Introduction to Lie algebras

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Motivation to Lie groups

We all know $M_n(\mathbb{R})$ and have seen this in many different ways.

1. Vector space: we can add matrices and multiply by complex scalars.
2. Ring: we can multiply square matrices.
3. Geometry: You can think it as literally $\mathbb{R}^{n^2}$, just with the linear 'look' of $\mathbb{R}^{n^2}$ being changed to a tabular form. It's possible to measure distances. Some examples:
   - $\|M\| = \sum_{1\leq i,j\leq n} |M_{ij}|$
   - $\|M\|_{p,q} = \left\{ \sum_{j=1}^{n} \left\{ \sum_{i=n}^{n} |M_{ij}|^p \right\}^{\frac{q}{p}} \right\}^{\frac{1}{q}}$
4. Manifold: umm...

What’s a manifold?

A place where you can do calculus

Elaborate please...

Think of placing 'charts' on a surface, so that there is no roughness or fold-lines on the chart - in other words, we should be able to pick up a small region on the surface and **smoothly** deform it into $\mathbb{R}^2$ in a one-one way. And these charts are 'stitched' smoothly.

Well, for a manifold, you would allow this to happen for $\mathbb{R}^n$ for any $n \geq 1$, not just $\mathbb{R}^2$. Since we know how to do calculus on $\mathbb{R}^n$, and we can easily (and smoothly) transition between charts and $\mathbb{R}^n$, we know how to do calculus in any manifold.

Definition (Submanifolds of $\mathbb{R}^n$)

A subset $M \subseteq \mathbb{R}^n$ is said to be an m-dimensional submanifold of $\mathbb{R}^n$ if $\forall \boldsymbol{x} \in M, \exists W \underset{\text{open}}{\subset} \mathbb{R}^n$ containing $\boldsymbol{x}$ such that $W \cap M$ is diffeomorphic to some $U \underset{\text{open}}{\subset} \mathbb{R}^m$.

The diffeomorphism $\psi : U \to W \cap M$ is called a *parameterization*.

Building manifolds

Lemma

An open subset of a manifold is itself a manifold.

Proof.

Let $M' \underset{\text{open}}{\subset} M$. Fix $x \in M' \subseteq M$. Denote the previous diffeomorphism by $\varphi_x : W_x \cap M \to U_x$. Then the restriction of φ_x to M' does the job. ∎

Building manifolds

A beautiful way to build manifolds is to look at graphs of functions...

Theorem (Implicit function theorem)

Let $m < n, \Omega \underset{\text{open}}{\subset} \mathbb{R}^n, \varphi \in \mathcal{C}^1(\Omega, \mathbb{R}^m)$*. Let* $M_{\boldsymbol{c}} = \varphi^{-1}(c)$ *be non-empty such that* $\mathcal{J}\varphi(\boldsymbol{x})$ *has full rank (namely,* m*)* $\forall \boldsymbol{x} \in M_{\boldsymbol{c}}$*. Then* $M_{\boldsymbol{c}}$ *is an* $(n-m)$*-dimensional submanifold of* $\mathbb{R}^n$*.*

Example: $\varphi(x, y) = y - e^x$

Let $m < n, \varphi \in \mathcal{C}^1(\Omega \subseteq \mathbb{R}^n, \mathbb{R}^m)$. Let $M_{\boldsymbol{c}} = \varphi^{-1}(c)$ be non-empty such that $\mathcal{J}\varphi(\boldsymbol{x})$ has full rank (namely, m) $\forall \boldsymbol{x} \in M_{\boldsymbol{c}}$. Then $M_{\boldsymbol{c}}$ is an $(n-m)$-dimensional submanifold of $\mathbb{R}^n$.

$\varphi \in \mathcal{C}^1(\mathbb{R}^2, \mathbb{R})$. So $n = 2, m = 1$ here. $M_0 = \{(x, e^x) : x \in \mathbb{R}\}$.
$\mathcal{J}\varphi(x, y) = \begin{bmatrix} -e^x & 1 \end{bmatrix}$. This has rank 1 always. The graph of this function is a manifold of dimension $n - m = 1$.

Example: $\varphi(x, y, z) = x^2 + y^2 + z^2$

> Let $m < n, \varphi \in \mathcal{C}^1(\Omega \subseteq \mathbb{R}^n, \mathbb{R}^m)$. Let $M_{\boldsymbol{c}} = \varphi^{-1}(c)$ be non-empty such that $\mathcal{J}\varphi(\boldsymbol{x})$ has full rank (namely, m) $\forall \boldsymbol{x} \in M_{\boldsymbol{c}}$. Then $M_{\boldsymbol{c}}$ is an $(n-m)$-dimensional submanifold of $\mathbb{R}^n$.

