Maximally Entangled States

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Abstract

In this article, we study entangled quantum states and their measure(s), not very rigorously though! Our primary focus is on showing that the Bell state $(|00\rangle + |11\rangle)/\sqrt{2}$ is a maximally entangled state. We also discuss about maximally entangled states of more than two qubits. We show how we may try to address the question of finding subsystem states such that their product is maximally close to some other state, under the "mutual information" measure of entanglement.

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We assume the reader's basic familiarity with quantum computing and quantum mechanics, especially the notation. (See [2] and [3]). So we begin this article with a brief introduction to entangled quantum states and entanglement measures. Then, we study a characterization of maximally entangled states, which shows that $(|00\rangle + |11\rangle)\sqrt{2}$ is indeed a maximally entangled state. Furthermore, we try to find " how close can we get to an entangled state by multiplying states ".

1 Quantum Entanglement

Quantum entanglement is a physical phenomenon that occurs when pairs or groups of particles are generated or interact in ways such that the quantum state of each particle cannot be described independently instead, a quantum state must be described for the system as a whole.

Measurements of physical properties such as position, momentum, spin, polarization, etc., performed on entangled particles are found to be appropriately correlated. For example, if a pair of particles are generated in such a way that their total spin is known to be zero, and one particle is found to have clockwise spin on a certain axis, then the spin of the other particle, measured on the same axis, will be found to be counterclockwise, as to be expected due to their entanglement.

Now let us speak mathematically. An *entangled system* is defined to be one whose quantum state cannot be factored as a product of states of its local constituents, that is, they are not individual particles but are an inseparable whole. If entangled, one constituent cannot be fully described without considering the other(s). Note that the state of a composite system is always expressible as a sum, or superposition, of products of states of local constituents; it is entangled if this sum necessarily has more than one term.

(See [4]).

A quantum state is defined to be entangled if it cannot be written as the product of states of its component systems, i.e., a state $|\psi\rangle$ is said to be entangled if $|\psi\rangle \neq |a\rangle \otimes |b\rangle$ for any two subsystem states $|a\rangle$ and $|b\rangle$. (Actually, this is called *global entanglement*[1]. (See [1]).)

A state that is not entangled is called a *separable state*. (See [7]).

2 Entanglement Measures

An important class of quantum operations will be important to us: *Local Operations with Classical Communication (LOCC)*. In this, a multipartite quantum system is distributed to various parties, and they are restricted to act locally on their respective subsystems by performing measurements and more general quantum operations. However in order to enhance their measurement strategies, the parties are free to communicate any classical data, which includes the sharing of randomness and previous measurement results. Quantum operations implemented in such a manner are known as LOCC. We shall not go into the details of LOCC, which can be found in [6].

[1] *Globally entangled state* refers to a state which admits no two subsystems into which it can be decomposed as a product.

Intuitively, *entanglement measures* measure the degree of entanglement of a quantum system.

Formally, an entanglement measure is a non-negative real-valued function of the quantum state, which does not increase under LOCC (this is called *monotonicity*, and we demand this because entanglement cannot be created by LOCC (see [5])), and is zero for separable states. (See [10].)

In fact, we can have a notion of one quantum state being "more entangled" than another. (The following subsections discuss this). Hence, we can ask whether there exists a *maximally entangled state*, i.e., a state that is "more entangled" than any other state. In this article, our main focus shall be on this.

Linden and Popescu ([8]) discuss that two states which can be transformed one into another by local operations (unitary transformations) are equivalent as far as their non-local properties are concerned. Consequently, a state that is local-unitarily equivalent to a maximally entangled state is itself a maximally entangled state. (See [9]).

There are quite a few notions of ordering quantum states based on the notion of one state being "more entangled" than another. We shall discuss one of them, which leads to the conclusion that the Bell state $(|00\rangle + |11\rangle)\sqrt{2}$ is a maximally entangled state in two-qubit systems. (Ref. [9] discusses some more characterizations of maximally entangled states, which are thereby compared with the characterization we present in this article).

3 Ordering Based on Mutual Information

We consider n qubit states, with n being an integer greater than or equal to 2. Let a_j denote the results of simultaneous measurements of all the qubits in the computational basis. A natural criterion for ordering states is to use the notion that a more entangled state should have greater *mutual information* among the n random variables a_j's than a less entangled state.

