de Sitter, extremal surfaces and time entanglement

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- de Sitter space, dS/CFT and de Sitter entropy
- dS extremal surfaces: future-past, no-boundary.
- Time-entanglement (EE, timelike separations) and Pseudo-entropy
- dS surfaces: analytic cont'ns, subregion duality, entropy relns, Lewkowycz-Maldacena

2210.12963, 2310.00320, KN; 2303.01307, KN, Saini (also 1501.03019, 1711.01107, 2002.11950, . . .)

[collaborations: Ritabrata Bhattacharya, Karan Fernandes, Kaberi Goswami, Dileep Jatkar, Kedar Kolekar, A. Manu, Partha Paul, Sourav Roy, Hitesh Saini]

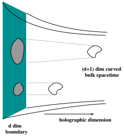
[Partial overlap: Doi,Harper,Mollabashi,Takayanagi,Taki,'22]

Holography and asymptotics

25+ yrs since AdS/CFT '97 Maldacena; '98 Gubser,Klebanov,Polyakov; Witten.

Holography: quantum gravity in $\mathcal{M} \leftrightarrow$ dual without gravity on $\partial\mathcal{M}$ ('t Hooft, Susskind).

(Witten@Strings'98, '01) Gauge/gravity duality and asymptotics —

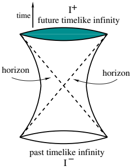


$\Lambda < 0$: *AdS* $\rightarrow$ asymptotics at spatial infinity.
Dual: unitary Lorentzian CFT, includes time.

$\Lambda = 0$: flat space $\rightarrow$ null infinity $\rightarrow$ S-matrix, symmetries...

$\Lambda > 0$: de Sitter space
Boundary at future/past timelike infinity $\mathcal{I}^{\pm}$.
Dual $\rightarrow$ Euclidean CFT ... Time emergent.

[note: gravity dual of ordinary Euclidean CFT $\longrightarrow$ Euclidean AdS]

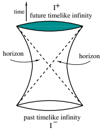


Cosmology, holography, entanglement ...

Time-dependence and cosmology in string theory and holography.
Might regard de Sitter as toy model for cosmology.

dS very different from AdS: a natural boundary in far future/past.
de Sitter space, holography with future boundary $\rightarrow dS/CFT$.
Many questions remain with spacelike holographic boundary ($\perp$ time).

de Sitter entropy = area of cosmological horizon (Gibbons,Hawking).
de Sitter entropy as some sort of (holographic) entanglement?
Generalizations of Ryu-Takayanagi?



de Sitter space and dS/CFT

dS/CFT: ('01 Strominger; Witten) future timelike infinity $\mathcal{I}^+$ as a natural dS boundary. Euclidean non-unitary CFT dual. Time emergent.

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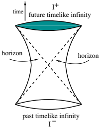
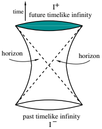


(Maldacena '02) AdS, analytic continuation → $\boxed{Z_{CFT} = \Psi_{dS}}$ Hartle-Hawking Wavefunction of the Universe

Bulk expectation values $\langle\varphi_k\varphi_{k'}\rangle \sim \int D\varphi\ \varphi_k\varphi_{k'}|\Psi|^2 \rightarrow$ dual ≡ two CFT copies.

Global/static dS from global AdS: other analytic continuations.

de Sitter space and dS/CFT



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$\left.\begin{array}{l} \underline{dS_4}, \text{ Poincare: } ds^2 = \frac{R_{dS}^2}{\tau^2}(-d\tau^2 + d\vec{x}^2) \\ r \rightarrow -i\tau, \quad R_{AdS} \rightarrow -iR_{dS}. \end{array}\right\}$ $\Psi_{dS}[\varphi] \sim e^{iS_{cl}[\varphi]} \sim e^{-\int_k R_{dS}^2 k^3 \varphi_{-k}^0\varphi_k^0 + \ldots}$

$\longrightarrow$ dual CFT: $\langle O_k O_{k'}\rangle \sim \frac{\delta^2 Z}{\delta\varphi_k^0\delta\varphi_{k'}^0}$.

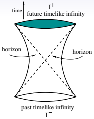
Energy-momentum $\langle TT\rangle$ 2-pt fn $\rightarrow$ $\mathcal{C}_3 \sim -\frac{R_{dS}^2}{G_4} < 0$, ghost-CFT.

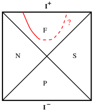
Anninos,Hartman,Strominger: higher-spin dS_4 dual to $Sp(N)$ ghost CFT_3, ...

