Programming Language Concepts: Lecture 21

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- Thus $\lambda x.(M\,x)$ behaves just like M
- New reduction rule η (when $x \notin FV(M)$)

$$\lambda x.(M\,x) \longrightarrow_{\eta} M$$

One step reduction

- Define a one step reduction inductively (where $x \in \{\beta, \eta, \ldots\}$)

$$\frac{M \longrightarrow_x M'}{M \longrightarrow M'}$$

$$\frac{M \longrightarrow M'}{MN \longrightarrow M'N} \qquad \frac{N \longrightarrow N'}{MN \longrightarrow MN'} \qquad \frac{M \longrightarrow M'}{\lambda x.M \longrightarrow \lambda x.M'}$$

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  - There is a sequence $M = M_0, M_1, \ldots, M_k = N$ such that for each $i < k$: either $M_i \longrightarrow M_{i+1}$ or $M_{i+1} \longrightarrow M_i$

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- $\Omega = (\lambda x.xx)(\lambda x.xx) \longrightarrow_\beta (\lambda x.xx)(\lambda x.xx) = \Omega$

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Does every term reduce to normal form?

- Consider the terms $\omega = \lambda x.xx$ and $\Omega = \omega\omega$
- $\Omega = (\lambda x.xx)(\lambda x.xx) \longrightarrow_\beta (\lambda x.xx)(\lambda x.xx) = \Omega$
  - Reduction never terminates

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- Consider the term $[\mathit{false}]\,\Omega = (\lambda yz.z)((\lambda x.xx)(\lambda x.xx))$
- **Outermost reduction**

$$(\lambda yz.z)((\lambda x.xx)(\lambda x.xx)) \longrightarrow_{\beta} \lambda z.z$$

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$$(\lambda yz.z)((\lambda x.xx)(\lambda x.xx)) \longrightarrow_\beta (\lambda yz.z)((\lambda x.xx)(\lambda x.xx))$$

- Choice of reduction strategy may determine whether a normal form can be reached, but can more than one normal form be reached?

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- **Yes!** We can do a breadth-first search of the reduction graph, and we are guaranteed to find a normal form eventually
- We could also reduce the term following the strategy of **leftmost outermost reduction**
- If a term has a normal form, leftmost outermost reduction will find it!

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- We have seen how to encode recursive functions in the λ-calculus
- We cannot in general determine if the computation of $f(n)$ terminates, given f and n
- But computing $f(n)$ is equivalent to asking if $[f][n]$ has a normal form
- So checking whether a given term has a normal form is **undecidable**

Church-Rosser theorem

Theorem (Church-Rosser)

If $M \longleftrightarrow N$ *there is a term* P *such that* $M \stackrel{*}{\longrightarrow} P$ *and* $N \stackrel{*}{\longrightarrow} P$

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  - Suppose a term M_0 reduces to two normal forms M and N

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  - Then $M \longleftrightarrow N$
  - Thus there is a P such that $M \xrightarrow{*} P$ and $N \xrightarrow{*} P$ (by Church-Rosser)
  - But since M and N are already in normal form, $M = P = N$ (upto renaming of bound variables)

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- **Recall**: $M \longleftrightarrow N$ iff there is a sequence $M = M_0, M_1, \ldots, M_k = N$ such that for all $i < k$: either $M_i \longrightarrow N$ or $N \longrightarrow M$

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  - If $M_{i+1} \longrightarrow M_i$, take $P_{i+1} = P_i$
  - If $M_i \longrightarrow M_{i+1}$, use the **Diamond property** to arrive at the desired P_{i+1}

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- We can talk of the Diamond property for any relation R
- R has the Diamond property if

$$(\forall a, b, c)[(aRb \wedge aRc) \Rightarrow (\exists d)(bRd \wedge cRd)]$$

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The proof is by induction on length of R-chains

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Unfortunately, $\longrightarrow$ does not have the Diamond property!

- Recall that $\omega = \lambda x.xx$ and $\mathbf{I} = \lambda x.x$

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- $\omega\mathbf{I} \longrightarrow \mathbf{II}$ but it takes **two** steps to go from $(\mathbf{II})(\mathbf{II})$ to $\mathbf{II}$!

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Solution: Define a new "parallel reduction" $\Longrightarrow$ as follows

$$M \Longrightarrow M \qquad \frac{M \Longrightarrow M'}{\lambda x.M \Longrightarrow \lambda x.M'}$$

$$\frac{M \Longrightarrow M' \quad N \Longrightarrow N'}{MN \Longrightarrow M'N'} \qquad \frac{M \Longrightarrow M' \quad N \Longrightarrow N'}{(\lambda x.M)N \Longrightarrow M'[x := N']}$$

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  - if $M \Longrightarrow N$ then $M \xrightarrow{*}_\beta N$
  - Hence $M \overset{*}{\Longrightarrow} N$ iff $M \xrightarrow{*}_\beta N$

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- Whenever $M \Longrightarrow N$, $N \Longrightarrow M^*$

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