Lie algebras

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Preliminaries

Introduction

Definition (Lie algebra)

A vector space L over a field F with an operation $[\cdot,\cdot] : L \times L \to L$ (so $(x,y) \mapsto [x,y]$ or $[xy]$) is called a Lie algebra over F if:

1. $[\cdot,\cdot]$ is bilinear.
2. $[xx] = 0 \forall x \in L$.
3. $[x\,[yz]] + [y\,[zx]] + [z\,[xy]] = 0 \forall x,y,z \in L$.

Some small observations:

1. $0 = [x+y, x+y] = [xx] + [xy] + [yx] + [yy] = [xy] + [yx] \implies [xy] = -[yx]$. In fact, this statement is equivalent to the second statement above, whenever $\operatorname{char} F \neq 2$.
2. $[x\,[yz]] + [y\,[zx]] + [z\,[xy]] = 0$
$\iff [x\,[yz]] = -[y\,[zx]] - [z\,[xy]] = [y, -[zx]] + [[xy]\,z] = [y\,[xz]] + [[xy]\,z]$

Example: $\mathbb{R}^3$

Endow $L = \mathbb{R}^3$ the usual cross product $(x, y, z) \times (a, b, c) = (yc - zb, za - xc, xb - ya)$.
Define $[\boldsymbol{uv}] = \boldsymbol{u} \times \boldsymbol{v}$.
Turns out that $\boldsymbol{u} \times (\boldsymbol{v} \times \boldsymbol{w}) = (\boldsymbol{u} \cdot \boldsymbol{v})\, \boldsymbol{w} - (\boldsymbol{u} \cdot \boldsymbol{w})\, \boldsymbol{v}$ (Exercise!).
This gives $\boldsymbol{u} \times (\boldsymbol{v} \times \boldsymbol{w}) + \boldsymbol{v} \times (\boldsymbol{w} \times \boldsymbol{u}) + \boldsymbol{w} \times (\boldsymbol{u} \times \boldsymbol{v}) = \mathbf{0}$.
It is not hard to see that $[\cdot, \cdot]$ is bilinear and $[\boldsymbol{uu}] = \boldsymbol{u} \times \boldsymbol{u} = \mathbf{0}$.

Example: $\mathfrak{gl}(V)$

Let V be a finite dimensional $(= n)$ F-vector space. Consider $L = \mathrm{End}(V)$, the associative algebra of endomorphisms of V, along with the commutator $[AB] = AB - BA$ for $A, B \in L$. Verify it's a Lie algebra:

1. Bilinearity is clear.
2. $[AA] = AA - AA = 0.$
3. $[A\,[BC]] + [B\,[CA]] + [C\,[AB]] =$ $[A, BC - CB] + [B, CA - AC] + [C, AB - BA] = ABC - ACB - BCA + CBA + BCA - BAC - CAB + ACB + CAB - CBA - ABC + BAC = 0.$

We shall use $\mathfrak{gl}(V)$ to distinguish the Lie algebra from the older associative algebra $\mathrm{End}(V)$. The notation is $\mathfrak{gl}$ because it turns out to be the tangent space of the Lie group GL at the identity.

A basis of $\mathfrak{gl}(n, F) = M_n(F)$ is $\left\{e_{ij} = e_i e_j^t\right\}_{1 \le i,j \le n}$.

Example: $\mathfrak{sl}(V)$

Let V be a finite dimensional $(= n)$ F-vector space. Consider $L = \mathfrak{sl}(V)$ $= \{T \in \mathfrak{gl}(V) : \mathrm{Tr}(T) = 0\}$, along with the commutator $[AB] = AB - BA$ for $A, B \in L$.
To verify it's a Lie algebra, we do exactly what we did in the previous page.
A basis of $\mathfrak{sl}(n, F)$ is $\{e_{i,j} : i \neq j, 1 \leq i, j \leq n\} \bigcup \{e_{i,i} - e_{i+1,i+1} : 1 \leq i \leq n-1\}$.
Here, $e_{i,j} = e_i e_j^t$.