[In general $\Psi = \Psi[g^3]$, final 3-metric is g^3; sum over final boundary condns for bulk vevs.]

dS, future boundary, extremal surfaces

[KN '15-'23] A natural generalization of Ryu-Takayanagi to de Sitter $\equiv$ bulk analog of setting up entanglement entropy in dual Eucl CFT $\rightarrow$ define some boundary Eucl time slice $\rightarrow$ codim-2 RT/HRT surfaces anchored at I^+, dipping into holographic (time) direction.





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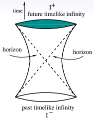

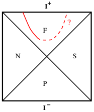

Extremization: surfaces anchored at future boundary I^+ $\rightarrow$
No real $I^+ \rightarrow I^+$ turning point (Lorentzian dS).
Surfaces do not return to I^+. *Interior boundary condns? Time contours?*

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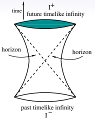

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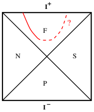

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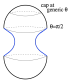
Entirely timelike so area has overall $-i$ (relative to AdS spacelike surfaces).



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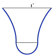
(top timelike part of f-p surface joined with real surface with turn-around in bottom hemisphere)



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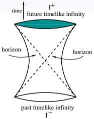

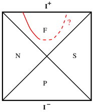

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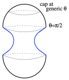

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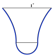


$AdS \rightarrow$ global/static dS surfaces: analytic continuation $\equiv$ *space* $\leftrightarrow$ *time* rotation.

Areas of these surfaces: new object $\rightarrow$ "Time entanglement". [time evol'n op, EE]
Entanglement-like structures with timelike separations $\Leftrightarrow$ Pseudo-entropy

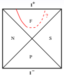
$\mathcal{T}^A_{F|I} = \mathrm{Tr}_B \left(\frac{|F\rangle\langle I|}{\mathrm{Tr}(|F\rangle\langle I|)} \right)$ [entropy of reduced transition matrix (Nakata,Takayanagi,Taki,Tamaoka,Wei,'20)]

de Sitter space, extremal surfaces

de Sitter extremal surfaces

<ins>dS (Poincare)</ins> : $ds^2_{d+1} = \frac{R^2_{dS}}{\tau^2}(-d\tau^2 + dw^2 + dx_i^2)$

$S_{dS} \propto \int \frac{d\tau}{\tau^{d-1}}\sqrt{1-(\partial_\tau x)^2} \;\longrightarrow\; (\partial_\tau x)^2 = \frac{B^2\tau^{2d-2}}{1+B^2\tau^{2d-2}}$ $[B^2 > 0]$



Bndry Eucl time $w{=}const$ strip @ I^+ $\rightarrow$ codim-2.

Sign diff. from AdS $\Rightarrow$ **No <ins>real</ins> $I^+ \rightarrow I^+$ "turning point".** KN '15

RT

$B^2 < 0$: Analytic cont'n $r \rightarrow -i\tau$, $R \rightarrow -iR_{dS}$ from AdS RT $\rightarrow$ Complex areas. KN '15; Sato; Miyaji,Takayanagi

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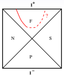

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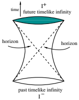
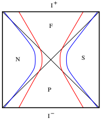


dSfp

dS (static) $ds^2 = -\frac{dr^2}{r^2/l^2-1} + (\frac{r^2}{l^2} - 1)dt^2 + r^2 d\Omega^2_{d-1}$. [KN ’17]

Bndry Eucl time slice, any S^{d-1} equatorial plane (OR $t = const$ slice).

Future-past (timelike) surfaces connecting I^+ to I^-



Hartman-Maldacena (AdS bh) rotated. [area scaling: dS entropy] timelike $\Rightarrow$ overall $-i$

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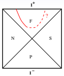

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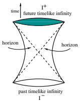

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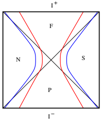

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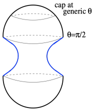


dS global: $ds^2_{d+1} = -d\tau^2 + l^2\cosh^2\frac{\tau}{l}\,d\Omega^2_d$.