The *mutual information* among the n random variables a_j's is defined as

$$I(\{a_j\}) \text{ (say)} := \left\{ \sum_{j=1}^{n} H(a_j) \right\} - H(a_1, a_2, \ldots, a_n)\,, \tag{3.1}$$

where

$$H(x) := -\mathbb{E}(\log(p(x))) = -\left\{\sum_x p(x)\log(p(x))\right\},$$

is the *entropy* function of a random variable x with probability distribution p.

And similarly we have the *joint entropy* function of n random variables $x_1, x_2, \ldots, x_n$ with joint probability distribution p, as

$$\begin{aligned} H(x_1, x_2, \ldots, x_n) :&= -\mathbb{E}(\log(p(x_1, x_2, \ldots, x_n))) \\ &= -\left\{\sum_{x_1, x_2, \cdots, x_n} p(x_1, x_2, \ldots, x_n)\log(p(x_1, x_2, \ldots, x_n))\right\}. \end{aligned}$$

So, (3.1) gives us

$$I(\{a_j\}) = \mathbb{E}(\log(p(a_1, a_2, \ldots, a_n))) - \sum_{j=1}^{n} \mathbb{E}\left(\log\left(p\left(a_j\right)\right)\right) . \tag{3.2}$$

We further have the *conditional entropy* function of $(n-1)$ random variables $x_2, \ldots, x_n$ given the value of another random variable x_1, with conditional probability distribution p, as

$$\begin{aligned} H(x_2, \ldots, x_n|x_1) :&= -\mathbb{E}(\log(p(x_2, \ldots, x_n|x_1))) \\ &= -\left\{\sum_{x_2, \cdots, x_n} p(x_2, \ldots, x_n|x_1)\log(p(x_2, \ldots, x_n|x_1))\right\}. \end{aligned}$$

Also, by the chain rule,

$$H(a_1, a_2, \ldots, a_n) = H(a_1) + H(a_2, \ldots, a_n|a_1).$$

(See [11] for a proof for the $n = 2$ case).

So, (3.1) becomes

$$I(\{a_j\}) \text{ (say)} := \left\{\sum_{j=2}^{n} H(a_j)\right\} - H(a_2, \ldots, a_n|a_1). \tag{3.3}$$

Remarks

This characterization of entanglement seems reasonable, because:

- *The mutual information is zero if the state is separable*: Because then the measurement results are independent random variables, and so, from (3.2), we have

$$\begin{aligned}
I(\{a_j\}) &= \mathbb{E}(\log(p(a_1, a_2, \ldots, a_n))) - \sum_{j=1}^{n} \mathbb{E}\left(\log\left(p\left(a_j\right)\right)\right) \\
&= \mathbb{E}(\log(p(a_1)p(a_2)\cdots p(a_n))) - \sum_{j=1}^{n} \mathbb{E}\left(\log\left(p\left(a_j\right)\right)\right) \\
&= \mathbb{E}(\log(p(a_1)) + \log(p(a_2)) + \cdots + \log(p(a_n))) \\
&\qquad - \sum_{j=1}^{n} \mathbb{E}\left(\log\left(p\left(a_j\right)\right)\right) \\
&= \mathbb{E}(\log(p(a_1))) + \mathbb{E}(\log(p(a_2))) + \cdots + \mathbb{E}(\log(p(a_n))) \\
&\qquad - \sum_{j=1}^{n} \mathbb{E}\left(\log\left(p\left(a_j\right)\right)\right) \\
&= 0.
\end{aligned}$$

- *The mutual information is non-negative real-valued*: Can be proved using Jensen's inequality and the concavity of the log function.

(See [11], [12] and [13]).

3.1 A Maximally Entangled State

A maximally entangled state should be such that the mutual information of the n random variables a_j's is maximum.

Here we consider only symmetric states, i.e., we assume that all the n qubits are equivalent.

By the symmetry among the a_j's, we get, from (3.3),

$$I(\{a_j\}) = (n-1)H(a_1) - H(a_2, \ldots, a_n | a_1).$$

So, the mutual information $I(\{a_j\})$ is maximal if and only if $H(a_1)$ is maximal and also $H(a_2, \ldots, a_n | a_1)$ is minimal. Now, $H(a_1)$ is maximal if both possible outcomes are equally probable, and $H(a_2, \ldots, a_n|a_1)$ is minimal if the outcome a_1 fully determines all the results $a_2, \ldots, a_n$.

For $n = 2$, these conditions are satisfied by the Bell state $|\psi_2^+\rangle = \dfrac{|00\rangle + |11\rangle}{\sqrt{2}}$.

