de Sitter, extremal surfaces and time entanglement

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- de Sitter space, dS/CFT and de Sitter entropy;
- dS extremal surfaces: future-past, no-boundary, time-entanglement/pseudo-entropy, analytic cont'ns, Lewkowycz-Maldacena/Wavefn
- QM: pseudo-entropy, time-entanglement (EE, timelike separations)
- dS surfaces, subregion duality; Slow-roll inflation, no-bndy surfaces

2210.12963, 2310.00320, KN; 2303.01307, KN, Saini (also 1501.03019, 1711.01107, 2002.11950, . . .)

2409.14208, Goswami, KN, Yadav, ongoing w/ Kanhu Nanda, Somnath Porey, Gopal Yadav

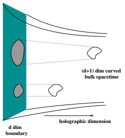
[collaborations: Kaberi Goswami, Dileep Jatkar, Kedar Kolekar, Hitesh Saini, Gopal Yadav]

Holography and asymptotics

25+ yrs since AdS/CFT '97 Maldacena; '98 Gubser,Klebanov,Polyakov; Witten.

Holography: quantum gravity in $\mathcal{M} \leftrightarrow$ dual without gravity on $\partial\mathcal{M}$ ('t Hooft, Susskind).

(Witten@Strings'98, '01) Gauge/gravity duality and asymptotics —



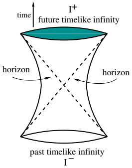
$\Lambda < 0$: AdS $\rightarrow$ asymptotics at spatial infinity.
Dual: unitary Lorentzian CFT, includes time.

$\Lambda = 0$: flat space $\rightarrow$ null infinity.
S-matrix, symmetries, Carroll/celestial ...

$\Lambda > 0$: de Sitter space
Boundary at future/past timelike infinity $\mathcal{I}^{\pm}$.
Dual $\rightarrow$ Euclidean CFT ... Time emergent.

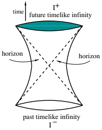
[note: gravity dual of ordinary Euclidean CFT $\longrightarrow$ Euclidean AdS]

Might regard de Sitter as toy model for cosmology.



de Sitter space, entropy, dS/CFT

de Sitter entropy = area of cosmological horizon (Gibbons,Hawking).
Some sort of (holographic) entanglement? Ryu-Takayanagi generalizations?



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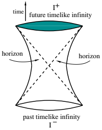

dS/CFT: ('01 Strominger; Witten) future timelike infinity $\mathcal{I}^+$ as a natural dS boundary. Euclidean non-unitary CFT dual. Time emergent.

[note: gravity dual of ordinary Eucl CFT $\rightarrow$ Eucl AdS]

(Maldacena '02) AdS, analytic continuation $\rightarrow$ $\boxed{Z_{CFT} = \Psi_{dS}}$ Hartle-Hawking Wavefunction of the Universe

Bulk expectation values $\langle\varphi_k\varphi_{k'}\rangle \sim \int D\varphi\ \varphi_k\varphi_{k'}|\Psi|^2 \rightarrow$ dual $\equiv$ two CFT copies.

Dual energy-momentum $\langle TT\rangle$ 2-pt fn $\rightarrow$ $\mathcal{C}$ negative/imaginary, ghost-CFT.

Anninos,Hartman,Strominger: higher-spin dS_4 dual to $Sp(N)$ ghost CFT_3, ...

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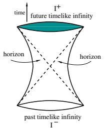

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$\underline{dS_4}$, Poincare: $ds^2 = \frac{R_{dS}^2}{\tau^2}(-d\tau^2 + d\vec{x}^2)$

$r \rightarrow -i\tau, \quad R_{AdS} \rightarrow -iR_{dS}$.

$\Psi_{dS}[\varphi] \sim e^{iS_{cl}[\varphi]} \sim e^{-\int_k R_{dS}^2 k^3 \varphi_{-k}^0 \varphi_k^0 + \ldots}$

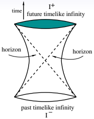
→ dual CFT: $\langle O_k O_{k'} \rangle \sim \frac{\delta^2 Z}{\delta \varphi_k^0 \delta \varphi_{k'}^0} \rightarrow \mathcal{C}_3 \sim -\frac{R_{dS}^2}{G_4}$.

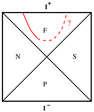
Global/static dS from global AdS: other analytic continuations.

de Sitter space, extremal surfaces

dS, future boundary, extremal surfaces

[KN '15-'23] A natural generalization of Ryu-Takayanagi to de Sitter $\equiv$ bulk analog of setting up entanglement entropy in dual Eucl CFT $\rightarrow$ define some boundary Eucl time slice $\rightarrow$ codim-2 RT/HRT surfaces anchored at I^+, dipping into holographic (time) direction.





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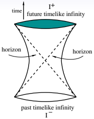

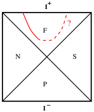

Extremization: surfaces anchored at future boundary $I^+ \rightarrow$ No real $I^+ \rightarrow I^+$ turning point (Lorentzian dS).
Surfaces do not return to I^+. *Interior boundary condns? Time contours?*

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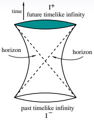

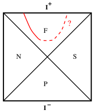

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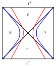
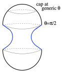
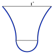


Future-past surfaces: stretch from I^+ to I^-, entirely Lorentzian dS.
Entirely timelike so area has overall $-i$ (relative to AdS spacelike surfaces).



No-boundary surfaces: Hartle-Hawking no-boundary dS. Complex area.
(top timelike part of f-p surface joined with real surface with turn-around in bottom hemisphere)



$AdS \rightarrow$ global/static dS surfaces: analytic continuation $\equiv$ *space* $\leftrightarrow$ *time* rotation.

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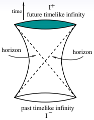

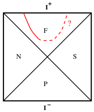

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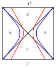

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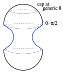

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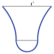

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Areas: new object $\rightarrow$ Pseudo entropy or "Time entanglement".
(EE-like structures, timelike separations) Doi,Harper,Mollabashi,Takayanagi,Taki; KN, '22

[LM replica on $\Psi_{dS} = Z_{CFT}$]

$\mathcal{T}^A_{F|I} = \mathrm{Tr}_B \left(\frac{|F\rangle\langle I|}{\mathrm{Tr}(|F\rangle\langle I|)} \right)$ [entropy of reduced transition matrix (Nakata,Takayanagi,Taki,Tamaoka,Wei,'20)]

“Entanglement” in ghost theories: ghost-spins

KN’16; Jatkar,KN’17; Jatkar,Kolekar,KN’18

- Replica arguments (Calabrese, Cardy) generalized to $c = -2$ ghost CFTs:
  twist operator 2-pt fn $\rightarrow$ $Re(S) < 0$. Subtleties. [$|\downarrow\rangle = |0\rangle$; $\langle -Q|T(z)|0\rangle = 0$]

- $\boxed{\begin{array}{c}\text{“Ghost-spin”} \rightarrow \text{2-state spin variable with indefinite norm.} \\ \langle\uparrow|\downarrow\rangle = \langle\downarrow|\uparrow\rangle = 1, \qquad \langle\uparrow|\uparrow\rangle = \langle\downarrow|\downarrow\rangle = 0\end{array}}$

  [ordinary spin: $\langle\uparrow|\uparrow\rangle = 1 = \langle\downarrow|\downarrow\rangle$]

  $|\pm\rangle \equiv \frac{1}{\sqrt{2}}(|\uparrow\rangle \pm |\downarrow\rangle)$; $\langle\pm|\pm\rangle = \gamma_{\pm\pm} = \pm 1$, $\langle+|-\rangle = \langle-|+\rangle = 0$

  Infinite ghost-spin chains, $\langle nn\rangle$-intns $\rightarrow$ continuum limit $\rightarrow$ bc-ghost CFT.

