PLC : Assignment 3

Set: April 19, 2021

Due: May 1, 2021, 23.55

General instructions:

1. Submit your solutions as a PDF or plain text file on Moodle. The file should be named `<un>-3.txt` (or `<un>-3.pdf`, if you are submitting a PDF), where `un` is your username. For example, I would submit a file named `spsuresh-3.txt`.

2. If you are submitting a text file, use the Haskell notation `\x -> M` for the $\lambda x.M$ when writing expressions in the lambda calculus. (For $\lambda x\,y\,z.M$, write `\x y z -> M`.) Use the notation `:=` for syntactic equality, and `-->` and `=` for (many-step) beta-reduction and beta-equality, respectively. Use `M[x <- N]` for substitution. Also use `<m>` for the Church encoding of m, and `x^{y}` for x^y.

3. Properly parenthesize your lambda expressions and use spacing to keep it readable.

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1. Recall the definition of *parallel reduction*. It is the relation $\Rightarrow$ over λ-terms defined by the rules given in Table 1.

$$\frac{}{M \Rightarrow M}$$

$$\frac{}{(\lambda x.M)N \Rightarrow M[x := N]}$$

$$\frac{M \Rightarrow M'}{\lambda x.M \Rightarrow \lambda x.M'}$$

$$\frac{M \Rightarrow M' \quad N \Rightarrow N'}{MN \Rightarrow M'N'}$$

$$\frac{M \Rightarrow M' \quad N \Rightarrow N'}{(\lambda x.M)N \Rightarrow M'[x := N']}$$

Table 1: Rules for $\Rightarrow$

Define M^* as follows:

$$
\begin{array}{rcl}
x^* & = & x \\
(\lambda x.M)^* & = & \lambda x.M^* \\
(MN)^* & = & M^* N^* \quad (M \text{ not of the form } \lambda x.P) \\
((\lambda x.P)N)^* & = & P^*[x := N^*]
\end{array}
$$

Prove the following:

(a) If $M \to_\beta N$, then $M \Rightarrow N$.

(b) If $M \Rightarrow N$, then $M \xrightarrow{*}_\beta N$.

(c) $M \xrightarrow{*}_\beta N$ if and only if $M \overset{*}{\Rightarrow} N$.

(d) If $M \Rightarrow N$, then $N \Rightarrow M^*$.

(e) If $M \Rightarrow P$ and $M \Rightarrow Q$, then there exists N such that $P \Rightarrow N$ and $Q \Rightarrow N$.

2. Are the following expressions typable? If so, what are the most general types? If not, explain why.

   (a) $\lambda f\, g\, x.f(g\, x)$

   (b) $\lambda x\, y.y x$

   (c) $\lambda f\, g\, x.g(f\, x)$

3. Recall the following standard encodings: $f^0 x = x$, $f^{n+1} x = f(f^n x)$, $[n] = (\lambda f\, x.f^n x)$, **true** $= (\lambda x\, y.x)$, **false** $= (\lambda x\, y.y)$, **pair** $= (\lambda x\, y\, w.w x y)$, **fst** $= (\lambda p.p$ **true**$)$, **snd** $= (\lambda p.p$ **false**$)$, **ite** $= (\lambda b\, x\, y.b\, x\, y)$, and **iszero** $= (\lambda x.(x(\lambda z.$**false**$))$ **true**$)$.

   Derive the most general types of each of the above expressions. If you feel that any of them is untypable, give a justification.