So, for two qubits, $|\psi_2^+\rangle$ is a maximally entangled state. In fact, any state that is local-unitarily equivalent to $|\psi_2^+\rangle$ is a maximally entangled state, even if it is not symmetric. (See [8] and [9]).

For $n \geq 3$, the GHZ state

$$\frac{|0\rangle^{\otimes n} + |1\rangle^{\otimes n}}{\sqrt{2}}$$

satisfies the above conditions, so it, and any state that is local-unitarily equivalent to it (see [8]), are maximally entangled states. (See [9]).

3.2 *Distance* between states ... How close can we get?

We use our mutual information measure as a measure of the "distance" between quantum states.

3.2.1 Two Qubits

Let $|\psi\rangle = a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle$ be a given two-qubit quantum state.

Then we have

$H\left(a_1\right)$ for the state $|\psi\rangle$

$$= H_1^{|\psi\rangle} \text{ (say)} \;=\; -\left(|a|^2 + |b|^2\right) \log\left(|a|^2 + |b|^2\right) - \left(|c|^2 + |d|^2\right) \log\left(|c|^2 + |d|^2\right).$$

And,

$H\left(a_2\right)$ for the state $|\psi\rangle$

$$= H_2^{|\psi\rangle} \text{ (say)} \;=\; -\left(|a|^2 + |c|^2\right) \log\left(|a|^2 + |c|^2\right) - \left(|b|^2 + |d|^2\right) \log\left(|b|^2 + |d|^2\right).$$

Also,

$$
\begin{aligned}
& H\left(a_2|a_1\right) \text{ for the state } |\psi\rangle \\
& = H_{2|1}^{|\psi\rangle} \text{ (say)} \\
& = -\frac{|a|}{\sqrt{|a|^2+|b|^2}}\log\left(\frac{|a|}{\sqrt{|a|^2+|b|^2}}\right)-\frac{|b|}{\sqrt{|a|^2+|b|^2}}\log\left(\frac{|b|}{\sqrt{|a|^2+|b|^2}}\right) \\
& -\frac{|c|}{\sqrt{|c|^2+|d|^2}}\log\left(\frac{|c|}{\sqrt{|c|^2+|d|^2}}\right)-\frac{|d|}{\sqrt{|c|^2+|d|^2}}\log\left(\frac{|d|}{\sqrt{|c|^2+|d|^2}}\right).
\end{aligned}
$$

Now, from (3.1) it follows that

$$I(\{a_j\}) \text{ for the state } |\psi\rangle = I^{|\psi\rangle} \text{ (say) } = H_2^{|\psi\rangle} - H_{2|1}^{|\psi\rangle}.$$

Now, suppose we want to get as *close*[2] as possible to $|\psi\rangle$, by multiplying (taking the tensor product of) two single-qubit states.

So let $|\phi_1\rangle = \alpha|0\rangle + \beta|1\rangle$ and $|\phi_2\rangle = \gamma|0\rangle + \delta|1\rangle$ be two single-qubit states. We analyze their product $|\phi\rangle$ (say) $= |\phi_1\rangle \otimes |\phi_2\rangle = (\alpha|0\rangle + \beta|1\rangle) \otimes (\gamma|0\rangle + \delta|1\rangle) = \alpha\delta|00\rangle + \alpha\gamma|01\rangle + \beta\delta|1\rangle + \beta\gamma|11\rangle$.

We have

$$
\begin{aligned}
H\left(a_1\right) \text{ for the state } |\phi\rangle = H_1^{|\phi\rangle} & \text{ (say)} \\
= & -|\alpha|^2\left(|\gamma|^2+|\delta|^2\right)\log\left(|\alpha|^2\left(|\gamma|^2+|\delta|^2\right)\right) \\
& -|\beta|^2\left(|\gamma|^2+|\delta|^2\right)\log\left(|\beta|^2\left(|\gamma|^2+|\delta|^2\right)\right).
\end{aligned}
$$

And,

$$
\begin{aligned}
H\left(a_2\right) \text{ for the state } |\phi\rangle = H_2^{|\phi\rangle} & \text{ (say)} \\
= & -|\delta|^2\left(|\alpha|^2+|\beta|^2\right)\log\left(|\delta|^2\left(|\alpha|^2+|\beta|^2\right)\right) \\
& -|\gamma|^2\left(|\alpha|^2+|\beta|^2\right)\log\left(|\gamma|^2\left(|\alpha|^2+|\beta|^2\right)\right).
\end{aligned}
$$

[2]As per our definition, the lesser the difference between the mutual informations of two states, *closer* they are.