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- $\rho = |\psi\rangle\langle\psi| \xrightarrow{\text{tr}_B} \text{RDM}_A$, remaining ghost-spin $\rightarrow$ von Neumann entropy.

  2 g.s., $\sum \psi^{ij}|ij\rangle$: $\quad \langle\psi|\psi\rangle = \gamma_{ik}\gamma_{jl}\psi^{ij}\psi^{kl*} = |\psi^{++}|^2 + |\psi^{--}|^2 - |\psi^{+-}|^2 - |\psi^{-+}|^2 = \pm 1$

  RDM: $(\rho_A)^{ik} = \gamma_{jl}\psi^{ij}\psi^{kl*}$; $\qquad$ EE: $S_A = -\gamma_{ij}(\rho_A \log \rho_A)^{ij}$ $\qquad$ [new patterns]

  - $\boxed{-ve \text{ norm} \leftrightarrow \text{Im}(S_A)}$
  - $+ve$ norm $|\psi\rangle \not\Rightarrow +ve$ RDM, EE.

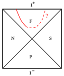
- 2 copies: entangle identical ghost-spins from each copy $\rightarrow$ $+ve$ norm, RDM, EE

  $|\psi\rangle = \psi^{++}|+\rangle|+\rangle + \psi^{--}|-\rangle|-\rangle \Rightarrow$ Positivity $\longrightarrow$ correlated ghost-spins

  Also true for 2 copies of general ghost-spin ensembles: $|\psi\rangle = \sum_{|\sigma_n\rangle} \psi^{\sigma_n,\sigma_n}|\sigma_n\rangle|\sigma_n\rangle \rightarrow$ Positivity.

dS future-past extremal surfaces

$\underline{dS \text{ (Poincare)}}$: $ds^2_{d+1} = \frac{R^2_{dS}}{\tau^2}(-d\tau^2 + dw^2 + dx_i^2)$



Bndry Eucl time w=$const$ strip @ I^+ $\rightarrow$ codim-2.

$S_{dS} \propto \int \frac{d\tau}{\tau^{d-1}} \sqrt{1-(\partial_\tau x)^2} \quad \longrightarrow \quad (\partial_\tau x)^2 = \frac{B^2 \tau^{2d-2}}{1+B^2 \tau^{2d-2}} \quad [B^2 > 0]$

Sign diff. from AdS $\Rightarrow$ $\boxed{\text{No } \underline{\text{real}}\ I^+ \rightarrow I^+ \text{ “turning point”.}}$

KN '15; Sato '15

$B^2 < 0$: Analytic cont'n $r \rightarrow -i\tau$, $R \rightarrow -iR_{dS}$ from AdS RT $\rightarrow$ Complex areas.

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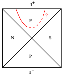

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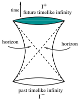
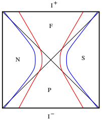


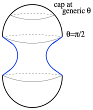
dS (static) $\quad ds^2 = -\frac{dr^2}{r^2/l^2-1} + (\frac{r^2}{l^2}-1)dt^2 + r^2 d\Omega^2_{d-1}$. KN '17



Bndry Eucl time slice, any S^{d-1} equatorial plane (OR $t=const$ slice).

$\boxed{\text{Future-past (timelike) surfaces connecting } I^+ \text{ to } I^-}$

Hartman-Maldacena (AdS bh) rotated. $\quad$ [area div $-i\frac{\pi l^2}{G_4}\frac{R_c}{l}$]



dS (global): $\quad ds^2_{d+1} = -d\tau^2 + l^2\cosh^2\frac{\tau}{l}\,d\Omega^2_d$.

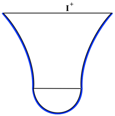
$\boxed{ds^2_{global}\Big|_{\theta_d=const} \ \equiv \ ds^2_{static}\Big|_{t=const} \quad [r=l\cosh\frac{\tau}{l}]}$

f-p surfaces connecting θ-caps at $I^{\pm}$, wrapping S^{d-2}. IR area $-i\frac{\pi l^2}{G_4}\frac{R_c}{l}$ $[dS_4]$

de Sitter no-boundary surfaces

Hartle-Hawking no-boundary proposal: Lorentzian dS evolves in time from a no-boundary Euclidean initial configuration. Cut global dS in middle ($\tau = 0$ slice), join top half with hemisphere in bottom half given by Euclidean continuation

$$ds^2 = l^2 d\tau_E^2 + l^2 \cos^2 \tau_E \, d\Omega_d^2 \,; \quad \tau = il\tau_E \,, \quad 0 \le \tau_E \le \tfrac{\pi}{2} \,.$$



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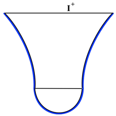

Some S^d equatorial plane (i.e. S^{d-1}) $\rightarrow$ timelike future-past surface at $\theta = \frac{\pi}{2}$ [IR limit]. Hits $\tau = 0$ mid-slice "vertically": join smoothly at $\tau = 0$ with surface going around bottom hemisphere. Smooth joining $\Leftrightarrow$ consistency of F-P with Hartle-Hawking.

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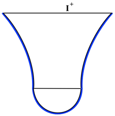

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IR bottom surface: $ds^2 = l^2 d\tau_E^2 + l^2 \cos^2\tau_E (d\theta^2 + \sin^2\theta \, d\Omega_{d-2}^2)\Big|_{\theta=\frac{\pi}{2}}$

$$\boxed{\text{Area} = \frac{l^{d-1}}{4G_{d+1}} V_{S^{d-2}} \int_0^{\pi/2} d\tau_E \; (\cos\tau_E)^{d-2} \; = \; \frac{1}{2} \, \frac{l^{d-1} \, V_{S^{d-1}}}{4G_{d+1}}}$$

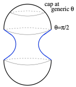
Precisely half dS entropy: emerges differently from area of cosmological horizon (static patch observers). [One hemisphere direction here is Euclidean continuation of time in future universe]

$$S_{dS_4} = -i\frac{\pi l^2}{2G_4}\frac{R_c}{l} + \frac{\pi l^2}{2G_4} . \qquad \text{Similarities with Wavefunction } \Psi_{dS} = e^{iS_{cl}} .$$

- Half dS entropy also emerges for no-bndry dS static $t = const$ surfaces.

[dS_3: Hikida,Nishioka,Takayanagi,Taki]; [KN'22]

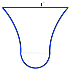
dS extremal surfaces $\rightarrow$ "Time-Entanglement"



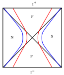
IR limit

$$\boxed{\begin{array}{l} S_{fp} = S_{nb} - S_{nb}^*, \qquad \mathrm{Re}(S_{nb}) = \frac{1}{2} \cdot dS \,\text{entropy}. \\ \text{Suggests} \quad S_{nb} \equiv \Psi_{dS}, \quad S_{fp} \equiv \Psi_{dS}, \Psi_{dS}^* \ (I^+ \cup I^-). \end{array}}$$

$$[dS_4] \quad S_{fp} = -i\, \frac{\pi l^2}{G_4} \frac{R_c}{l}; \qquad S_{nb} = -i\, \frac{\pi l^2}{2G_4} \frac{R_c}{l} + \frac{\pi l^2}{2G_4}$$



dS extremal surfaces at I^+ & areas $\equiv$ space-time rotations from AdS.
e.g. dS future-past surfaces $\equiv$ rotated Hartman-Maldacena surfaces (AdS BH).



dS extremal surfaces: no $I^+ \rightarrow I^+$ returns $\rightarrow$ timelike components necessarily.

