Programming Language Concepts: Lecture 23

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Curry typing: typability

Definition (Typability problem)

Given a term M of the untyped λ-calculus, check whether it can be given a time (assuming some types for free variables)

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Theorem

Typability and type inference for simply typed λ-calculus is solvable in polynomial time

Type inference

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- Type variables in E_M and τ_M
  - **main** – p_x, for $x \in FV(M)$ (if $x \neq y$, $p_x \neq p_y$)
  - **auxiliary** – not of the form p_x

Type inference …

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  - Define $\tau_M = p$
  - $E_M = E_P \cup E_Q \cup \{\tau_P = \tau_Q \rightarrow p\}$

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  - Choose a fresh auxilliary type variable p
  - $E_M = E_P[p_x := p]$

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- M is $\lambda x \cdot P$
  - Choose a fresh auxilliary type variable p
  - $E_M = E_P[p_x := p]$
  - Define $\tau_M = p \rightarrow \tau_P[p_x := p]$

Type inference: example

- $M = \lambda x\, y\, z \cdot N$ where $N = x(y\, z)$

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- $M = \lambda x y z \cdot N$ where $N = x(yz)$
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  - $\tau_{yz} = p, E_{yz} = \{p_y = p_z \rightarrow p\}$

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  - $\tau_{yz} = p, E_{yz} = \{p_y = p_z \rightarrow p\}$
  - $\tau_N = q, E_N = \{p_y = p_z \rightarrow p, p_x = p \rightarrow q\}$

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  - $\tau_N = q, E_N = \{p_y = p_z \rightarrow p, p_x = p \rightarrow q\}$
  - $\tau_{\lambda z \cdot N} = r \rightarrow q, E_{\lambda z \cdot N} = \{p_y = r \rightarrow p, p_x = p \rightarrow q\}$

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  - $\tau_{\lambda yz \cdot N} = s \to r \to q, E_{\lambda yz \cdot N} = \{s = r \to p, p_x = p \to q\}$

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  - $\tau_M = t \rightarrow s \rightarrow r \rightarrow q, E_M = \{s = r \rightarrow p, t = p \rightarrow q\}$

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  - $\tau_x = p_x, \tau_y = p_y, \tau_z = p_z$
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  - $\tau_M = t \to s \to r \to q, E_M = \{s = r \to p, t = p \to q\}$
- A minimal solution for E_M is $S = \{s := r \to p, t := p \to q\}$

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- A minimal solution for E_M is $S = \{s := r \rightarrow p, t := p \rightarrow q\}$
- The principal type of M: $S(\tau_M) = (p \rightarrow q) \rightarrow (r \rightarrow p) \rightarrow (r \rightarrow q)$

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- A minimal solution for E_M is $S = \{p := q \rightarrow q, r := q \rightarrow q\}$
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Type inference: richer typing

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Type inference: richer typing

- $M = (\lambda x \cdot x)(\lambda x \cdot x)$ has principal type $q \rightarrow q$
- Let M_1 be $\mathbf{let}\ y = \lambda x \cdot x\ \mathbf{in}\ y\,y$

Type inference: richer typing

- $M = (\lambda x \cdot x)(\lambda x \cdot x)$ has principal type $q \to q$
- Let M_1 be $\textbf{let } y = \lambda x \cdot x \textbf{ in } y\,y$
- Let M_2 be $(\lambda y \cdot y\,y)(\lambda x \cdot x)$

Type inference: richer typing

- $M = (\lambda x \cdot x)(\lambda x \cdot x)$ has principal type $q \rightarrow q$
- Let M_1 be $\mathbf{let}\ y = \lambda x \cdot x\ \mathbf{in}\ y\,y$
- Let M_2 be $(\lambda y \cdot y\,y)(\lambda x \cdot x)$
- M_1 is equivalent to M and has the same principal type

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- Let M_1 be $\mathbf{let}\ y = \lambda x \cdot x\ \mathbf{in}\ y\,y$
- Let M_2 be $(\lambda y \cdot y\,y)(\lambda x \cdot x)$
- M_1 is equivalent to M and has the same principal type
- M_2 is not typable, because $\lambda y \cdot y\,y$ is not typable

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- Let M_1 be $\textbf{let } y = \lambda x \cdot x \textbf{ in } y\,y$
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- M_2 is not typable, because $\lambda y \cdot y\,y$ is not typable
- M_1 is typable despite the occurrence of $y\,y$

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- M_1 is equivalent to M and has the same principal type
- M_2 is not typable, because $\lambda y \cdot y\,y$ is not typable
- M_1 is typable despite the occurrence of $y\,y$
    - **variable defined by local definition** – treated differently

Type inference: non-recursive local definitions

- M is **let** $\{x_1 = M_1 \ ; \ \cdots \ ; \ x_n = M_n\}$ **in** N

Type inference: non-recursive local definitions

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  - Set τ_{x_i} to be τ_{M_i}, and $E_{x_i} = \emptyset$