Also,

$$\begin{aligned}
&H\left(a_2|a_1\right) \text{ for the state } |\phi\rangle \\
&= H_{2|1}^{|\phi\rangle} \text{ (say)} \\
&= -\frac{|\alpha\delta|}{\sqrt{|\alpha|^2\left(|\gamma|^2+|\delta|^2\right)}} \log\left(\frac{|\alpha\delta|}{\sqrt{|\alpha|^2\left(|\gamma|^2+|\delta|^2\right)}}\right) \\
&- \frac{|\alpha\gamma|}{\sqrt{|\alpha|^2\left(|\gamma|^2+|\delta|^2\right)}} \log\left(\frac{|\alpha\gamma|}{\sqrt{|\alpha|^2\left(|\gamma|^2+|\delta|^2\right)}}\right) \\
&- \frac{|\beta\delta|}{\sqrt{|\beta|^2\left(|\gamma|^2+|\delta|^2\right)}} \log\left(\frac{|\beta\delta|}{\sqrt{|\beta|^2\left(|\gamma|^2+|\delta|^2\right)}}\right) \\
&- \frac{|\beta\gamma|}{\sqrt{|\beta|^2\left(|\gamma|^2+|\delta|^2\right)}} \log\left(\frac{|\beta\gamma|}{\sqrt{|\beta|^2\left(|\gamma|^2+|\delta|^2\right)}}\right).
\end{aligned}$$

Now, by (3.1), we have

$$I(\{a_j\}) \text{ for the state } |\phi\rangle = I^{|\phi\rangle} \text{ (say) } = H_2^{|\phi\rangle} - H_{2|1}^{|\phi\rangle}.$$

So, our goal is to find $(\alpha, \beta, \gamma, \delta) \in \mathbb{C}^4$ such that

$$f_{|\psi\rangle}(|\alpha|, |\beta|, |\gamma|, |\delta|) := |I^{|\phi\rangle} - I^{|\psi\rangle}|$$

is minimal, where $|\phi\rangle = (\alpha|0\rangle + \beta|1\rangle) \otimes (\gamma|0\rangle + \delta|1\rangle)$. However, we have the constraints that $|\alpha|^2 + |\beta|^2 = 1$ and $|\gamma|^2 + |\delta|^2 = 1$.

We can try to achieve our goal using Lagrange multipliers, by treating $f_{|\psi\rangle}$ as a function from $[0,\infty)^4 \subseteq \mathbb{R}^4$ to $[0,\infty) \subseteq \mathbb{R}$.

3.2.2 Two or More Qubits — A Generalization

Let

$$|\psi\rangle = \sum_{j\in\{0,1\}^n} \mu_j |j\rangle$$

be a given n-qubit (n is an integer ≥ 2) quantum state.

Let $I^{|\psi\rangle}$ be its mutual information content.

Now, we try to get as *close* as possible to $|\psi\rangle$, by taking products of m states, where $2 \leq m \leq n$ is an integer.

We consider all ways of writing n as a sum of positive integers, with the order of the terms in the summation being important (e.g., for $n = 5$, we consider $2 + 3 = 5$ and $3 + 2 = 5$ both, not just one of them).

Suppose $n = t_1 + t_2 + \cdots + t_m$ is such a partition. (Here $2 \leq m \leq n$ is an integer, and $t_1, t_2, \ldots, t_m$ are positive integers). Then, we analyze the tensor product $|\phi\rangle$ (say) $= |\phi_1\rangle \otimes |\phi_2\rangle \otimes \cdots \otimes |\phi_m\rangle$, where $|\phi_j\rangle$ is a general t_j-qubit state, for all $2 \leq j \leq m$. We write variable coefficients for each of these m states (hence we have called them "general states"), expand out their product (using the distributive laws), and find an expression for the mutual information content, say $I^{|\phi\rangle}$, of the state $|\phi\rangle$, in terms of the coefficients of the individual $|\phi_j\rangle$ states ($2 \leq j \leq m$). And then, our goal is to minimize

$$f := |I^{|\phi\rangle} - I^{|\psi\rangle}| \, .$$

In fact, our goal is to minimize f over all possible (ordered) partitions of n into m positive integers ($2 \leq m \leq n$).

We can try to do this by Lagrange multiplier method, by treating f as a non-negative real-valued function of the absolute values of the coefficients of the $|\phi_j\rangle$ states ($2 \leq j \leq m$). However, it definitely becomes harder as the number of these coefficients increases.

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