$\varphi \in \mathcal{C}^1(\mathbb{R}^3, \mathbb{R})$. So $n = 3, m = 1$ here. $M_1 = \left\{(x, y, z) : x^2 + y^2 + z^2 = 1\right\}$. $\mathcal{J}\varphi(x, y, z) = \begin{bmatrix} 2x & 2y & 2z \end{bmatrix}$. This has rank 1 always. The graph of this function is a manifold of dimension $n - m = 2$. We popularly know this as the $2-$sphere or $\mathbb{S}^2$.

Special mention: $\varphi(\boldsymbol{x}) = 0$

Let $m < n, \varphi \in \mathcal{C}^1(\Omega \subseteq \mathbb{R}^n, \mathbb{R}^m)$. Let $M_{\boldsymbol{c}} = \varphi^{-1}(c)$ be non-empty such that $\mathcal{J}\varphi(\boldsymbol{x})$ has full rank (namely, m) $\forall \boldsymbol{x} \in M_{\boldsymbol{c}}$. Then $M_{\boldsymbol{c}}$ is an $(n-m)$-dimensional sub-manifold of $\mathbb{R}^n$.

$\varphi \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R}^0)$. So $m = 0$ here. $M_0 = \{\boldsymbol{x} \in \mathbb{R}^n : \varphi(\boldsymbol{x}) = 0\} = \mathbb{R}^n$.
$\mathcal{J}\varphi(x, y, z) = \begin{bmatrix} 0 & 0 & \cdots & 0 \end{bmatrix}$. This has rank 0 (full!) always. The graph of this function is a manifold of dimension $n - m = n$.

Back to Lie Groups

Let F be a field.

Lie Group	**Function**	**Set**
$M_n(\mathbb{R})$	-	-
$GL_n(\mathbb{R})$	$\det$	$\det^{-1}(\mathbb{R} \setminus \{0\})$
$SL_n(\mathbb{R})$	$\det$	$\det^{-1}(1)$
Strictly upper Δ matrices	Projection π onto lower Δ matrices	$\pi^{-1}(0)$
$O_n(\mathbb{R})$	$\varphi = \pi \circ (A \mapsto A^t A)$	$\varphi^{-1}(I)$[1]

[1] $(A \mapsto A^t A)^{-1}(I) = \varphi^{-1}(I) : A \in \varphi^{-1}(I) \implies e_i^t A^t A e_j = e_j^t A^t A e_i = 0$. $A^t A$ triangular $\implies A^t A$ diagonal.

Tangent spaces

How to compute tangents?

Let's consider $\varphi(x,y,t) = (x-y, y-t^2)$ along with the level set $M_{(1,0)} = \{(x,y,t) : x-y=1, y=t^2\} = \{(1+t^2, t^2, t) : t \in \mathbb{R}\}$. Call this curve $\boldsymbol{\gamma}$.
We all know how to compute the tangent at, say, $\boldsymbol{p} = (2,1,-1)$.

We first 'solve' for $m=2$ coordinates in terms of the other 'free' $n-m=1$ coordinates: $x = 1+t^2, y = t^2$. Then $\dot{x} = 2t, \dot{y} = 2t, \dot{t} = 1$ so that the required direction of the 'velocity' at the point $(2,1,-1)$ is just $\boldsymbol{v} = (-2,-2,1)$.
But we want the line to be passing through $\boldsymbol{p}$. So we say that out line is just given by $\{\boldsymbol{p} + \boldsymbol{v}s : s \in \mathbb{R}\}$. One thing to note is that $\boldsymbol{\gamma}(-1+s) = \boldsymbol{\gamma}(-1) + \dot{\boldsymbol{\gamma}}(-1)s + \cdots$.