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  - Find principal types of $M_1, \ldots, M_n$
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  - Find the type of N as usual, using the above definition for τ_{x_i}'s

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  - Each occurrence of x_i in N will get a different instance of τ_{x_i} as its type

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  - Each occurrence of x_i in N will get a different instance of τ_{x_i} as its type
  - All auxilliary type variables in τ_{x_i} will be renamed to fresh variables

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  - Main type variables of the form p_x will not be renamed

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  - Each occurrence of x_i in N will get a different instance of τ_{x_i} as its type
  - All auxilliary type variables in τ_{x_i} will be renamed to fresh variables
  - Main type variables of the form p_x will not be renamed
- x_i's are used in N as **polymorphic expressions**

Type inference: non-recursive local definitions

- Consider **let** $y = \lambda x \cdot x$ **in** $y\, y$

Type inference: non-recursive local definitions

- Consider $\textbf{let}\ y = \lambda x \cdot x\ \textbf{in}\ y\,y$
  - $\tau_y = p \rightarrow p$, for some auxilliary type variable p

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- Consider $\textbf{let}\ y = \lambda x \cdot x\ \textbf{in}\ y\,y$
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  - $y\,y$ is of the form $P\,Q$

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  - $\tau_y = p \rightarrow p$, for some auxilliary type variable p
  - $y\,y$ is of the form $P\,Q$
  - We rename auxilliary type variables in τ_Q

Type inference: non-recursive local definitions

- Consider **let** $y = \lambda x \cdot x$ **in** $y\, y$
  - $\tau_y = p \rightarrow p$, for some auxilliary type variable p
  - $y\, y$ is of the form $P\, Q$
  - We rename auxilliary type variables in τ_Q
  - Type of the first y is $p \rightarrow p$

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  - $\tau_y = p \rightarrow p$, for some auxilliary type variable p
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  - Type of the first y is $p \rightarrow p$
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  - Now solve as usual!

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  - Type of the first y is $p \rightarrow p$
  - Type of second y is $q \rightarrow q$
  - Now solve as usual!
- Let M be **let** $\{f = \lambda y \cdot x\ ;\ g = \lambda x \cdot x\}$ **in** $g\,f$

Type inference: non-recursive local definitions

- Consider **let** $y = \lambda x \cdot x$ **in** $y\, y$
  - $\tau_y = p \rightarrow p$, for some auxilliary type variable p
  - $y\, y$ is of the form $P\, Q$
  - We rename auxilliary type variables in τ_Q
  - Type of the first y is $p \rightarrow p$
  - Type of second y is $q \rightarrow q$
  - Now solve as usual!
- Let M be **let** $\{f = \lambda y \cdot x\ ;\ g = \lambda x \cdot x\}$ **in** $g\, f$
  - $\tau_f = p \rightarrow p_x$

Type inference: non-recursive local definitions

- Consider **let** $y = \lambda x \cdot x$ **in** $y\,y$
  - $\tau_y = p \to p$, for some auxilliary type variable p
  - $y\,y$ is of the form $P\,Q$
  - We rename auxilliary type variables in τ_Q
  - Type of the first y is $p \to p$
  - Type of second y is $q \to q$
  - Now solve as usual!
- Let M be **let** $\{f = \lambda y \cdot x\ ;\ g = \lambda x \cdot x\}$ **in** $g\,f$
  - $\tau_f = p \to p_x$
  - $\tau_g = q \to q$

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  - $\tau_f = p \rightarrow p_x$
  - $\tau_g = q \rightarrow q$
  - $\tau_{g\,f} = r$, $E_{g\,f} = \{q \rightarrow q = (p \rightarrow p_x) \rightarrow r\}$

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- Consider **let** $y = \lambda x \cdot x$ **in** $y\,y$
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  - Type of second y is $q \rightarrow q$
  - Now solve as usual!
- Let M be **let** $\{f = \lambda y \cdot x\ ;\ g = \lambda x \cdot x\}$ **in** $g\,f$
  - $\tau_f = p \rightarrow p_x$
  - $\tau_g = q \rightarrow q$
  - $\tau_{g\,f} = r$, $E_{g\,f} = \{q \rightarrow q = (p \rightarrow p_x) \rightarrow r\}$
  - $S = q := p \rightarrow p_x, r := p \rightarrow p_x$ is a solution

Type inference: non-recursive local definitions

- Consider **let** $y = \lambda x \cdot x$ **in** $y\,y$
  - $\tau_y = p \rightarrow p$, for some auxilliary type variable p
  - $y\,y$ is of the form $P\,Q$
  - We rename auxilliary type variables in τ_Q
  - Type of the first y is $p \rightarrow p$
  - Type of second y is $q \rightarrow q$
  - Now solve as usual!
- Let M be **let** $\{f = \lambda y \cdot x\ ;\ g = \lambda x \cdot x\}$ **in** $g\,f$
  - $\tau_f = p \rightarrow p_x$
  - $\tau_g = q \rightarrow q$
  - $\tau_{g\,f} = r$, $E_{g\,f} = \{q \rightarrow q = (p \rightarrow p_x) \rightarrow r\}$
  - $S = q := p \rightarrow p_x, r := p \rightarrow p_x$ is a solution
  - Principal type of M is $p \rightarrow p_x$