$ds^2_{global}|_{\theta_d=const} = -d\tau^2 + l^2\cosh^2\frac{\tau}{l}\,d\Omega^2_{d-1} \equiv ds^2_{static}|_{t=const}$ $[r = l\cosh\frac{\tau}{l}]$

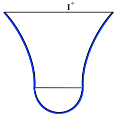
f-p surfaces connecting θ-caps at $I^\pm$, wrapping S^{d-2}. IR area $-i\frac{\pi l^2}{G_4}\frac{1}{\epsilon}$ $[dS_4]$

[t = Killing time in AdS BH analogy before rotating to dS; also Killing time in static patch.]

de Sitter no-boundary surfaces

Hartle-Hawking no-boundary proposal: Lorentzian dS evolves in time from a no-boundary Euclidean initial configuration. Cut global dS in middle ($\tau = 0$ slice), join top half with hemisphere in bottom half given by Euclidean continuation

$$ds^2 = l^2 d\tau_E^2 + l^2 \cos^2 \tau_E \, d\Omega_d^2 \, ; \quad \tau = il\tau_E \, , \quad 0 \le \tau_E \le \tfrac{\pi}{2} \, .$$



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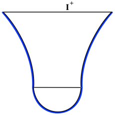

Some S^d equatorial plane (*i.e.* S^{d-1}) $\rightarrow$ timelike future-past surface at $\theta = \frac{\pi}{2}$ [IR limit]. Hits $\tau = 0$ mid-slice "vertically": join smoothly at $\tau = 0$ with surface going around bottom hemisphere. Smooth joining $\Leftrightarrow$ consistency of F-P with Hartle-Hawking.

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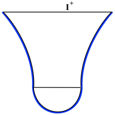

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IR bottom surface: $ds^2 = l^2 d\tau_E^2 + l^2 \cos^2 \tau_E (d\theta^2 + \sin^2 \theta \, d\Omega_{d-2}^2)\Big|_{\theta=\frac{\pi}{2}}$

$$\text{Area} = \frac{l^{d-1}}{4G_{d+1}} V_{S^{d-2}} \int_0^{\pi/2} d\tau_E \, (\cos \tau_E)^{d-2} = \frac{1}{2} \frac{l^{d-1} V_{S^{d-1}}}{4G_{d+1}}$$

Precisely half dS entropy: emerges differently from area of cosmological horizon (static patch observers). [One hemisphere direction here is Euclidean continuation of time in future universe]

$$S_{dS_4} = -i \frac{\pi l^2}{4G_4} \frac{l}{T_c} + \frac{\pi l^2}{2G_4} \, . \quad \text{Similarities with Wavefunction } \Psi_{dS} = e^{iS_{cl}} \, .$$

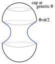
- Half dS entropy also emerges for no-bndry static patch $t = const$ surfaces.

[dS_3: Hikida,Nishioka,Takayanagi,Taki]; [KN]

dS extremal surfaces → "Time-Entanglement"

dS extremal surfaces: no $I^+ \to I^+$ returns → timelike components necessarily.

Future-past surfaces: entirely Lorentzian dS. Surfaces start at I^+, end at I^-.



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No-boundary surfaces: Hartle-Hawking no-boundary dS. Complex area.

(top timelike part of f-p surface joined with real surface with turn-around in bottom hemisphere)

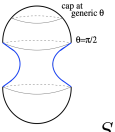
Note: timelike geodesic length has overall $-i$ relative to spacelike geodesic length.
We call this timelike length as "time" rather than "$-i$·space".
These extremal surface areas with timelike components ≡ new object,
"time entanglement" or pseudo-entropy. [entanglement-like structures, timelike separations]

Time-entanglement/Pseudo-entropy in QM [$\tau^A_{F|I} = \mathrm{Tr}_B \left(\frac{|F\rangle\langle I|}{\mathrm{Tr}(|F\rangle\langle I|)} \right)$] — later.

dS, future boundary, extremal surfaces

No $I^+ \to I^+$ returns $\to$ future-past, or no-boundary surfaces.

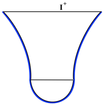
Areas: $S_{fp} = -2iS_0\, I[\tau_{cF}, \tau_*]$; $\quad S_{nb} = -iS_0\, I[\tau_{cF}, \tau_*] + \frac{S_0}{2}$ [$I=$ 'reduced' time integral].



IR limit

$S_{fp} = S_{nb} - S_{nb}^*$, $\quad \mathrm{Re}(S_{nb}) = \frac{1}{2} \cdot dS$ entropy.

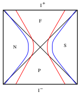
Suggests $S_{nb} \equiv \Psi_{dS}$, $\quad S_{fp} \equiv \Psi_{dS}, \Psi_{dS}^*$ $(I^+ \cup I^-)$.



$S_{nb} \equiv$ time entanglement entropy in one copy $\Psi_{dS} = Z_{CFT}$.

dS extremal surfaces at I^+ & areas $\equiv$ space-time rotations from AdS.

e.g. dS future-past surfaces $\equiv$ rotated Hartman-Maldacena surfaces (AdS BH).



