enough information to uniquely determine χ.

Let's take an example with $U = V = \mathbb{R}^2, W = \mathbb{R}$. Let $\boldsymbol{e}_1, \boldsymbol{e}_2$ be the canonical basis of $\mathbb{R}^2$. Specifying the images of $\mathcal{B} = \{(\boldsymbol{e}_1, \boldsymbol{e}_1), (\boldsymbol{e}_2, \boldsymbol{e}_1), (\boldsymbol{e}_1, \boldsymbol{e}_2), (\boldsymbol{e}_2, \boldsymbol{e}_2)\}$ would be enough to uniquely determine χ. Well, it would also be uniquely determined by the vectors in $\mathcal{D} = \{(\boldsymbol{e}_1, \boldsymbol{e}_1), (\boldsymbol{e}_2, \boldsymbol{e}_2), (\boldsymbol{e}_1 + \boldsymbol{e}_2, \boldsymbol{e}_1 + \boldsymbol{e}_2), (\boldsymbol{e}_1 + \boldsymbol{e}_2, \boldsymbol{e}_1 + 2\boldsymbol{e}_2)\}$.

But specifying the images of vectors in $\mathcal{E} = \{(\boldsymbol{e}_1, \boldsymbol{e}_1), (\boldsymbol{e}_2, \boldsymbol{e}_2), (\boldsymbol{e}_1+\boldsymbol{e}_2, \boldsymbol{e}_1+\boldsymbol{e}_2), (\boldsymbol{e}_1+\boldsymbol{e}_2, \boldsymbol{e}_1-\boldsymbol{e}_2)\}$ is not a good idea! $c_1 = \chi(\boldsymbol{e}_1, \boldsymbol{e}_1)$ and $c_2 = \chi(\boldsymbol{e}_2, \boldsymbol{e}_2)$ gives information, totally 'unrealted' from each other. $c_3 = \chi(\boldsymbol{e}_1 + \boldsymbol{e}_2, \boldsymbol{e}_1 + \boldsymbol{e}_2)$ gives new information: $\chi(\boldsymbol{e}_1 + \boldsymbol{e}_2, \boldsymbol{e}_1 + \boldsymbol{e}_2)$. But nothing new is conveyed by $\chi(\boldsymbol{e}_1 - \boldsymbol{e}_2, \boldsymbol{e}_1 - \boldsymbol{e}_2) = 2c_1 + 2c_2 - c_3$. This suggests $\mathcal{B}, \mathcal{D}$ are 'like' linearly independent sets, but $\mathcal{E}$ is not. In fact, it also makes sense to say $\mathcal{B}, \mathcal{D}$ are 'like' bases, in the sense that they *determine* (analog of spanning) the bilinear map *uniquely* (analog of linear independence).

Exercise 10
Try out more examples with $V = U = W = \mathbb{R}$.

But what we are trying to imply here is something independent of χ. Sure, these above four vectors in $\mathcal{E}$ do not give you the entire information about χ. But, note that if one replaces χ with any other bilinear map $U \times V \to W$, the statement still remains the same. Hence we want to make sense of this 'linear independence' so that it is independent of any bilinear map. One way could be to exchange the roles of χ and $(\boldsymbol{x}, \boldsymbol{y})$. What I mean is: we declare $(\boldsymbol{x}, \boldsymbol{y})$ to be a function on a bilinear map, say denoted by $[\boldsymbol{x}, \boldsymbol{y}]$ and defined by $[\boldsymbol{x}, \boldsymbol{y}]\,(\phi) = \phi(\boldsymbol{x}, \boldsymbol{y})$ for each $\phi \in B(U, V, W)$. And now we see that it lives inside a vector space (perhaps a subspace of the vector space that we immediately see from this discussion) and hence it makes sense to talk about span, linear independence and basis. The vector space would be the space spanned by all functions $[\boldsymbol{x}, \boldsymbol{y}]$, quotiented by the evaluation map (restriction that the image of a bilinear map under $[\boldsymbol{x}, \boldsymbol{y}]$ is the function evatuated at $(\boldsymbol{x}, \boldsymbol{y})$).

3.2 Treat bilinear maps as two successive linear maps

Now for each $\boldsymbol{u} \in U$, the map $\chi_{\boldsymbol{u}} := \chi(\boldsymbol{u}, \cdot) = (\boldsymbol{v} \mapsto \chi(\boldsymbol{u}, \boldsymbol{v}))$ is linear. Similarly the map $\chi'_{\boldsymbol{v}} := \chi(\cdot, \boldsymbol{u}) = (\boldsymbol{u} \mapsto \chi(\boldsymbol{u}, \boldsymbol{v}))$ is linear for each $\boldsymbol{v} \in V$.

Clearly $\chi_{\boldsymbol{u}} \in \mathrm{Hom}(V, W)\ \forall\ \boldsymbol{u} \in U$ and $\chi'_{\boldsymbol{v}} \in \mathrm{Hom}_{\mathbb{K}}(U, W)\ \forall\ \boldsymbol{v} \in V$. Now consider the homomorphisms $\psi : U \to \mathrm{Hom}(V, W)$ given by $\boldsymbol{u} \overset{\psi}{\mapsto} \chi_{\boldsymbol{u}}$ and $\psi' : V \to \mathrm{Hom}(U, W)$ given by $\boldsymbol{v} \overset{\psi'}{\mapsto} \chi_{\boldsymbol{v}}$.