An example to keep in mind is $\mathfrak{sl}(2, F)$. This will come up later. An ordered basis is

$$x = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, y = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

Abstract Lie algebras

Let L be a finite dimensional F-vector space with basis $\{u_i\}_{i=1}^n$. And suppose we know the commutator $[\cdot,\cdot]$ on L (i.e., we know $[u_i, u_j]$). At a more atomic level, if we say that $[u_i, u_j] = \sum_{k=1}^{n} a_{ij}^k u_k$, then we know all the n^3 numbers $\left\{a_{ij}^k : 1 \leq i, j, k \leq n\right\}$. In fact, those a_{ij}^k's, for which $i \geq j$, can be deduced from the others because $[xy] = -[yx]$. You might observe that these are the only numbers which can uniquely determine my commutator (of course, not arbitrary numbers: for example, taking $a_{11}^k = 1$ is absurd).

Proposition

Consider a set of n^3 numbers $\left\{a_{ij}^k \in F : 1 \leq i, j, k \leq n\right\}$ (indexed by i, j, k) satisfying

$$a_{ii}^k = a_{ij}^k + a_{ji}^k = 0 \qquad 1 \leq i, j, k \leq n$$

$$\sum_{k=1}^{n} \left(a_{ij}^k a_{kj}^m + a_{jl}^k a_{ki}^m + a_{li}^k a_{kj}^m\right) = 0 \qquad 1 \leq i, j, m \leq n$$

uniquely determines the commutator of a Lie algebra.

A small classification problem

Let's determine all Lie algebras (upto isomorphism) of dimension ≤ 2.

1. For dimension 1: If we have $L = Fx$ then clearly $[a, b] = 0 \forall a, b \in L$.
2. For dimension 2: Suppose a basis of L is x, y. The commutator of any two vectors is just a multiple of $[x, y]$. Look at $[xy] = \alpha x + \beta y, \because [xx] = [yy] = 0$.
   We either have $\alpha = \beta = 0$, in which case L is just abelian.
   Otherwise, we define $x' = [xy]$ and let $y' \in L$ be independent of x'. This will give us $[x'y'] = \lambda [x, y] = \lambda x', \lambda \neq 0$. Now take $y'' = \lambda^{-1} y'$ to finally get $[x'y''] = x'$.
   So, upto isomorphism, there is **atmost** one non-abelian L. We ensure that atleast one such exists.

2-dimensional non-abelian Lie algebra

Take $F = \mathbb{R}, L = \mathbb{R}^2, [\boldsymbol{u}, \boldsymbol{v}] = \left(\det \begin{bmatrix} \boldsymbol{u} & \boldsymbol{v} \end{bmatrix}\right) \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

Indeed, letting $x = \boldsymbol{e}_1, y = \boldsymbol{e}_2$, we have $[x, y] = \left(\det \begin{bmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_2 \end{bmatrix}\right) \boldsymbol{e}_1 = \boldsymbol{e}_1 = x$.

Subspaces

Morphisms and subalgebras

Let L, L' be F-Lie algebras with commutators, $[\cdot,\cdot]\,, [\cdot,\cdot]'$.

Definition (Homomorphism and isomorphism)

$\varphi : L \to L'$ is said to be a **homomorphism** of Lie algebras if φ is a homomorphism of vector spaces such that $\varphi\left([x, y]\right) = [\varphi(x), \varphi(y)]'$.
Further if φ is an isomorphism of vector spaces, then φ is an **isomorphism** of Lie algebras.

Definition (Subalgebra)

A vector subspace $K \subseteq L$ is said to be a **subalgebra** if $x, y \in K \implies [x, y] \in K$.

Derivations

Note that the Jacobi identity really gives $[x\,[yz]] = [y\,[xz]] + [[xy]\,z]$.
For $a \in L$, write $D_a = [a, \cdot]$. Just for this slide let's write ab instead of $[ab]$.
Then this is what the Jacobi identity gives us: $D_x(yz) = y(D_x z) + (D_x y)z$.
This is just a derivation!

