PLC : Assignment 4

Set: March 31, 2021

Due: April 10, 2021, 23.55

General instructions:

1. Submit your solutions as a PDF or plain text file on Moodle. The file should be named `<un>-4.txt` (or `<un>-4.pdf`, if you are submitting a PDF), where `un` is your username. For example, I would submit a file named `spsuresh-4.txt`.

2. If you are submitting a text file, use the Haskell notation `\x -> M` for the $\lambda x.M$ when writing expressions in the lambda calculus. (For $\lambda x\,y\,z.M$, write `\x y z -> M`.) Use the notation `:=` for syntactic equality, and `-->` and `=` for (many-step) beta-reduction and beta-equality, respectively. Use `M[x <- N]` for substitution. Also use `<m>` for the Church encoding of m, and `x^{y}` for x^y.

3. Properly parenthesize your lambda expressions and use spacing to keep it readable.

4. Recall that the Church encoding of n, denoted $[n]$, is the expression $\lambda f\,x.f^n x$, where $f^0 x$ is defined to be just x, and $f^{i+1}x := f(f^i x)$.

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1. Let $\mathbf{exp} := \lambda pq.pq$. Prove that for all $m \geq 0$ and $n \geq 1$,

$$\mathbf{exp}[n][m] \longrightarrow [m^n].$$

   Hint: Prove the following claims in order:

   (a) For $k, l \geq 0$, $(\lambda z.x^k z)^l y \longrightarrow x^{kl} y$.

   (b) For $m \geq 0, n \geq 1$, $(\lambda g\,y.g^m y)^n x \longrightarrow (\lambda y.x^{m^n} y)$.

   (c) From the above, show that for all $m \geq 0$ and $n \geq 1$, $[n][m] \longrightarrow [m^n]$.

   (d) Conclude that $\mathbf{exp}[n][m] \longrightarrow [m^n]$.

2. What is the normal form of $[5](\mathbf{exp}[2])[2]$? What is the size (number of applications) of the normal form?

3. (a) Find a lambda-expression F such that for all M, $FM = F$.

   (b) Find a lambda-expression F such that for all M, $FM = MF$.

4. Prove that every expression in normal form M is of the form $\lambda x_1 \cdots \lambda x_n.yM_1M_2\cdots M_l$, where y is a variable and $M_1, \ldots, M_l$ are themselves in normal form.

5. Find an encoding for the predecessor function in lambda calculus. The predecessor function is given by: $pred(0) = 0$ and $pred(n+1) = n$.

6. Find an encoding for the *Pow* function in lambda calculus. It is given by:

$$Pow(m,n) = \begin{cases} \textbf{true} & \text{if } \exists k : m^k = n \\ \textbf{false} & \text{otherwise} \end{cases}$$

7. **Combinatory logic** is a related system which uses function applications, but no function abstraction. A CL term is either a variable or the constant **S** or the constant **K** or the application of term t to term t', denoted $(t\,t')$. The behaviour of **S** and **K** is given by the following rules:

$$\begin{array}{rcl} \mathbf{S}xyz & \longrightarrow & xz(yz) \\ \mathbf{K}xy & \longrightarrow & x \end{array}$$

A **combinator** is a variable-free CL term. Find combinators (expressions involving **S**, **K**, and previously defined combinators, but no variables) with the following behaviour:

(a) **I** such that $\mathbf{I}x \longrightarrow x$

(b) **T** such that $\mathbf{T}xy \longrightarrow yx$

(c) **B** such that $\mathbf{B}xyz \longrightarrow x(yz)$

(d) **M** such that $\mathbf{M}x \longrightarrow xx$

8. For a variable x and any CL term M, define $[x]M$ as follows:

$$\begin{array}{rcl} [x]x & = & I \\ [x]y & = & \mathbf{K}y \quad (y \neq x) \\ [x](MN) & = & \mathbf{S}([x]M)([x]N) \end{array}$$

Prove that for any CL term M, x does not occur in $[x]M$ and $([x]M)N \longrightarrow M[x \leftarrow N]$.

From the above $[x]M$ behaves just like $\lambda x.M$. So we have the following translation of lambda terms to CL terms.

$$\begin{array}{rcl} CL(x) & := & x \\ CL(MN) & := & CL(M)CL(N) \\ CL(\lambda x.M) & := & [x](CL(M)) \end{array}$$

Find combinatory logic terms corresponding to the lambda terms $\lambda f.(\lambda x.f(xx))(\lambda x.f(xx))$ and $\lambda f.(\lambda x.f(fx))$.