Note: timelike geodesic length has overall $-i$ relative to spacelike geodesic length.
We call this timelike length as "time" rather than "$-i$·space".
These extremal surface areas with timelike components $\equiv$ new object,
"time entanglement" or pseudo-entropy. [entanglement-like structures, timelike separations]

Time-entanglement/Pseudo-entropy in QM $[\tau^A_{F|I} = \mathrm{Tr}_B \big(\frac{|F\rangle\langle I|}{\mathrm{Tr}(|F\rangle\langle I|)}\big)]$ — later.

dS_3, 2-dim CFT; timelike intervals

Future-past surfaces, entirely Lorentzian global dS_3. [some S^2 equatorial plane]

Area $S_{fp} = -i\frac{l}{G_3}\log\frac{l}{\varepsilon} \equiv 2(\frac{c}{3}\log\frac{l}{\varepsilon})$ with $c_{dS_3} = -i\frac{3l_{dS}}{2G}$ [2 copies].

No-boundary dS_3 surface: area $S_{nb} = -i\frac{l}{2G_3}\log\frac{l}{\varepsilon} + \frac{\pi l}{4G_3} \equiv \frac{c}{3}\log\frac{l}{\varepsilon} + \frac{c}{6}(i\pi)$.

$\mathrm{Im}(S_{nb}) \equiv \frac{c}{3}\log\frac{l}{\varepsilon}$ for half-size interval (IR) in Eucl CFT on circle.

$\mathrm{Re}(S_{nb})$ from deep interior Euclideanization $\leftrightarrow$ "interior regularity" in Eucl CFT dual (no time; bulk time emergent).

$\left[S_{nb}\right.$ is overall $-i$ times EE for timelike interval in AdS_3 with $c = \frac{3l_{AdS}}{2G_3}$.$\left.\right]$

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Ordinary unitary 2-dim CFTs: EE is $S = \frac{c}{6}\log\frac{\Delta^2}{\epsilon^2} = \frac{c}{6}\log\frac{-(\Delta t)^2+(\Delta x)^2}{\epsilon^2}$.
Ordinary spacelike intervals $\Delta^2 > 0 \ \rightarrow\ S = \frac{c}{3}\log\frac{\Delta x}{\epsilon}$.
Entirely timelike interval, width Δt so $\Delta^2 < 0$: $S = \frac{c}{3}\log\frac{\Delta t}{\epsilon} + \frac{c}{6}(i\pi)$.

[Quantum extremal surfaces, dS Poincare: bulk $c > 0$ matter entropy timelike separations Chen,Gorbenko,Maldacena, '20, also Goswami,KN,Saini,'21]. qes

[Usual replica formulation in Euclidean CFT: pick interval $\Delta x \equiv [u, v]$ on Eucl time slice $\tau_E = const$ $\rightarrow n$ replicas glued at interval endpts $\rightarrow \mathrm{Tr}\rho_A^n \rightarrow$ twist op 2-pt fn $\rightarrow S_A = -\lim_{n\rightarrow 1}\partial_n \mathrm{Tr}\rho_A^n$. Timelike interval $\Delta t \equiv [u_t, v_t]$ on Eucl time slice $x = const$: continue to Lorentzian time rotating (u_t, v_t), to $(-iu_t,\ -iv_t)$ so $\Delta^2 = -(v_t - u_t)^2 = -(\Delta t)^2$.]

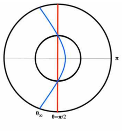
dS no-boundary surfaces, analytic cont'n

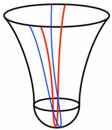
<u>Analytic cont'n $\equiv$ space$\leftrightarrow$time rotation</u>: AdS RT surface from $r \to \infty$ (boundary) to $r = 0$ (and back) $\longrightarrow$ IR dS RT/HRT surface from $r \to \infty$ (future boundary) to $r = l$ (Lorentzian dS) going around Eucl hemisphere ($r = l$ to $r = 0$) (& back to $\mathcal{I}^+$).

dS RT/HRT surfaces on bndry Eucl time slice $[r = l \cosh \frac{\tau}{l}]$

$[r > l]$ $\quad ds^2 = -\frac{dr^2}{\frac{r^2}{l^2} - 1} + r^2 d\Omega_{d-1}^2 \xrightarrow{l \to iL} ds^2 = \frac{dr^2}{1 + \frac{r^2}{L^2}} + r^2 d\Omega_{d-1}^2$

$[r < l]$ $\quad ds^2 = \frac{dr^2}{1 - \frac{r^2}{l^2}} + r^2 d\Omega_{d-1}^2 \xrightarrow{l \to iL} ds^2 = \frac{dr^2}{1 + \frac{r^2}{L^2}} + r^2 d\Omega_{d-1}^2$





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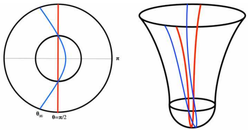


IR: max subregion

$$\frac{V_{S^{d-2}}}{4G_{d+1}} \int_0^{R_c} \frac{r^{d-2}dr}{\sqrt{1 + \frac{r^2}{L^2}}} \xrightarrow{L \to -il} \frac{V_{S^{d-2}}}{4G_{d+1}} \left(\int_0^l \frac{r^{d-2}dr}{\sqrt{1 - \frac{r^2}{l^2}}} + \int_l^{R_c} r^{d-2} \sqrt{\frac{dr^2}{-(\frac{r^2}{l^2} - 1)}} \right)$$

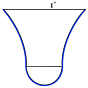
(blue: generic θ_∞)

$$= \frac{1}{2} \frac{l^{d-1} V_{S^{d-1}}}{4G_{d+1}} - i\# \frac{l^{d-1}}{4G_{d+1}} \frac{R_c^{d-2}}{l^{d-2}} + \ldots$$

$\left[dS_4: \ \frac{\pi L^2}{2G_4}\left(\frac{R_c}{L} - 1\right) \to -i\frac{\pi l^2}{2G_4}\frac{R_c}{l} + \frac{\pi l^2}{2G_4}\right]$ $\quad$ $\left[dS_3: \ \frac{2L}{4G_3} \log \frac{R_c}{L} \to -i\frac{l}{2G_3} \log \frac{R_c}{l} + \frac{\pi l}{4G_3}\right]$

dS no-bndry surfaces, Lewkowycz-Maldacena

Hartle-Hawking Wavefunction of the Universe: amplitude (transition matrix) for creating universe (final bndry condns) from "nothing" (satisfying HH no-boundary condition).



Semiclassically $\Psi_{dS} \sim e^{iS^{(r>l)}} e^{S_E^{(r<l)}}$ (fixed dS).

Top Lorentzian (real $S^{(r>l)}$), pure phase. Bottom hemisphere: $iS_{cl} \longrightarrow$ Eucl gravity action $S_E^{(r<l)} = -\int_{nbp} \sqrt{g}\,(R-2\Lambda) \longrightarrow \frac{1}{2}\,\frac{l^4 V_{S^4}}{16\pi G_4}\,\frac{6}{l^2} = \frac{\pi l^2}{2G_4}$ for dS_4 (nbp is $\tau_E = \frac{\pi}{2}$).

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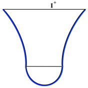

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Lewkowycz-Maldacena: bulk AdS replica dual to boundary replica EE argument.
$Z_{CFT} = Z_{bulk} \Rightarrow$ boundary entanglement entropy = bulk entanglement entropy.
Replica quotient space $\tilde{\mathcal{B}}_n = \mathcal{B}_n/\mathbb{Z}_n$: conical singularities smoothed by codim-2 cosmic brane source (Dong).

