Lie algebras

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Recap

- If L is a Lie algebra over F then $\mathrm{ad}_x = (y \mapsto [xy]) \in \mathrm{End}(L)$. In fact, it is a derivation: $\mathrm{ad}_x\,[ab] = [a\,(\mathrm{ad}_x\,b)] + [(\mathrm{ad}_x\,a)\,b]$.
- The set of all derivations on L is denoted by $\mathrm{Der}(L)$. It turns out to be a subalgebra of $\mathrm{End}(L)$.
- The map $\mathrm{ad} : L \to \mathrm{ad}(L)$ given by $x \mapsto \mathrm{ad}_x$ is a homomoprhism of Lie algebras: $\mathrm{ad}_{[xy]} = [\mathrm{ad}_x, \mathrm{ad}_y]$. This is called the **adjoint representation** of L.
- A subspace I of a Lie algebra L over F is called an **ideal** of L if $[xy] \in L \forall x \in L, y \in I$.
- A non-abelian Lie algebra L (i.e., $[LL] \neq 0$) is said to be **simple** if it has no nontrivial proper ideals.
- $\mathfrak{sl}(2)$. Ordered basis $x = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, y = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$.

$$\mathrm{ad}_x = \begin{bmatrix} 0 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \mathrm{ad}_h = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix}, \mathrm{ad}_y = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix}$$

Automorphisms

An automorphism of L is an isomorphism $L \to L$ of Lie algebras. The group of automorphisms of L is denoted by $\mathrm{Aut}(L)$.

Definition (Exponential map)

Let $\delta \in \mathrm{Der}(L)$ be nilpotent, i.e., $\delta^n = 0$ for some n. We define

$$\exp(\delta) = \sum_{i=0}^{n-1} \frac{\delta^i}{i!}$$

Claim: $\delta \in \mathrm{Der}(L)$ and $\delta^k = 0 \implies \exp(\delta) \in \mathrm{Aut}(L)$.

Verify that $\frac{\delta^n}{n!}[x,y] = \sum_{i=0}^{n} \left[\frac{\delta^i x}{i!}, \frac{\delta^{n-i} y}{(n-i)!}\right]$. Using this Leibniz rule, one can show that $[\exp\delta(x), \exp\delta(y)] = \exp\delta\,[x,y]$. So $\exp\delta \in \mathrm{End}(L)$.

The inverse of $\exp\delta$ is given by $\sum_{j=0}^{k-1} (1-\exp\delta)^j$ (Check!).

Solvability

Definition (Derived series and solvability)

Let L be a Lie algebra. Define the **derived series** of L as follows:

$$D^0(L) = L$$

$$D^{n+1}(L) = [D^n(L), D^n(L)]$$

We say L is **solvable** if $D^k(L) = 0$ for some k.

Example (Derived series)

Consider $L = \mathfrak{t}(n, F)$, the Lie algebra of all (non-strict) upper triangular matrices, with the commutator $[AB] = AB - BA$. It is not hard to see that the diagonal elements of $AB - BA$ are all 0 whenever $A, B \in L$. It follows that $D^1(L) = [L, L] = \mathfrak{n}(n, F)$ the algebra of strictly upper triangular matrices.
In fact, for a matrix $A = (a_{ij}) \in L$ define $\min \{j - i : a_{ij} \neq 0\}$ to be the level of A. Denote the set of all matrices of level l by $\mathfrak{t}_l(n, F)$ and $\mathfrak{t}_k = 0 \forall k \geq n$. So $\mathfrak{t}_0 = \mathfrak{t}$ and $\mathfrak{t}_1 = \mathfrak{n}$. Turns out that $D^l(\mathfrak{t}) = \mathfrak{t}_{2^{l-1}}$ for $l \geq 1$. Note that these are all ideals of $\mathfrak{t}_0$.

Remark: $D^k(L)$ are ideals of L, in general.

Solvability: properties

Proposition

1. All subalgebras and homomorphs of a solvable Lie algebra are solvable.
2. If I is a solvable ideal of a Lie algebra L such that L/I is solvable, then L is solvable.
3. Sum of solvable ideals of a Lie algebra is solvable.

Proof.

1. If L' is a subalgebra of L then $D^i(L') \subseteq D^i(L)$.
   If $\varphi : L \to L'$ is a surjective homomorphism, then $\varphi(D^i(L)) = D^i(L')$.
2. First note that $D^i(D^j(L)) = D^{i+j}(L)$. Consider the natural projection $\pi : L \twoheadrightarrow L/I$. Say k, l are such that $D^k(L/I) = 0, D^l(I) = 0$. By the second statement in the proof of 1, we have $\pi\left(D^k(L)\right) = D^k(L/I) = 0 \implies D^k(L) \subseteq \ker \pi = I \implies D^{k+l}(L) \subseteq D^l(I) = 0$.
3. By an isomorphism theorem, $(I + J)/J \cong I/(I \cap J)$. The RHS is the image of I under the natural projection $\varpi : I \twoheadrightarrow I/(I \cap J)$, and thus solvable by 1. Since J is solvable, conclude by 2 that $I + J$ is solvable.

