Programming Language Concepts: Lecture 14

S P Suresh

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λ-calculus

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  - How are outputs computed from inputs?

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- Assume a countably infinite set *Var* of variables

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where $x \in Var$ and $M, N \in \Lambda$.

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- Can also apply functions to non-meaningful data, but the result has no significance

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  - Cannot do anything with terms like xx or $(y(\lambda x.x))(\lambda y.y)$

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  - Warning: Possible for a variable to be both in $FV(M)$ and $BV(M)$

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  - There is a sequence $M_0, M_1, \ldots, M_k$ such that

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- In λ-calculus, we encode n by the number of times we apply a function (**successor**) to an element (**zero**)

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  - ...
- $[n]\, g\, y = (\lambda f\, x. f(\cdots(f\, x)\cdots))\, g\, y \xrightarrow{*}_{\beta} g(\cdots(g\, y)\cdots) = g^n y$