[Smooth action $I_n = nI_1 + I_{brane} = nI_1 + \frac{n-1}{n}\frac{A}{4G}$]

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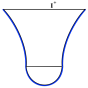

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Analytic cont'n (semicl.): $Z_n^{AdS} \sim e^{-I_n} \longrightarrow \Psi_n^{dS} \sim e^{iS_n}$; $-I_n \rightarrow iS_n = iS_n^{(r>l)} + S_E^{(r<l)}$;
Pseudo-entropy (IR surface): $S_t = \lim_{n\to 1}(1 - n\partial_n)\log \Psi_n = \lim_{n\to 1}(1 - n\partial_n)\,\frac{1-n}{n}\,\frac{A_{brane}}{4G} = \frac{S_{dS}^{IR}}{4G}$

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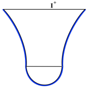

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Cosmic brane not spacelike $\leftrightarrow$ Euclidean + timelike no-bndry dS extremal surface.
LM replica formulation: entropy = area of cosmic brane created from "nothing".
Amplitude for this process divergent if Lorentzian part (going all the way to I^+) were real. Here timelike part = pure phase cancels in probability (finite: bounded real part from hemisphere, set by dS entropy). [KN'23]

“Time-Entanglement”/Pseudo-entropy: QM entanglement with timelike separations

(i) time-evolution operator as generalized density operator $\rightarrow$ partial trace

$\rightarrow$ RTE op $\rightarrow$ complex von Neumann entropy ... Pseudo-entropy.

(ii) future-past entangled state & its density matrix: positive entropy $EE > 0$.

"Time-Entanglement": reduced time evol^n op

dS extremal surfaces anchored at future boundary I^+ do not return: extra data required on boundary conditions in far past. [Witten '01, $dS \equiv$ past-future amplitudes]
Like scattering amplitudes: final states from initial states; equivalently time evolution.

$\rightarrow$ Entanglement-like structures from time evolution operator $\mathcal{U}(t)$ after partial trace over environment: *i.e.* "reduced transition amplitudes" and entropy.

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$$\mathcal{U}(t) = e^{-iHt} \;\rightarrow\; \rho_t(t) \equiv \frac{\mathcal{U}(t)}{\mathrm{Tr}\,\mathcal{U}(t)} \quad\rightarrow\quad \rho_t^A = tr_B\,\rho_t \quad\rightarrow\quad S_A = -tr(\rho_t^A \log \rho_t^A)$$

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$$\rightarrow\quad \rho_t^A = \sum_i p_i'\, |i'\rangle_P \langle i'|_P \quad\rightarrow\quad S_A = -\sum_i p_i' \log p_i'$$

[alternative normalization, $t=0$: $\rho_t(t) \equiv \frac{\mathcal{U}(t)}{\mathrm{Tr}\,\mathcal{U}(0)}$]

Resemble usual finite temp entanglement: but imaginary temperature ($\beta = it$).

[Related quantities: time-evolution op with projection onto some state, *i.e.* $\mathcal{U}(t)|I\rangle\langle I| = |F_I(t)\rangle\langle I|$.]

$\Leftrightarrow$ Pseudo-entropy [entropy of reduced transition matrix (Nakata,Takayanagi,Taki,Tamaoka,Wei,'20)].

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[KN, Saini, '23]

Time Evol'n operator ρ_t	$\rightarrow$	“Components” of time Evol'n op $\rho_t^{\vert I\rangle} = \rho_t\vert I\rangle\langle I\vert$: ρ_t with projection onto initial state $\vert I\rangle$	$\leftarrow$	Pseudo-entropy from reduced transition matrix $\mathcal{T}^A_{F\vert I} = \mathrm{Tr}_B\left(\frac{\vert F\rangle\langle I\vert}{\mathrm{Tr}(\vert F\rangle\langle I\vert)}\right)$

“Time-Entanglement”, examples: 2-qubits etc

2-state system: $H|k\rangle = E_k|k\rangle$, $(k = 1, 2;\ \langle 1|2\rangle = 0)$; $\quad |k\rangle_F \equiv |k(t)\rangle = e^{-iE_k t}|k\rangle_P$.

$\rho_t = \frac{1}{1+e^{i\theta}}\left(|1\rangle\langle 1| + e^{i\theta}|2\rangle\langle 2|\right)$, $\quad \theta = -(E_2 - E_1)t$; 2-spin analogy: $|1\rangle \equiv |{+}{+}\rangle$, $|2\rangle \equiv |{-}{-}\rangle$

$$\xrightarrow{\mathrm{Tr}_B} \quad \rho_t^A \quad \rightarrow \quad \boxed{\text{entropy} \quad S_A^\theta = -\mathrm{tr}(\rho_t^A \log \rho_t^A) = -\frac{1}{1+e^{i\theta}} \log \frac{1}{1+e^{i\theta}} - \frac{1}{1+e^{-i\theta}} \log \frac{1}{1+e^{-i\theta}}}$$

Real-valued, oscillating in time, periodicity $\sim \frac{1}{\Delta E}$; unbounded at $t = \frac{(2n+1)\pi}{\Delta E}$; min $S_A^\theta = \log 2$ at $t = \frac{2n\pi}{\Delta E}$.

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General 2-qubit Hamiltonian $H = E_{11}|11\rangle\langle 11| + E_{22}|22\rangle\langle 22| + E_{12}(|12\rangle\langle 12| + |21\rangle\langle 21|)$

$$\rho_t = \mathcal{N}_t \sum_{i,j} e^{-iE_{ij}t}\, |ij\rangle\langle ij| = \frac{\left(|11\rangle\langle 11| + e^{i\theta_1}|22\rangle\langle 22| + e^{i\theta_2}(|12\rangle\langle 12| + |21\rangle\langle 21|)\right)}{1+e^{i\theta_1}+2e^{i\theta_2}} \qquad \left[\xrightarrow{t=0} \tfrac{1}{4}\hat{\mathbf{1}}\right]$$

$$\xrightarrow{Tr_2} \boxed{\rho_t^A = \frac{1}{1+e^{i\theta_1}+2e^{i\theta_2}}\Big((1+e^{i\theta_2})|1\rangle\langle 1| + (e^{i\theta_1}+e^{i\theta_2})|2\rangle\langle 2|\Big)} \qquad \begin{aligned}\theta_1 &= -(E_{22}-E_{11})t,\\ \theta_2 &= -(E_{12}-E_{11})t.\end{aligned}$$

Generically complex-valued von Neumann entropy. (mixed EE, imaginary temp $\beta = it$)

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$$\xrightarrow{\mathrm{Tr}_B} \rho_t^A \rightarrow \boxed{\text{entropy}\quad S_A^\theta = -\mathrm{tr}(\rho_t^A \log \rho_t^A) = -\frac{1}{1+e^{i\theta}}\log\frac{1}{1+e^{i\theta}} - \frac{1}{1+e^{-i\theta}}\log\frac{1}{1+e^{-i\theta}}}$$

Real-valued, oscillating in time, periodicity $\sim \frac{1}{\Delta E}$; unbounded at $t=\frac{(2n+1)\pi}{\Delta E}$; min $S_A^\theta = \log 2$ at $t=\frac{2n\pi}{\Delta E}$.