■

Semisimplicity

Let L be a Lie algebra. Suppose S is a maximal solvable ideal of L, i.e., if T is a solvable ideal containing S, then $S = T$. Let $I \subseteq L$ be any solvable ideal. It follows that $S + I$ is solvable. Further, it contains S. So $S = S + I$. It follows that $I \subseteq S$. This proves the following

Lemma

Every Lie algebra has a unique maximal solvable ideal.

The maximal solvable ideal of L is called the **radical of** L and denoted by $\mathrm{Rad}\, L$. L is said to be **semisimple** if $\mathrm{Rad}\, L = 0$.

Example

Say L is simple, i.e., L has exactly two ideals, namely, 0 and L. Now $[L, L] = D^1(L)$ is an ideal of L , and nonzero (as L is non-abelian, by definition). This forces $D^k(L) = L \forall k$. The only solvable ideal of L is thus 0, which means L is semisimple.

A different characterization

A clear characterization

L is semisimple iff L has no nonzero solvable ideals.

Another characterization

L is semisimple iff L has no nonzero abelian ideals.

Proof.

Say L is semisimple. If I is an abelian ideal of L, then $D^1(I) = 0$ whence I is solvable. It follows that $I = 0 \because I \subseteq \operatorname{Rad} L = 0$.
Say L has no nonzero abelian ideals. For any solvable ideal I, there is no nonzero term in the derived series, else the last nonzero term would be abelian. So $I = 0$. ■

Nilpotency

Definition (Central series and nilpotency)

Let L be a Lie algebra. Define the **central series** of L as follows:

$$C^0(L) = L$$

$$C^{n+1}(L) = [L, C^n(L)]$$

We say L is **nilpotent** if $C^k(L) = 0$ for some k.

Nilpotency: properties

Proposition

1. All subalgebras and homomorphs of a nilpotent Lie algebra are nilpotent.
2. If $L/Z(L)$ is nilpotent, then L is nilpotent.
3. If $L \neq 0$ is nilpotent, then $Z(L) \neq 0$

Proof.

1. Exactly as in solvability.
2. Consider the natural projection $\pi : L \twoheadrightarrow L/I$ where $I = Z(L)$. Say k is such that $C^k(L/Z(L)) = 0$. Now $\pi(C^k(L)) = C^k(L/Z(L)) = 0 \implies C^k(L) \subseteq \ker \pi = Z(L) \implies C^{k+1}(L) \subseteq [L, Z(L)] = 0$.
3. Let k be least such that $C^k(L) = 0$. $L \neq 0 \implies k \geq 1$. Then $C^{k-1}(L) \neq 0$ and $\left[C^{k-1}(L), L\right] = 0 \implies 0 \neq C^{k-1}(L) \subseteq Z(L) \implies Z(L) \neq 0$.

■

Engel's theorem

Nilpotency is defined by looking at the central series. At an elemental level, we are really looking at terms $[x, y] = \mathrm{ad}_x(y)$ where $x \in L, y \in C^k(L)$. If L is nilpotent, we can conclude that there is some n for which $\mathrm{ad}_{x_n} \cdots \mathrm{ad}_{x_1}(y) = 0 \forall x_1, \cdots, x_n, y \in L$. In particular, $(\mathrm{ad}_x)^n = 0 \forall x \in L$. In other words,

Lemma

If L is nilpotent, then all elements of L are ad-nilpotent.

It turns out that the converse of the above lemma is also true and we call it Engel's theorem:

Theorem (Engel)

If all elements of L are ad-nilpotent, then L is nilpotent.

An intermediate theorem

Theorem

Let $V \neq 0$ be a finite dimensional vector space and $L \subseteq \mathfrak{gl}(V)$ be a Lie subalgebra consisting of only nilpotent elements. Then there is a common eigenvector for all elements of L. In other words, $\exists \vec{v} \in V, \vec{v} \neq 0$ such that $L\vec{v} = 0$.

Proof of Engel's theorem.