$[dS_4]$ $\quad S_{fp} = -i\,\frac{\pi l^2}{2G_4}\frac{l}{\epsilon}$; $\quad S_{nb} = -i\,\frac{\pi l^2}{4G_4}\frac{l}{\epsilon} + \frac{\pi l^2}{2G_4} = -i\left(\frac{\pi l^2}{4G_4}\frac{l}{\epsilon} + i\frac{\pi l^2}{2G_4}\right)$ $\quad [S_0 = \frac{\pi l^2}{G_4}]$.

$[dS_3]$ $\quad S_{fp} = -i\frac{l}{G_3}\log\frac{l}{\epsilon}$; $\quad S_{nb} = -i\left(\frac{c}{3}\log\frac{l}{\epsilon} + \frac{c}{6}i\pi\right)$ $\quad [c = \frac{3l}{2G_3}]$.

[overall $-i \equiv$ space$\leftrightarrow$time rotation; $\quad (\ldots) =$ EE for timelike interval, AdS.]

dS_3, 2-dim CFT, timelike intervals

Entirely Lorentzian global dS_3: future-past surfaces on some S^2 equatorial plane.
Area $S_{fp} = -i\frac{l}{G_3}\log\frac{l}{T_c}$. Matches $2(\frac{c}{3}\log\frac{l}{T_c})$ with $c_{dS_3} = -i\frac{3l_{dS}}{2G}$ [2 copies].

No-boundary dS_3 surface: area $S_{nb} = -i\frac{l}{2G_3}\log\frac{l}{T_c} + \frac{\pi l}{4G_3} \equiv \frac{c}{3}\log\frac{l}{T_c} + \frac{c}{6}(i\pi)$.

Im(S_{nb}) matches $\frac{c}{3}\log\frac{l}{T_c}$ for half-size interval (IR) in Eucl CFT on circle.

Re(S_{nb}) from deep interior Euclideanization $\leftrightarrow$ "interior regularity" in Eucl CFT dual (no time; bulk time emergent).

[S_{nb} is overall $-i$ times EE for timelike interval in AdS_3 with $c = \frac{3l_{AdS}}{2G_3}$.]

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Ordinary unitary 2-dim CFTs: EE is $S = \frac{c}{6} \log \frac{\Delta^2}{\epsilon^2} = \frac{c}{6} \log \frac{-(\Delta t)^2 + (\Delta x)^2}{\epsilon^2}$.
Ordinary spacelike intervals $\Delta^2 > 0 \rightarrow S = \frac{c}{3} \log \frac{\Delta x}{\epsilon}$.
Entirely timelike interval, width Δt so $\Delta^2 < 0$: $S = \frac{c}{3} \log \frac{\Delta t}{\epsilon} + \frac{c}{6}(i\pi)$.

[Usual replica formulation in Euclidean CFT: pick interval $\Delta x \equiv [u, v]$ on Eucl time slice $\tau_E = const$ $\rightarrow n$ replicas glued at interval endpts $\rightarrow \mathrm{Tr}\rho_A^n \rightarrow$ twist op 2-pt fn $\rightarrow S_A = -\lim_{n\to 1} \partial_n \mathrm{Tr}\rho_A^n$.
Timelike interval $\Delta t \equiv [u_t, v_t]$ on Eucl time slice $x = const$: continue to Lorentzian time rotating (u_t, v_t), to $(-iu_t, -iv_t)$ so $\Delta^2 = -(v_t - u_t)^2 = -(\Delta t)^2$.]

[See also studies of quantum extremal surfaces for dS Poincare: bulk $c > 0$ matter entropy with timelike separations Chen,Gorbenko,Maldacena, '20, also Goswami,KN,Saini,'21]. qes

“Time-Entanglement”/Pseudo-entropy: QM entanglement with timelike separations

(i) time-evolution operator as generalized density operator $\rightarrow$ partial trace
$\rightarrow$ RTE op $\rightarrow$ complex von Neumann entropy . . . Pseudo-entropy.

(ii) future-past entangled state & its density matrix: positive entropy $EE > 0$.

“Time-Entanglement”: reduced time evol^n op

In AdS, specifying boundary data fixes RT/HRT extremization problem.
dS extremal surfaces anchored at future boundary I^+ do not return: extra data required on boundary conditions in far past. [Witten ’01, $dS \equiv$ past-future amplitudes]
Like scattering amplitudes: final states from initial states; equivalently time evolution.

→ Entanglement-like structures from time evolution operator $\mathcal{U}(t)$ after partial trace over environment: *i.e.* “reduced transition amplitudes” and entropy.