General 2-qubit Hamiltonian $H = E_{11}|11\rangle\langle 11| + E_{22}|22\rangle\langle 22| + E_{12}(|12\rangle\langle 12| + |21\rangle\langle 21|)$

$$\rho_t = \mathcal{N}_t \sum_{i,j} e^{-iE_{ij}t}\, |ij\rangle\langle ij| = \frac{(|11\rangle\langle 11| + e^{i\theta_1}|22\rangle\langle 22| + e^{i\theta_2}(|12\rangle\langle 12| + |21\rangle\langle 21|))}{1+e^{i\theta_1}+2e^{i\theta_2}} \qquad \left[\xrightarrow{t=0} \tfrac{1}{4}\hat{\mathbf{1}}\right]$$

$$\xrightarrow{Tr_2} \boxed{\rho_t^A = \frac{1}{1+e^{i\theta_1}+2e^{i\theta_2}}\Big((1+e^{i\theta_2})|1\rangle\langle 1| + (e^{i\theta_1}+e^{i\theta_2})|2\rangle\langle 2|\Big)} \qquad \begin{array}{l}\theta_1 = -(E_{22}-E_{11})t,\\ \theta_2 = -(E_{12}-E_{11})t.\end{array}$$

Generically complex-valued von Neumann entropy. (mixed EE, imaginary temp $\beta = it$)

$\frac{\rho_t|I\rangle\langle I|}{\mathrm{Tr}(\rho_t|I\rangle\langle I|)}$: Projection onto Thermofield-double initial states $|I\rangle = \sum_{1,2} c_{ii}|ii\rangle$

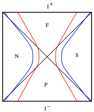
$$\xrightarrow{Tr_2} \boxed{\rho_t^{|I\rangle,A} = \frac{1}{|c_{11}|^2+|c_{22}|^2 e^{i\theta}}\Big(|c_{11}|^2|1\rangle\langle 1| + |c_{22}|^2 e^{i\theta}|2\rangle\langle 2|\Big)} \qquad [\theta = -(E_{22}-E_{11})t]$$

$\equiv$ reduced transition matrix for $|I\rangle$ and $|F\rangle = \sum c_{ii} e^{-iE_{ii}t}|ii\rangle$ ($\longrightarrow$ pseudo-entropy).

Max. entangled (Bell-pair) states $|c_{11}|^2 = |c_{22}|^2 = \frac{1}{2} \longrightarrow S_A^\theta$ (2-state above). Min $S_A^\theta = \log 2 = \mathrm{EE}(|I\rangle)$.

Future-past surfaces, f-p "entanglement"

dS future-past surfaces connecting I^+ to I^-.
(Hartman-Maldacena (AdS bh) rotated)
Suggests future-past entanglement (betw $I^\pm$).



Recall eternal AdS bh dual to $CFT_L \times CFT_R$ in TFD state (Maldacena)

Speculation: (Lorentzian) dS_4 dual to $CFT_F \times CFT_P$ in thermofield-double entangled state $|\psi^{tfd}_{fp}\rangle = \sum \psi^{i^F_n, i^P_n} |i^F_n\rangle |i^P_n\rangle$?

[KN '17; also Arias,Diaz,Sundell,'19]

Tracing fp-dm over past copy gives mixed state at I^+.

$\boxed{\text{2 copies of future-past entangled states \& density matrices: positive entropy } EE > 0.}$

$|\psi_{fp}\rangle \ \rightarrow$ f-p density matrix $\rho_{fp} \equiv |\psi\rangle_{fp}\langle\psi|_{fp} \xrightarrow{Tr_P}$ positive structures.
Connectedness of fp-entangled states & timelike entanglement $\leftrightarrow$ emergence of time?

van Raamsdonk: space emerges from entanglement.
Factorized fp-states $|\psi^{(1)}_f\rangle|\psi^{(2)}_P\rangle$: $\mathrm{Tr}_P \rho_{fp} \rightarrow$ pure.
Entangled fp-states: reduced transition matrix $\equiv$ time evolution operator.

$[\mathcal{U}(t) = Tr_2(|\psi_{fp}\rangle\langle\psi_I|)]$ Time evol'n $\equiv$ f-p EE. Timelike ER=EPR?

"Time-Entanglement": future-past EE

dS extremal surfaces at I^+ & areas $\equiv$ space-time rotations from AdS.
e.g. dS future-past surfaces $\leftrightarrow$ rotated Hartman-Maldacena surfaces (AdS bh).

Recall f-p surfaces suggest future-past entanglement $|\psi\rangle_{fp} = \sum \psi^{i_n^F, i_n^P} |i_n\rangle_F |i_n\rangle_P$.

f-p density matrix $|\psi\rangle_{fp}\langle\psi|_{fp} \xrightarrow{Tr_P}$ red. d.m., nontrivial EE.

Example, 2-state QM: $H|k\rangle = E_k|k\rangle, \quad k = 1, 2; \quad |k\rangle_F \equiv |k(t)\rangle = e^{-iE_k t}|k\rangle_P. \quad [\langle 1|2\rangle = 0]$

$$|\psi\rangle_{fp} = \frac{1}{\sqrt{2}}|1\rangle_F|1\rangle_P + \frac{1}{\sqrt{2}}|2\rangle_F|2\rangle_P = \frac{1}{\sqrt{2}}e^{-iE_1 t}|1\rangle_P|1\rangle_P + \frac{1}{\sqrt{2}}e^{-iE_2 t}|2\rangle_P|2\rangle_P$$

fp-density matrix $\rho = |\psi\rangle_{fp}\langle\psi|_{fp} \xrightarrow{\mathrm{Tr}_P} \delta_{ij}\psi_{fp}^{ki}(\psi_{fp}^*)^{lj} \to$ time-evol'n phases cancel $\to$

$$\rho_{fp} = \mathrm{Tr}_P|\psi\rangle_{fp}\langle\psi|_{fp} = \frac{1}{2}|1\rangle_F\langle 1|_F + \frac{1}{2}|2\rangle_F\langle 2|_F$$

Now imagine 2-spin analogy with $|1\rangle = |++\rangle$, $|2\rangle = |--\rangle$: partial trace over second component $\to \ Tr_2\rho_{fp} = \frac{1}{2}|+\rangle_F\langle +|_F + \frac{1}{2}|-\rangle_F\langle -|_F \to$ positive entropy $\log 2$.

[Similar positive structures with ghost-spins]

Future-past TFD state with timelike separation quite different in principle from usual TFD. Positive structures in f-p d.m. despite timelike separation.

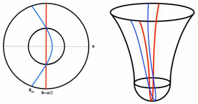
Further aspects:

dS surfaces, subregion duality.

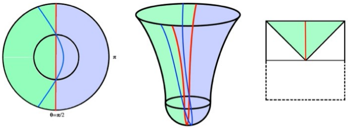
Slow-roll inflation, no-boundary surfaces

dS surfaces, subregion duality, geometrically

IR surface, $t = const$ slice, maximal subregion $\rightarrow$ red surface; Generic subregion, blue: tilted "great circle" in hemisphere, joining with tilted timelike surface in Lorentzian top half. dS_3 explicitly solvable; dS_{d+1}, perturbatively analysed.

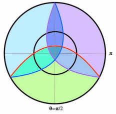


Time-entanglement/Pseudo-entanglement wedge: Max subregion, $t = const$ slice: green bulk region bounded by (red) IR surface and boundary subregion. (Violet complement region)



Including t-direction $\rightarrow$ top wedge (containing future of IR surface on vertical $t = const$ slice), bounded by I^+ subregion $\equiv$ analytic continuation from AdS. Space-time rotation from AdS EE wedge. (dS/CFT via relative entropy, modular flow etc?)

Multiple disjoint boundary subregions: red, violet, blue no-bndry dS extremal surfaces. Complex areas so quite different from AdS EE. Bulk subregions not disjoint: except for IR (maximal) subregions.



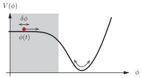
[Maybe other possibilities: subregion duality in Lorentzian dS via future-past surfaces..]

Slow-roll inflation

$\rightarrow$ Inflation: brief period of exponential expansion phase in early universe.

[Standard Big-Bang Cosmology, horizon/flatness (antipodal points in causal contact in past; universe flattens out)]

Nearly de Sitter: departures from dS (slow-roll parameters ϵ, η) driven by inflaton scalar field slowly rolling down its potential.



(Baumann,McAllister'14)

[End of inflation $\equiv$ reheating surface, transition to standard Big-Bang phase. Inhomogeneous quantum fluctuations $\rightarrow$ seeds for structure formation]

Inflationary perturbations to no-boundary global dS: preserve spherical symmetry. No-boundary HH regularity at nbp for both inflaton and metric; no singularities.