Let L be a Lie algebra in which all elements are ad-nilpotent. So all elements of $\mathrm{ad}(L) = \{\mathrm{ad}_x : x \in L\} \subseteq \mathfrak{gl}(L)$ are nilpotent, by definition. It follows (by the above theorem) that $\exists v \in L \setminus \{0\}$ such that $\mathrm{ad}_x(v) = 0 \forall x \in L$, i.e., $[v, L] = 0$. So $v \in Z(L)$. Note that $L' = L/Z(L)$ is a Lie algebra of smaller dimension $(\because Z(L) \neq 0)$ and all elements of L' are ad-nilpotent. By induction, that L' is nilpotent, whence by an earlier proposition, L is nilpotent. ■

A followup

Theorem

$(F = \overline{F})$ Let L be a solvable Lie subalgebra of $\mathfrak{gl}(V)$ with $V \neq 0$ finite dimensional vector space. V contains a common eigenvector for all endomorphisms in L. In other words, there is a $v \in L \setminus \{0\}$ and a functional $\lambda : L \to F$ such that $xv = \lambda(x)v \forall x \in L$.

Theorem (Lie' theorem)

$(F = \overline{F})$ Let $V \neq 0$ be a finite dimensional vector space. Let L be a solvable Lie subalgebra of $\mathfrak{gl}(V)$. Then $\exists$ a flag $0 = V_0 \subseteq V_1 \subseteq \cdots \subseteq V_n = V$ (i.e., $\dim V_i = i$) and is stable under L (i.e., $[L, V_i] \subseteq V_i \forall i$).
(In other words, the matrices of L relative to a suitable basis of V are all upper triangular).

Proof.

Let v be an eigenvector as stated in the previous theorem and let $V_1 = Fv$. Then induct by looking at V/V_1. Details in writeup. ∎

Cartan's criterion

Theorem

Let $A \subseteq B \subseteq \mathfrak{gl}(V)$ *with* V *being a finite dimensional vector space. Consider* $M = \{x \in \mathfrak{gl}(V) : [x, B] \subseteq A\}$*. Suppose* $x \in M$ *satisfies* $\mathrm{Tr}(xy) = 0 \forall y \in M$*. Then* x *is nilpotent.*

Theorem (Cartan's criterion)

Let L *be a subalgebra of* $\mathfrak{gl}(V)$*, with* V *being a finite dimensional vector space. If* $\mathrm{Tr}(xy) = 0 \forall x \in [LL], y \in L$ *then* L *is solvable.*

Proof.

Use the previous theorem with $A = [LL], B = L, V$ as given, along with the following associativity: $\mathrm{Tr}([xy]\, z) = \mathrm{Tr}(x\, [yz])$ if x, y, z are endomorphisms of some finite dimensional vector space. ■

Representations

Recap

- Let L be a Lie algebra over field F. A **representation** of L is a homomorphism $\rho : L \to \mathfrak{gl}(V)$ along with some vector space V/F.
- Let L be a Lie algebra over field F. An L**-module** is a vector space V endowed with an operation $L \times V \to V$ (denoted $(\boldsymbol{x}, \vec{v}) \mapsto \boldsymbol{x}\vec{v}$) such that the following hold $\forall a, b \in F, \boldsymbol{x}, \boldsymbol{y} \in L, \vec{u}, \vec{v} \in V$:
  1. $(a\boldsymbol{x} + b\boldsymbol{y})v = a(\boldsymbol{x}\vec{v}) + b(\boldsymbol{y}\vec{v})$
  2. $\boldsymbol{x}(a\vec{u} + b\vec{v}) = a(\boldsymbol{x}\vec{u}) + b(\boldsymbol{x}\vec{v})$
  3. $[\boldsymbol{x}\boldsymbol{y}]\,\vec{v} = \boldsymbol{x}\,(\boldsymbol{y}\vec{v}) - \boldsymbol{y}\,(\boldsymbol{x}\vec{v})$
-
  - For a representation $\rho : L \to \mathfrak{gl}(V)$, V is an L-module via the action $\boldsymbol{x}\vec{v} = \rho(\boldsymbol{x})\vec{v}$.
  - If V is an L-module (action $(\boldsymbol{x}, \vec{v}) \mapsto \boldsymbol{x}\vec{v}$), then $\rho : L \to \mathfrak{gl}(V)$ is a representation by defining $\rho(\boldsymbol{x}) = (\vec{v} \mapsto \boldsymbol{x}\vec{v})$.
- An L-module V is said to be **irreducible** iff V has precisely two L-submodules, namely 0 and V.
- (Schur's lemma) Let V, W be irreducible representations of a Lie algebra L, everything over field F. Let $\psi : V \to W$ be a homomorphism of L-modules. Then we have
  1. $\psi = 0$ or ψ is an isomorphism.
  2. $\left(F = \overline{F}\right)$ For $V = W$, we have $\psi = \lambda \cdot I$ for some $\lambda \in F$.
- (New today) A corollary: Let $\varphi : L \to \mathfrak{gl}(V)$ be irreducible. If $\psi \in \mathfrak{gl}(V)$ is such that $[\psi, \varphi_{\boldsymbol{x}}] = 0 \forall x \in L$ then ψ is a scalar.
  Proof: Write $\varphi_{\boldsymbol{x}}(\vec{v})$ as $\boldsymbol{x} \cdot \vec{v}$. $[\psi, \varphi_{\boldsymbol{x}}] = 0$ simply means that $\psi \in \mathrm{End}_L(V)$.