Definition (F-algebra)

An F-vector space $\mathfrak{U}$ is said to be an F-algebra if it comes with a bilinear operation $\mathfrak{U} \times \mathfrak{U} \to \mathfrak{U}$.
(We do not ask for associativity.)

Definition (Derivation)

A derivation δ of an F-algebra $\mathfrak{U}$ is a linear map $\mathfrak{U} \to \mathfrak{U}$ s.t. $\delta(ab) = a\delta(b) + \delta(a)b$.

The collection of all derivations of $\mathfrak{U}$ is denoted by $\mathrm{Der}(\mathfrak{U})$.

Derivations

Let L be an F-Lie algebra.
It is clearly seen that $\mathrm{Der}(L)$ is a *subset* of $\mathfrak{gl}(\mathfrak{V})$. Indeed, it is something more (exercise!):

1. If $\delta_1, \delta_2 \in \mathrm{Der}(L)$, then $[\delta_1, \delta_2] \in \mathrm{Der}(L)$.
2. If $\delta_1, \delta_2 \in \mathrm{Der}(L), a \in F$, then $\delta_1 + a\delta_2 \in \mathrm{Der}(L)$.

$\mathrm{Der}(L)$ is a subalgebra of $\mathrm{End}(L)$.
However, (exercise!) The ordinary product of two derivations need not be a derivation.

We have already see that $D_x \in \mathrm{Der}(L) \forall x \in L$. In literature, D_x is usually written as $\mathrm{ad}\, x$ or ad_x and read as the **adjoint of** x. Since this derivation comes from *inside* the algebra, we call all such ad_x's as **inner derivations**. Others are called **outer derivations**.

An example: ad

Recall $\mathfrak{sl}(2, F)$. Take the ordered basis $x = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, y = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$.

Check that $[xy] = h, [hx] = 2x, [yh] = 2y$. To determine, say, ad_x, it is enough to look at its action on the basis and then use the coefficients to build up the vector w.r.t. the above basis.

For example $\mathrm{ad}_x(x) = (0, 0, 0), \mathrm{ad}_x(h) = (-2, 0, 0), \mathrm{ad}_x(y) = (0, 1, 0)$ in the matrix form. So this will mean that $\mathrm{ad}_x = \begin{bmatrix} 0 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$.

After computing for y, h the final result is

$$\mathrm{ad}_x = \begin{bmatrix} 0 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \qquad \mathrm{ad}_h = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix} \qquad \mathrm{ad}_y = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix}$$

Which subspaces allow quotienting?

A subspace I of a Lie algebra L over F is called an **ideal** of L if $[xy] \in L \forall x \in L, y \in I$. These are the subspaces of our interest which arise as kernels of homoprhisms. We can note that every ideal is a subalgebra, but not conversely.

Examples

1. $0, L$ are ideals of L.
2. The center $Z(L) = \{x \in L : [xz] = 0 \forall z \in L\} = \{x \in L : [xL] = 0\}$. Consider the [1]map $\mathrm{ad} : L \to \mathfrak{gl}(L)$ given by $x \mapsto \mathrm{ad}_x$. The kernel of this map is precisely $\{x \in L : \mathrm{ad}_x = 0\} = \{x \in L : [xL] = 0\} = Z(L)$.
3. The derived algebra $[LL] = \left\{ \sum_{i \in I} [x_i y_i] : I \text{ finite and } x_i, y_i \in L \forall i \in I \right\}$.
   This can be realized as a special case of the fact that if I, J are ideals, so is $[IJ]$ (and $I + J$).

[1] This is very special, an example of a **representation**. From now, we will call this the adjoint representation.

Adjoint representation

Lemma

$\mathrm{ad} : L \to \mathfrak{gl}(L)$ *is a homomorphism of Lie algebras.*

Proof.