Type inference: recursive local definitions

- M is **letrec** $\{x_1 = M_1 \; ; \; \cdots \; ; \; x_n = M_n\}$ **in** N

Type inference: recursive local definitions

- M is **letrec** $\{x_1 = M_1 \; ; \; \cdots \; ; \; x_n = M_n\}$ **in** N
  - Build each τ_{M_i} and E_{M_i}, treating each x_j as a free variable with type p_{x_j}

Type inference: recursive local definitions

- M is **letrec** $\{x_1 = M_1 \;;\; \cdots \;;\; x_n = M_n\}$ **in** N
  - Build each τ_{M_i} and E_{M_i}, treating each x_j as a free variable with type p_{x_j}
  - Ensure that the auxilliary variables in the E_{M_i}'s and τ_{M_i}'s are all distinct

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    - $\sigma_i := \tau_{M_i}[p_{x_1} := q_1, \ldots, p_{x_n} := q_n]$

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    - Solve $E := E_1 \cup \cdots \cup E_n \cup \{q_1 = \sigma_1, \ldots, q_n = \sigma_n\}$

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  - Let S be the most general solution to E

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    - Let S be the most general solution to E
    - $\tau_{x_i} := S(q_i)$

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  - Solve $E := E_1 \cup \cdots \cup E_n \cup \{q_1 = \sigma_1, \ldots, q_n = \sigma_n\}$
  - Let S be the most general solution to E
  - $\tau_{x_i} := S(q_i)$
  - Find the type of N as usual, using the above τ_{x_i}'s

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- Consider **letrec** $x = \lambda f \cdot f(x\,f)$ **in** x

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  - Let M_1 be $\lambda f \cdot f(x f)$
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  - Let M_1 be $\lambda f \cdot f(x f)$
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  - Now $\sigma_1 = p \rightarrow r$ and $E_1 = \{q_1 = p \rightarrow q, p = q \rightarrow r\}$

Type inference: recursive local definitions

- Consider **letrec** $x = \lambda f \cdot f(x f)$ **in** x
  - Let M_1 be $\lambda f \cdot f(x f)$
  - $\tau_{M_1} = p \to r$ and $E_{M_1} = \{p_x = p \to q, p = q \to r\}$
  - Now $\sigma_1 = p \to r$ and $E_1 = \{q_1 = p \to q, p = q \to r\}$
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  - Now $\sigma_1 = p \rightarrow r$ and $E_1 = \{q_1 = p \rightarrow q, p = q \rightarrow r\}$
  - $E = \{q_1 = p \rightarrow q, p = q \rightarrow r, q_1 = p \rightarrow r\}$
  - Solution for E is $\{q := r, p := r \rightarrow r, q_1 := (r \rightarrow r) \rightarrow r\}$

Type inference: recursive local definitions

- Consider **letrec** $x = \lambda f \cdot f(xf)$ **in** x
  - Let M_1 be $\lambda f \cdot f(xf)$
  - $\tau_{M_1} = p \rightarrow r$ and $E_{M_1} = \{p_x = p \rightarrow q, p = q \rightarrow r\}$
  - Now $\sigma_1 = p \rightarrow r$ and $E_1 = \{q_1 = p \rightarrow q, p = q \rightarrow r\}$
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  - Solution for E is $\{q := r, p := r \rightarrow r, q_1 := (r \rightarrow r) \rightarrow r\}$
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  - Let M_1 be $\lambda f \cdot f(x f)$
  - $\tau_{M_1} = p \rightarrow r$ and $E_{M_1} = \{p_x = p \rightarrow q, p = q \rightarrow r\}$
  - Now $\sigma_1 = p \rightarrow r$ and $E_1 = \{q_1 = p \rightarrow q, p = q \rightarrow r\}$
  - $E = \{q_1 = p \rightarrow q, p = q \rightarrow r, q_1 = p \rightarrow r\}$
  - Solution for E is $\{q := r, p := r \rightarrow r, q_1 := (r \rightarrow r) \rightarrow r\}$
  - $\tau_x := (r \rightarrow r) \rightarrow r$
- The type of **letrec** $x = \lambda f \cdot f(x f)$ **in** x is thus $(r \rightarrow r) \rightarrow r$

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- constant terms and constant functions
    - $0 : Int$, $True : Bool$
    - $cons : a \to [a] \to [a]$
    - $if : Bool \to a \to a \to a$

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- **Polymorphic** – each occurrence of *if*, *cons*, etc. is given a fresh instance of the types

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  - $cons : a \to [a] \to [a]$
  - $if : Bool \to a \to a \to a$
- **Polymorphic** – each occurrence of *if*, *cons*, etc. is given a fresh instance of the types
- The type inference algorithm is more or less unchanged!