Symmetric bilinear forms

Consider a symmetric bilinear form $\beta : L \times L \to F$.
Its radical is defined to be $S = \{x \in L : \beta(x, y) = 0 \forall L\} = \{x \in L : \beta(x, L) = 0\}$. If the radical is 0 we say β is **nondegenerate**.

A different way to see nondegeneracy is as follows. Fix a basis $\mathcal{B} = (e_1, \cdots, e_n)$ of L. β is nondegenerate iff the matrix $M = [\beta(e_i, e_j)]_{ij}$ is nonsingular.
This can be seen as follows: $S \neq 0 \iff \beta(x, L) = 0$ for some $x \neq 0 \iff \exists x \in L \setminus \{0\}$ such that $\beta(e_i, x) = 0 \forall i \iff M[x]_{\mathcal{B}} = 0$ for some $x \neq 0 \iff \det M = 0$.

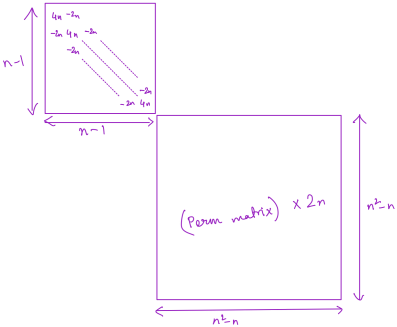
Trace form

Let L be a semisimple Lie algebra along with a faithful $(1-1)$ representation $\varphi : L \to \mathfrak{gl}(V)$, for some finite dimensional vector space V, denoted by $x \mapsto \varphi_x$.
We say that the traceform of a representation is the bilinear map $\beta : L \times L \to F$ given by $(x, y) \mapsto \mathrm{Tr}(\varphi_x \varphi_y)$.
One can check that this is associative, in the following sense: $\beta([x, y], z) = \beta(x, [y, z])$.
Further, faithfulness of the representation implies the nondegeneracy of β.

Example: ad representation

Let's look at the special case when $\varphi = \mathrm{ad}$. In this case, the trace form is called the **killing form** and usually denoted by κ. It is a result due to Cartan that L is semisimple iff κ is nondegenrate.

As a further special example, take $L = \mathfrak{sl}(n, F)$. Choose an ordered basis $x_1, x_2 \cdots$ as $e_{11} - e_{22}, \cdots, e_{n-1,n-1} - e_{nn}, e_{12}, e_{13}, \cdots, e_{1n}, e_{21}, \cdots, e_{2n}, \cdots, e_{n-1,n}$). The matrix of $\kappa(x_i, x_j)$ looks as below. The modulus of the determinant is $n^{n^2} \times 2^{n^2-1}$.



Casimir element of a representation $(F = \overline{F})$

Let γ be any nondegenerate symmetric associative bilinear form on L. Consider the fixed basis $\mathcal{B} = (e_i)_{i=1}^{n}$ of L. And let $f_1, \cdots, f_n$ be a basis dual to $\mathcal{B}$ with respect to γ (this makes sense because γ is nondegenerate). For any representation $\rho : L \to \mathfrak{gl}(V)$ define $c_\rho(\gamma) = \sum_i \rho(e_i)\rho(f_i)$.

Definition

For a faithful representation $\varphi : L \to \mathfrak{gl}(V)$ denoted by $x \mapsto \varphi_x$, with trace form $\beta(x, y) = \mathrm{Tr}(\varphi_x \varphi_y)$, the map $c_\varphi(\beta)$ defined above is called the **Casimir element** of the representation φ with respect to the chosen bases. Since the information of β is encoded in φ, we simply call this c_φ.

Turns out that this is has a very beautiful structure for irreducible representations.

Lemma

For $x \in L$ define a_{ij}, b_{ij} to be such that $[x, e_i] = \sum_j a_{ij} e_j$ and $[x, f_i] = \sum_j b_{ij} f_j$. Then $a_{ij} + b_{ji} = 0 \forall 1 \leq i, j \leq n$.

If V is irreducible, then $c_\rho = \frac{n}{k} I_k$, where $k = \dim V, n = \dim L$. This can be seen by Schur's lemma (in combination with the above lemma). Such c_ρ is independent of the chosen basis.

Finally

We end by stating an important theorem, without proof, in light of the above discussion:

Theorem (Weyl)

Let $\varphi : L \to \mathfrak{gl}(V)$ *be a finite dimensional representation of a semisimple Lie algebra* L. *Then* φ *is completely reducible.*

[0] Look at the writeup for more details on some machinery required for proving this theorem.