This suggests that $B(U, V, W)$ is somehow related to $\mathrm{Hom}(U, \mathrm{Hom}(V, W))$ because we have shown a linear map, say Φ, given by $\chi \overset{\Phi}{\mapsto} \psi' = (\boldsymbol{u} \mapsto \chi(\boldsymbol{u}, \cdot))$. The first isomorphism theorem could give us some strong result if we could find the kernel of Φ. In fact, if $\Phi(\chi)$ is the zero map, it follows that $\chi(\boldsymbol{u}, \cdot)$ is the zero operator for each $\boldsymbol{u} \in U$ and hence $\chi(\boldsymbol{u}, \boldsymbol{v})$ is zero (in W) for each $\boldsymbol{u} \in U, \boldsymbol{v} \in V$, which by definition, suggests that χ is the zero map. Hence, the kernel is trivial and gives us that $B(U, V, W) \cong \mathrm{Hom}\,(U, \mathrm{Hom}(V, W))$! One might as well say that $B(U, V, W) \cong \mathrm{Hom}\,(V, \mathrm{Hom}(U, W))$.

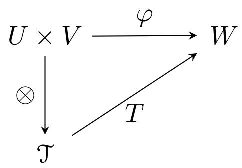
So the end result of this discussion is

$$B(U, V, W) \cong \mathrm{Hom}\,(U, \mathrm{Hom}(V, W)) \cong \mathrm{Hom}\,(V, \mathrm{Hom}(U, W))$$

3.3 Universal property and definining tensor product

Definition 3.1. Let U, V be vector spaces and $\otimes : U \times V \to \mathcal{T}$ be a bilinear map, where $\mathcal{T}$ is another vector space. We say that $\otimes$ has the *universal property* if it satisfies the following two conditions:

1. The vectors $\boldsymbol{x} \otimes \boldsymbol{y}$ generate $\mathcal{T}$ (that is, $\langle \mathrm{Im}(\otimes) \rangle = \mathcal{T}$).
2. If $\varphi : U \times V \to W$ is a bilinear map (where W is a vector space) then there exists a linear map $T : \mathcal{T} \to W$ so that the following diagram commutes(that is, $\varphi = T \circ \otimes$).



EXERCISE 11
Show that the above two conditons is equivalent to the following single condition: For every bilinear map $\varphi : U \times V \to W$, there exists a unique linear map $T : \mathcal{T} \to W$ so that $\varphi = T \circ \otimes$.

Lemma 3.2. *The vector space $\mathcal{T}$, described as above, exists and is unique.*

Definition 3.3. The space $\mathcal{T}$ corresponding to U, V is called the *tensor product* of U and V and denoted by $U \otimes V$.

EXERCISE 12
Let U and V be finite dimensional $\mathbb{C}-$vector spaces. Considering the map $\varphi : U^* \times V \to \mathrm{Hom}(U, V)$ given by $\varphi(\boldsymbol{x^*}, \boldsymbol{y}) = (\boldsymbol{t} \mapsto \boldsymbol{x^*}(\boldsymbol{t})\boldsymbol{y})$, show that $U^* \otimes V \cong \mathrm{Hom}(U, V)$

EXERCISE 13
Tensor product is associative.

EXERCISE 14
Let H_1 and H_2 be finite dimensional Hilbert spaces. Consider the map $\varphi : H_1 \times H_2 \to \mathrm{Hom}(H_1, H_2)$ given by $\varphi(\xi, \eta) = (\alpha \mapsto \langle \xi | \alpha \rangle \, \eta)$. Show that $H_1 \otimes H_2$ is (conjugate linearly) isomorphic to $\mathrm{Hom}(H_1, H_2)$

EXAMPLE.
1. $U^* \otimes V$ is the tensor popularly known as the space of linear operators $U \to V$.
2. $U^* \otimes V^*$ is the tensor popularly known as the space of bilinear forms $U \times V \to \mathbb{K}$.
3. $V_1^* \otimes \cdots \otimes V_n^*$ is the tensor popularly known as the space of $n-$forms $V_1 \times \cdots \times V_n \to \mathbb{K}$.

Definition 3.4. Let $T \in \mathrm{End}(H)$ for some finite dimensional Hilbert space H, and $\mathcal{B} = \{\xi_i\}_{i=1}^n$ an orthonormal basis. The trace of T with respect to $\mathcal{B}$ is given by $Tr_{\mathcal{B}}(T) = \sum_{i=1}^n \langle \xi_i | T\xi_i \rangle$

Proposition 3.5. *If $\mathcal{B}_1$ and $\mathcal{B}_2$ be orthonormal bases of finite dimensional Hilbert space H, and let $T \in End(H)$. Then $Tr_{\mathcal{B}_1}(T) = Tr_{\mathcal{B}_2}(T)$.*

4 Notation

In quantum computation, we generally use the finite dimensional Hilbert space $H = \mathbb{C}^n$. Since this has a canonical basis, say $(\boldsymbol{e}_i)_{i=1}^n$, we can freely talk about a vector or linear operator interchangibly with their matrix forms. And this gives the natural isomorphism $\displaystyle\sum_{i=1}^n v_i \boldsymbol{e}_i \mapsto \begin{bmatrix} v_1^* & \dots & v_n^* \end{bmatrix}$, thus H and H^* are conjugate linearly isomorphic. The image of a vector ξ, under the above isomorphism, is called its dual and denoted by ξ^* in mathematics. However for the sake of quantum mechanics, we will denote the vector by $|\xi\rangle$ (read as *ket of xi*) and its dual is denoted by $\langle \xi|$ (read as *bra of xi*), that is, $\langle \xi|^* = |\xi\rangle$.This is Dirac's bra-ket notation.
Also for an indexed subset $\{\xi_\alpha\}_{\alpha \in A}$ of H we will simply write $|\alpha\rangle$ instead of $|\xi_\alpha\rangle$.