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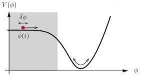

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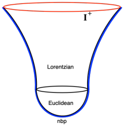


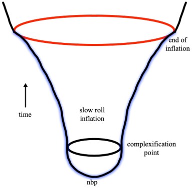
small wiggles around dS $\longrightarrow$

No-boundary slow-roll inflation:

$$ds^2 = -dt^2 + a(t)^2 d\Omega_d^2 = g_{aa} da^2 + a^2 d\Omega_d^2 \,.$$

Slow-roll inflation: $g_{aa} = \frac{1}{1-r^2}\big(1 + 2\epsilon\,\beta_>(r)\big)$

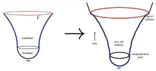
$$\epsilon \equiv \frac{{V_*'}^2}{2V_*^2}\,, \qquad 3H^2 \sim V(\phi), \quad 3H\dot{\phi} \sim -V'(\phi)\,.$$



dS: $g_{aa} = \frac{1}{1-r^2} < 0$, $r > 1$ Lorentzian region. $r < 1$ Euclidean hemisphere, $g_{aa} > 0$.

$\beta_>(r)$ is for $r > 1$; continue to $r < 1$ hemisphere region $\rightarrow$ $\beta_<(r)$

Slow-roll no-boundary extremal surfaces



$$g_{aa} = \frac{1}{1-r^2}\left(1 + 2\epsilon\,\beta_>(r)\right)$$

$$\beta_>(r) = \frac{8-9r^4+4ir^2\sqrt{r^2-1}+8i\sqrt{r^2-1}-6ir^4\sqrt{r^2-1}+r^6\left(6\log\left(1-i\sqrt{r^2-1}\right)-1+3i\pi\right)}{6r^4\left(r^2-1\right)}$$

IR no-boundary extremal surface area (hemisphere + Lorentzian) $= S^{r<1}_{d+1} + S^{r>1}_{d+1}$

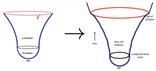
$$S_{sr_{d+1}} = \frac{V_{S^{d-2}}\,l^{d-1}}{4G_{d+1}}\left(\int_0^1 \frac{r^{d-2}\sqrt{1+2\epsilon\,\beta_<(r)}}{\sqrt{1-r^2}}\,dr \;+\; (-i)\int_1^{R_c/l} \frac{r^{d-2}\sqrt{1+2\epsilon\,\beta_>(r)}}{\sqrt{r^2-1}}\,dr\right)$$

Expand to $O(\epsilon)$: $\quad S_{sr_4} \simeq \frac{\pi\,l^2}{2G_4}\left(-i\int_1^{R_c/l}\frac{1+\epsilon\,\beta_>(r)}{\sqrt{r^2-1}}\,r\,dr + \int_0^1 \frac{1+\epsilon\,\beta_<(r)}{\sqrt{1-r^2}}\,r\,dr\right)$

Note: extra singular terms at complexification point $r=1$ from poles in $\beta(r)$ terms

(similar features in no-boundary Wavefunction also).

Slow-roll no-boundary extremal surfaces



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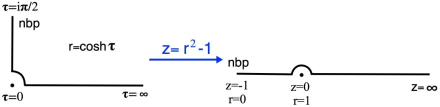
Note: extra singular terms at complexification point $r=1$ from poles in $\beta(r)$ terms

(similar features in no-boundary Wavefunction also).

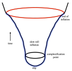
Define as complex-time-plane integral + appropriate time-contour (avoid $r=1$ pole).
Normalize with leading dS results.

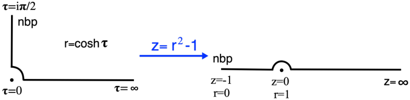
New coord $z = r^2 - 1 \quad \Rightarrow$
(nbp) $\; r = 0 \;\equiv\; z = -1$;
(complexification) $\; r = 1 \;\equiv\; z = 0$.



$$C \;=\; [-1,-\delta] \;\cup\; \left[z = \delta e^{i\theta}\,;\theta = \pi,\;\; \theta = 0\right] \;\cup\; [\delta, z_c]\,.$$

Slow-roll no-boundary extremal surfaces



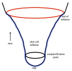


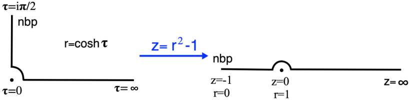
$$\boxed{\begin{aligned} S \;&=\; \frac{\pi\, l^2}{2G_4}\int_C \frac{1+\epsilon\,\beta_>(r)}{2\sqrt{1-r^2}}\; r\,dr \\ &\equiv\; \frac{\pi\, l^2}{2G_4}\int_C \frac{1+\epsilon\,\beta_>(z)}{2\sqrt{-z}}\; dz\,. \end{aligned}}$$

$$\longrightarrow\quad \begin{aligned} S \;=\; \frac{\pi\, l^2}{2G_4}\Bigg[&1-\sqrt{\delta}\;+\;\epsilon\left[\left(\log 4-\frac{7}{6}+i\pi\right)-\left(-\frac{2-3i\pi}{6\sqrt{\delta}}+\frac{5}{3}\right)\right]\;+\;\sqrt{\delta}(-i+1) \\ &+\;\epsilon\left[\frac{2-3i\pi}{6i\sqrt{\delta}}-\frac{2-3i\pi}{6\sqrt{\delta}}\right]+\;\frac{1}{i}(\sqrt{z_c}-\sqrt{\delta})+\;\epsilon\left[\left(1+i\frac{7}{6}\sqrt{z_c}-i\sqrt{z_c}\log\sqrt{z_c}\right)-\left(\frac{2-3i\pi}{6i\sqrt{\delta}}+\frac{5}{3}\right)\right]\Bigg] \end{aligned}$$

Various cancellations as expected. Details of regulating semicircle contour unimportant.

Slow-roll no-boundary extremal surfaces





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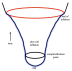
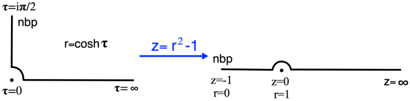
- Divergent parts pure imaginary. Vindicates finite cosmic brane creation probability, set by size of maximal hemisphere ($\equiv dS$ entropy $+$ slow-roll corrections $[<0]$).
- No clean separation betw real/imaginary parts of area, slow-roll corrections mix all.

Finite terms in particular arise from entire surface, both timelike and hemisphere parts.

Slow-roll no-boundary extremal surfaces





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- No clean separation betw real/imaginary parts of area, slow-roll corrections mix all.

Finite terms in particular arise from entire surface, both timelike and hemisphere parts.

Finite parts above (cosmic brane probability) match those in dS_4 Wavefunction:

$$iI_{sr4} = \frac{\pi\, l^2}{2G_4}\bigg[1+\epsilon\Big(\log 4-\frac{7}{2}+i\pi\Big)-i\Big(r_c^3-\frac{3}{2}r_c\Big)+i\epsilon\,\Big(r_c^3\Big(\log r_c-\frac{1}{6}\Big)+\frac{r_c}{4}\,(6\log r_c-11)\Big)\bigg]$$

(Maldacena '24) $\Psi\sim e^{iI_{sr4}}$, obtained by evaluating on-shell action via ADM formulation.

- AdS BH: IR RT surface wraps horizon, $S^{fin}\sim$ BH entropy $\leftarrow$ action $\equiv$ partition fn

Conclusions, questions

- dS future boundary: no $I^+ \to I^+$ turning point. Surfaces do not return to I^+.

(a) Future-past surfaces, end at past boundary I^-. Pure imaginary area. $CFT_F \times CFT_P$ f-p "entanglement".

(b) No-boundary surfaces, top timelike f-p joins Eucl surface in bottom hemisphere. Real part = half dS entropy.

Pseudo-entropy: AdS analytic cont'n $\equiv$ space $\leftrightarrow$ time rotation, Lewkowycz-Maldacena.