Generalizing derivatives

Definition (Derivative)

Let $\boldsymbol{f} : \Omega \underset{\text{open}}{\subset} \mathbb{R}^m \to \mathbb{R}^n$ be a function and $\boldsymbol{p} \in \Omega$. We say $\boldsymbol{f}$ is differentiable at $\boldsymbol{p}$ if there is a linear map $T : \mathbb{R}^m \to \mathbb{R}^n$ satisfying

$$\lim_{\boldsymbol{h} \to \boldsymbol{0}} \frac{\|\boldsymbol{f}(\boldsymbol{p}+\boldsymbol{h}) - \boldsymbol{f}(\boldsymbol{p}) - T\boldsymbol{h}\|}{\|\boldsymbol{h}\|} = 0$$

We say $T = D\boldsymbol{f}(\boldsymbol{p}) = \boldsymbol{f}'(\boldsymbol{p})$

Smooth maps bring up the question of derivatives. We define directional derivatives as

$$D_{\boldsymbol{v}}(\boldsymbol{f})(\boldsymbol{p}) = \lim_{t \to 0} \frac{\boldsymbol{f}(\boldsymbol{p}+t\boldsymbol{v}) - \boldsymbol{f}(\boldsymbol{v})}{t}$$

It turns out that for sufficiently 'nice' functions, $D_{\boldsymbol{v}}(\boldsymbol{f})(\boldsymbol{p}) = \langle \boldsymbol{v}, \boldsymbol{f}'(\boldsymbol{p}) \rangle$.

Consider the map $\det : M_n(\mathbb{R}) \to \mathbb{R}$ given by $A \mapsto \det A$. It can be proven that this is a 'smooth' map.

Smooth maps. Hmm... Derivatives!

Exercise

Show that $D_H \det(A) = \mathrm{Tr}(\mathrm{adj}(A) \cdot H)$

Turns out that the above fact can be used to prove that the 'tangent space' of $SL_n(\mathbb{R})$ at $I \in SL_n(\mathbb{R})$ is just $\{M \in M_n(\mathbb{R}) : \mathrm{Tr}(M) = 0\}$. We call this $\mathfrak{sl}_n(\mathbb{R})$.

Derivations

Recall the multiplication rule for derivatives: $D(uv) = (Du)v + u(Dv)$. We also know that the derivative is a linear. We shall try to generalize this notion.

Let $\boldsymbol{p} \in \mathbb{R}^n$. Consider the set of all pairs (U, f) where $U \subseteq \mathbb{R}^n$ is an open neighbourhood of $\boldsymbol{p}$, and $f \in \mathcal{C}^\infty(U, \mathbb{R})$. We define an equivalence relation $\overset{\boldsymbol{p}}{\sim}$ by: $(U, f) \overset{\boldsymbol{p}}{\sim} (V, g) \iff \exists W \underset{\text{open}}{\subset} U \cap V, a \in W$ such that $f|_W = g|_W$.

If $h \in C^\infty(A, \mathbb{R})$ with $\boldsymbol{a} \in A \underset{\text{open}}{\subset} \mathbb{R}^n$ then the germ of h is the equivalence class of h under $\overset{\boldsymbol{a}}{\sim}$. The set of all germs ($\mathcal{C}^\infty$ maps) at a particular point $\boldsymbol{a} \in \mathbb{R}^n$ (i.e., set of all such equivalence classes) is denoted by $\mathcal{C}^\infty_{\boldsymbol{a}}$.

This is really an $\mathbb{R}$-algebra.

Definition (Derivation)

An $\mathbb{R}$-linear map $D : \mathcal{C}^\infty_{\boldsymbol{a}} \to \mathbb{R}$ is said to be a derivation if $D(uv) = (Du) \cdot v(\boldsymbol{a}) + u(\boldsymbol{a}) \cdot (Dv)$.

Derivations

Definition (Derivation)

An $\mathbb{R}$-linear map $D : \mathcal{C}^{\infty}_{\boldsymbol{a}} \to \mathbb{R}$ is said to be a derivation if $D(uv) = (Du) \cdot v(\boldsymbol{a}) + u(\boldsymbol{a}) \cdot (Dv)$.

Example

$D_{i,\boldsymbol{a}} = \left.\dfrac{\partial}{\partial x_i}\right|_{\boldsymbol{a}}$ is a derivation on $\mathcal{C}^{\infty}_{\boldsymbol{a}}$.

Why am I saying all this?

For a smooth curve $\boldsymbol{\gamma}$ passing through $\boldsymbol{a}$, define

$$D_{\boldsymbol{\gamma},\boldsymbol{a}}(f_{\boldsymbol{a}}) = \frac{d}{dt}(f \circ \boldsymbol{\gamma})(t)\bigg|_{\boldsymbol{a}} \left(= \left\langle \nabla(f)(\boldsymbol{a}), \dot{\boldsymbol{\gamma}(0)} \right\rangle\right)$$

This is injective!