It's not hard to see that ad is a VS-homomorphism.
Say $x, y \in L$. Then $\mathrm{ad}_{[xy]}(u) = [[xy]\,u] = -\,[u\,[xy]] = -\,[[ux]\,y] - [x\,[uy]] = [y\,[ux]] - [x\,[uy]] = [x\,[yu]] - [y\,[xu]] = (\mathrm{ad}_x \mathrm{ad}_y - \mathrm{ad}_y \mathrm{ad}_x)\,(u) = [\mathrm{ad}_x, \mathrm{ad}_y]\,(u)$. So $\mathrm{ad}_{[xy]} = [\mathrm{ad}_x, \mathrm{ad}_y]$. ■

Note that we can also treat $(V\ =)L$ as a 'linear space' over $(K\ =)L$: The 'scalar' multiplication is (for $x \in K, v \in V$) $x \cdot v = [xv] = \mathrm{ad}_x(v)$.

Definition (Simple Lie algebras)

A non-abelian Lie algebra L (i.e., $[LL] \neq 0$) is said to be **simple** if it has no nontrivial proper ideals.

Representations

Two languages

Definition (Representation)

Let L be a Lie algebra over field F. A **representation** of L is a vector space V/F along with a homomorphism $\rho : L \to \mathfrak{gl}(V)$.

Definition (Module)

Let L be a Lie algebra over field F. An L**-module** is a vector space V endowed with an operation $L \times V \to V$ (denoted $(\boldsymbol{x}, \vec{v}) \mapsto xv$) such that the following hold $\forall a, b \in F, \boldsymbol{x}, \boldsymbol{y} \in L, \vec{u}, \vec{v} \in V$:

1. $(a\boldsymbol{x} + b\boldsymbol{y})\vec{v} = a(\boldsymbol{x}\vec{v}) + b(\boldsymbol{y}\vec{v})$
2. $\boldsymbol{x}(a\vec{u} + b\vec{v}) = a(\boldsymbol{x}\vec{u}) + b(\boldsymbol{x}\vec{v})$
3. $[\boldsymbol{xy}]\,\vec{v} = \boldsymbol{x}\,(\boldsymbol{y}\vec{v}) - \boldsymbol{y}\,(\boldsymbol{x}\vec{v})$

These are exactly the same

For a representation $\rho : L \to \mathfrak{gl}(V)$, V is an L-module via the action $\boldsymbol{x}\vec{v} = \rho(\boldsymbol{x})\vec{v}$.
If V is an L-module (action $(\boldsymbol{x}, \vec{v}) \mapsto \boldsymbol{x}\vec{v}$), then $\rho : L \to \mathfrak{gl}(V)$ is a representation by defining $\rho(\boldsymbol{x}) = (\vec{v} \mapsto \boldsymbol{x}\vec{v})$.

An important lemma

Definition (Irreducible representation)

An L-module V is said to be **irreducible** iff V has precisely two nontrivial proper L-submodules, namely 0 and V.

Theorem (Schur's lemma)

Let V, W be irreducible representations of a Lie algebra L, everything over field F. Let $\psi : V \to W$ be a homomorphism of L-modules. Then we have

1. $\psi = 0$ *or* ψ *is an isomorphism.*
2. $(F = \overline{F})$ *For* $V = W$, *we have* $\psi = \lambda \cdot I$ *for some* $\lambda \in F$.

Proof.

1. $\ker \psi$ is an L-submodule. So $\ker \psi = 0$ or $\ker \psi = V$.
   $\ker \psi = 0 \implies \psi(V) \cong V$. $\psi(V)$ is a L-submodule of W. It follows that by irreducibility of W that $W \cong \psi(V) \cong V$.
   $\ker \psi \neq 0 \implies \ker \psi = V \implies \psi = 0$.
2. $\psi \in \mathrm{End}(V)$. Let $\lambda \in F = \overline{F}$ be an eigenvalue of ψ. So $\ker(\psi - \lambda \cdot I) \neq 0 \implies \psi = \lambda \cdot I$.

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