QM & Time-entanglement/Pseudo-entropy: EE-like structures, timelike separations:

(i) reduced time evolution operator, mixed state EE + imaginary temperature
$\leftrightarrow$ reduced transition amplitudes, pseudo-entropy, ...

(ii) positivity in future-past entangled states & density matrices.

- Slow-roll inflation: no-boundary areas must be defined carefully via complex time plane integrals with appropriate contours avoiding potential poles.

Finite slow-roll corrections arise from entire Lorentzian+hemisphere regions.
Real parts of IR areas control maximal cosmic brane creation probability
($= dS_4$ entropy + slow-roll corrections $[< 0]$): match those from Wavefunction.

Dual CFT understanding/interpretations?
Time-contours for more general cosmologies?

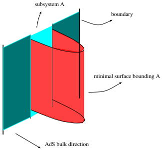
Pseudo-entropy meta-observables $\leftrightarrow$ standard Big-Bang cosmology observers/observables?

Holographic entanglement entropy

Entanglement entropy: entropy of reduced density matrix of subsystem.

EE for spatial subsystem A, $S_A = -tr\rho_A \log \rho_A$, with partial trace $\rho_A = tr_B\,\rho$.

Ryu-Takayanagi: $EE = \frac{A_{min.surf.}}{4G}$



[$\sim$ black hole entropy] Area of codim-2 minimal surface in gravity dual.

Non-static situations: extremal surfaces (Hubeny, Rangamani, Takayanagi).

Lewkowycz-Maldacena: bulk replica $\rightarrow$ RT $\equiv$ cosmic brane.

Operationally: Const time slice, boundary subsystem $\rightarrow$ bulk slice, codim-2 extremal surface.

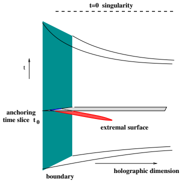
Ex.: CFT$_d$ ground state = empty AdS_{d+1}, $ds^2 = \frac{R^2}{r^2}(dr^2 - dt^2 + dx_i^2)$. $S_A \propto \int \frac{dr}{r^{d-1}}\sqrt{1+(\partial_r x)^2}$

Bulk surface $x(r)$. Turning point r_*

$[AdS_3/CFT_2]$ $S_A = \frac{R}{2G_3}\log\frac{l}{\epsilon} \equiv \frac{c}{3}\log\frac{l}{\epsilon}$

CFT thermal state (AdS black hole): minimal surface wraps horizon. $S^{fin} \sim BH\ entropy$

Entanglement wedge $\Rightarrow$ Subregion duality: boundary subregion $\leftrightarrow$ bulk subregion.



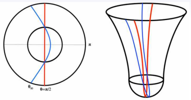
AdS Kasner: spacelike extremal surfaces in reliable semiclassical region far from singularity.

Surface lies almost on $t = const$ slice and bends away from singularity.

(Manu,KN,Paul,'20)

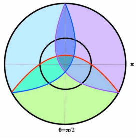
Back

dS surfaces, entropy relations/inequalities



dS_3 : $S_t^{\theta_\infty} = -i\frac{l}{2G_3}\log\frac{R_c}{l} - i\frac{l}{4G_3}\log(\sin^2\theta_\infty) + \frac{\pi l}{4G_3}$

IR, $\theta_\infty = \frac{\pi}{2}$: $S_t^{IR} = -i\frac{l}{2G_3}\log\frac{R_c}{l} + \frac{\pi l}{4G_3}$



Two adjacent disjoint subregions A, B $(2\theta_\infty = \frac{\pi}{2})$; $A \cup B \equiv (2\theta_\infty = \pi)$.

"Mutual time-information" or "mutual pseudo-information":

$I_t[A,B] = S[A] + S[B] - S[A\cup B] = -i\frac{l}{2G_3}\log\frac{R_c}{l} + i\frac{l}{2G_3}\log 2 + \frac{\pi l}{4G_3}$

$\Rightarrow$ $\text{Re}\, I_t \geq 0$, $\text{Im}\, I_t \leq 0$. (antipodal subregions, $I_t = 0$)

Tripartite time-information: 3 disjoint adjacent quadrant subregions A, B, C $(2\theta_\infty = \frac{\pi}{2})$.

$A\cup B$, $B\cup C$ maximal (IR) subregions. $A\cup C$, antipodal quadrants (extr. surf. = "inner" $(\equiv B)$ + "outer").

$S_A = S_B = S_C = S_t^{\pi/4}$, $S_{AB} = S_{BC} = S_t^{\pi/2}$, $S_{AC} = S_t^{\pi/4} + S_t^{\pi/4}$, $S_{ABC} = S_t^{\pi/4}$;

$I_3^t[A,B,C] = S_A + S_B + S_C - S_{AB} - S_{BC} - S_{AC} + S_{ABC} = i\frac{l}{2G_3}\log 2$ $\Rightarrow$ $\text{Im}\, I_3^t \geq 0$.

Strong subadditivity: $\left.\begin{array}{l} S_{AB} + S_{BC} - S_{ABC} - S_B = -i\frac{l}{2G_3}\log 2, \\ S_{AB} + S_{BC} - S_A - S_C = -i\frac{l}{2G_3}\log 2. \end{array}\right\}$ $\begin{array}{l} \text{Re}\, SSB_{1,2}^t \geq 0, \\ \text{Im}\, SSB_{1,2}^t \leq 0 \end{array}$

dS area/entropy relations special (relative to qubit system pseudo-entropies).

Note: AdS analytic continuation $il \to -L$ $\Rightarrow$ $MI \geq 0$, $I_3 \leq 0$, $SSB^{1,2} \geq 0$.

Consistent with AdS RT/HRT areas which are also special Hayden,Headrick,Maloney,'11.

Qubits, pseudo-entropy inequalities

Pseudo-entropy $\rho_t = \frac{|F\rangle\langle I|}{\mathrm{Tr}(|F\rangle\langle I|)} = \frac{\mathcal{U}(t)|I\rangle\langle I|}{\mathrm{Tr}(\mathcal{U}(t)|I\rangle\langle I|)}$ [= time evoln op $\mathcal{U}(t) = e^{-iHt}$ with projection]

for TFD-type initial state $|I\rangle$ and its time-evolved final state $|F\rangle = \mathcal{U}(t)|I\rangle$.

2-qubits: $|I\rangle = c_{11}|11\rangle + c_{22}|22\rangle\,, \quad |F\rangle = c_{11}e^{-iE_{11}t}|11\rangle + c_{22}e^{-iE_{22}t}|22\rangle$

$\left[|c_{11}|^2 + |c_{22}|^2 = 1; \quad |c_{11}|^2 \equiv x; \quad \theta = -(E_{22} - E_{11})t\right]$

$$\rho_t^1 = \mathrm{Tr}_2\,\rho_t\,, \quad \rho_t^2 = \mathrm{Tr}_1\,\rho_t\,, \quad \rho_t^2 = \rho_t^1 = \frac{1}{x+(1-x)e^{i\theta}}\left(x|1\rangle\langle 1| + (1-x)e^{i\theta}|2\rangle\langle 2|\right),$$

$$S_t^2 = S_t^1 = -\frac{x}{x+(1-x)e^{i\theta}}\log\frac{x}{x+(1-x)e^{i\theta}} - \frac{(1-x)e^{i\theta}}{x+(1-x)e^{i\theta}}\log\frac{(1-x)e^{i\theta}}{x+(1-x)e^{i\theta}}$$

Near $t=0$: $\quad S_t^1(t) \sim S_t^1(0) + \frac{d}{dt}S_t^1(0)\,t \equiv S_0\,,$

$$S_t^1(0) = -x\log x - (1-x)\log(1-x)\,, \qquad \frac{d}{dt}S_t^1(0) = -i\,\Delta E\,x(1-x)\log\frac{x}{1-x}$$

Mutual pseudo-information: $I_t[1,2] = S_t^1 + S_t^2 - S_t \sim 2S_0; \quad \mathrm{Re}I_t > 0\,, \quad \mathrm{Im}I_t \gtrless 0$

3-qubits: $|I\rangle = c_{111}|111\rangle + c_{222}|222\rangle\,, \quad |F\rangle = c_{111}e^{-iE_{111}t}|111\rangle + c_{222}e^{-iE_{222}t}|222\rangle$

$$\rho_t^{123} = \frac{|F\rangle\langle I|}{\mathrm{Tr}(|F\rangle\langle I|)}\,, \quad \rho_t^1 = \mathrm{Tr}_{23}\,\rho_t^{123} = \frac{1}{x+(1-x)e^{i\theta}}\left(x|1\rangle\langle 1| + (1-x)e^{i\theta}|2\rangle\langle 2|\right), \quad \rho_t^2 = \rho_t^3 = \rho_t^1\,,$$

$$\rho_t^{12} = \mathrm{Tr}_3\,\rho_t^{123} = \frac{1}{x+(1-x)e^{i\theta}}\left(x|11\rangle\langle 11| + (1-x)e^{i\theta}|22\rangle\langle 22|\right), \quad \rho_t^{23} = \rho_t^{13} = \rho_t^{12}$$

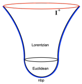
Tripartite pseudo-information: $I_3^t[1,2,3] = S_t^1 + S_t^2 + S_t^3 - S_t^{23} - S_t^{13} - S_t^{12} + S_t^{123} = 0$

SSB: $SSB_1^t = S_t^{12} + S_t^{23} - S_t^{123} - S_t^2 = S_t^1; \quad SSB_2^t = S_t^{12} + S_t^{23} - S_t^1 - S_t^3 = 0$

$\mathrm{Re}\,SSB_1^t > 0\,, \quad \mathrm{Im}\,SSB_1^t \gtrless 0 \quad (x \neq \frac{1}{2})$

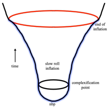
[Specific TFD states above; more general states?]

No-boundary slow-roll inflation



$\longrightarrow$

Small perturbation wiggles around dS



Slow-roll approx'n: $\epsilon \equiv \frac{{V_*'}^2}{2V_*^2}$.

$3H^2 \sim V(\phi), \quad 3H\dot{\phi} \sim -V'(\phi)$.

Convenient to parametrize as: $ds^2 = -dt^2 + a(t)^2 d\Omega_d^2 = g_{aa} da^2 + a^2 d\Omega_d^2$.

$$dS: \quad a(t) = l \cosh \tau \equiv l\, r\,, \quad \tau = \frac{t}{l}\,.$$

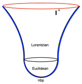
$g_{aa} = \frac{1}{1-r^2} < 0\,,\ \ r > 1$ Lorentzian region. $r < 1$ Euclidean hemisphere, $g_{aa} > 0$.

$$\boxed{\text{Slow-roll inflation:} \quad g_{aa} = \frac{1}{1-r^2}\big(1 + 2\epsilon\, \beta_>(r)\big)}$$

$$\beta_>(r) = \frac{8-9r^4+4ir^2\sqrt{r^2-1}+8i\sqrt{r^2-1}-6ir^4\sqrt{r^2-1}+r^6\left(6\log\left(1-i\sqrt{r^2-1}\right)-1+3i\pi\right)}{6r^4\left(r^2-1\right)}\,.$$

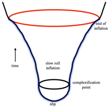
This is for $r > 1$; continue to $r < 1$ hemisphere region $\rightarrow\ \beta_<(r)$

No-boundary slow-roll inflation



$\longrightarrow$

Small perturbation wiggles around dS



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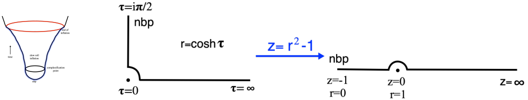
This is for $r > 1$; continue to $r < 1$ hemisphere region $\rightarrow \beta_<(r)$

ADM: $g_{aa} = \frac{3-\frac{1}{2}(a\,\partial_a\phi)^2}{3-V(\phi)\,a^2}$; inflaton $\phi = \phi_* + \varphi(\tau)$ & $V(\phi) = V_* + {V_*}'\varphi(\tau) \rightarrow$

$\beta(r) = -\frac{1}{6}\left(r\frac{\partial\tilde{\varphi}}{\partial r}\right)^2 + \frac{\tilde{\varphi}\, r^2}{1-r^2}$. $\leftrightarrow \ \varphi(r) = \sqrt{2\epsilon}\,\tilde{\varphi}(r)\,, \quad \tilde{\varphi}(r) = \frac{1+i\sqrt{r^2-1}}{r^2} - \log\left(1 - i\sqrt{r^2-1}\right) - \frac{i\pi}{2}$

3-dim Slow-roll, no-bndry extremal surfaces



$$H^2 \sim V(\phi),$$
$$2H\dot{\phi} \sim -V'(\phi)\,.$$

Inflaton perturbation $\varphi(\tau) = \sqrt{2\epsilon}\Big[\big(\frac{i\pi}{4} - \frac{\tau}{2}\big)\tanh\tau + \frac{\log 2}{2} - \frac{i\pi}{4}\Big] \equiv \sqrt{2\epsilon}\,\tilde{\varphi}(\tau)\,.$

$$g_{rr} = \frac{1-\frac{1}{2}(r\partial_r\varphi)^2}{1-Vr^2} \simeq \frac{1}{1-r^2}\,(1+2\epsilon\beta_>(r))\,;\quad 2\epsilon\beta_>(r) \equiv \frac{V_*'^2}{V_*^2}\left(-\frac{1}{2}(r\partial_r\tilde{\varphi})^2 + \frac{\tilde{\varphi}\ r^2}{1-r^2}\right)$$

$$S_{sr_3} \simeq \frac{l}{2G_3}\left(\int_\delta^{z_c}\frac{1+\epsilon\,\beta_>(z)}{2i\sqrt{z(1+z)}}\,dz + \int_\delta^1\frac{1+\epsilon\,\beta_<(z)}{2\sqrt{z(1-z)}}\,dz\right) + \frac{l}{2G_3}I_\epsilon^\theta$$

$$S_{sr_3} = \frac{l}{2G_3}\left(\frac{\pi}{2} - i\log\frac{R_c}{l}\right)$$
$$+\,\epsilon\,\frac{l}{2G_3}\left(-\frac{\pi}{16}(1+\log 16) + \frac{i}{16}\left(2\log\frac{R_c}{l} - 4\big(\log\frac{R_c}{l}\big)^2 + 3\pi^2 + 4(\log 2)^2 - 6\log 2\right)\right).$$

Cosmic brane creation probability $\Leftrightarrow$ $\mathrm{Re}S = \frac{\pi l}{4G_3} - \epsilon\,\frac{l}{2G_3}\frac{\pi}{16}(1+\log 16).$

Matches real finite terms from Wavefunction $\Psi_{dS3sr} \sim e^{iI_{sr3}}$ for dS_3 inflation:

$$iI_{sr3} = \frac{\pi l}{4G_3} - \frac{il}{G_3}\left(\frac{r_c^2}{2} - \frac{1}{2}\log r_c - \frac{1}{2}\log 2 - \frac{1}{4}\right)$$
$$+\frac{\epsilon l}{2G_3}\left[-\frac{\pi}{16}(1+\log 16) + \frac{i}{16}\left(8r_c^2\log r_c - 2r_c^2 + 4(\log r_c)^2 - 6\log r_c + \pi^2 - 1 - 4(\log 2)^2 - 2\log 2\right)\right]$$

Imaginary finite parts differ from $\Psi_{dS} = Z_{CFT}$: CFT_2 on even dim sphere. Anomalies?