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$$\mathcal{U}(t) = e^{-iHt} \;\rightarrow\; \rho_t(t) \equiv \frac{\mathcal{U}(t)}{\mathrm{Tr}\,\mathcal{U}(t)} \quad\rightarrow\quad \rho_t^A = tr_B\, \rho_t \quad\rightarrow\quad S_A = -tr(\rho_t^A \log \rho_t^A)$$

$$\rho_t(t) \equiv \frac{\mathcal{U}(t)}{\mathrm{Tr}\,\mathcal{U}(t)} \quad\Rightarrow\quad \rho_t(t) = \sum_i p_i\, |i\rangle_P \langle i|_P \;, \qquad p_i = \frac{e^{-iE_i t}}{\sum_j e^{-iE_j t}} \;,$$

$$\rightarrow \quad \rho_t^A = \sum_i p_i'\, |i'\rangle_P \langle i'|_P \quad\rightarrow\quad S_A = -\sum_i p_i' \log p_i'$$

[alternative normalization, $t=0$: $\rho_t(t) \equiv \frac{\mathcal{U}(t)}{\mathrm{Tr}\,\mathcal{U}(0)}$]

Resemble usual finite temp entanglement: but imaginary temperature ($\beta = it$).

[Related quantities: time-evolution op with projection onto some state, *i.e.* $\mathcal{U}(t)|I\rangle\langle I| = |F_I(t)\rangle\langle I|$.]

$\Leftrightarrow$ Pseudo-entropy [entropy of reduced transition matrix (Nakata,Takayanagi,Taki,Tamaoka,Wei,'20)].

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[KN, Saini, '23]

Time Evol'n operator ρ_t	$\rightarrow$	"Components" of time Evol'n op $\rho_t^{\vert I\rangle} = \rho_t \vert I\rangle\langle I\vert$: ρ_t with projection onto initial state $\vert I\rangle$	$\leftarrow$	Pseudo-entropy from reduced transition matrix $\mathcal{T}^A_{F\vert I} = \mathrm{Tr}_B\left(\frac{\vert F\rangle\langle I\vert}{\mathrm{Tr}(\vert F\rangle\langle I\vert)}\right)$

"Time-Entanglement": reduced time evol^n op

[KN, Saini, '23]

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Bipartite system with Hamiltonian eigenstates $|i,i'\rangle$ and energies $E_{i,i'}$.

- Time evolution operator (normalized) and partial trace over $B \equiv \{i'\}$:

$$\rho_t = \mathcal{N}_t \sum\ e^{-iE_{i,i'}t}\,|i,i'\rangle\langle i,i'| \quad\rightarrow\quad \rho_t^A = \mathcal{N}_t\big(\textstyle\sum_{i'} e^{-iE_{i,i'}t}\big)\,|i\rangle\langle i| \quad \big[\mathcal{N}_t^{-1} = \textstyle\sum_{i,i'} e^{-iE_{i,i'}t}\big]$$

  - Pseudo-entropy: entropy of reduced transition matrix $\mathcal{T}^A_{F|I} = \mathrm{Tr}_B\Big(\frac{|F\rangle\langle I|}{\mathrm{Tr}(|F\rangle\langle I|)}\Big)$, with $|I\rangle = c_{i,i'}|i,i'\rangle$ and $|F\rangle = c'_{i,i'}|i,i'\rangle$. $[\mathcal{N}_{F|I}^{-1} = \sum\ c'_{i,i'}c^*_{i,i'}]$

$$\mathcal{T}_{F|I} = \mathcal{N}_{F|I}\sum\ c'_{i,i'}c^*_{j,j'}\,|i,i'\rangle\langle j,j'| \quad\rightarrow\quad \mathcal{T}^A_{F|I} = \mathcal{N}_{F|I}\big(\textstyle\sum_{i'} c'_{i,i'}c^*_{j,i'}\big)\,|i\rangle\langle j|$$

- Time evolution operator with projection onto state $|I\rangle = c_{k,k'}|k,k'\rangle$:

$\equiv$ pseudo-entropy reduced transition matrix for $|F\rangle$ being time-evolved from $|I\rangle$.

$$\rho_t^{|I\rangle} = \frac{\rho_t|I\rangle\langle I|}{\mathrm{Tr}(\rho_t|I\rangle\langle I|)} = \mathcal{N}_t^{|I\rangle}\sum c_{i,i'}c^*_{j,j'}e^{-iE_{i,i'}t}|i,i'\rangle\langle j,j'| \quad\rightarrow$$

$$\rho_t^{|I\rangle,A} = \mathcal{N}_t^{|I\rangle}\textstyle\sum_{i,j}\big(\sum_{i'} c_{i,i'}c^*_{j,i'}e^{-iE_{i,i'}t}\big)|i\rangle\langle j| \qquad \big[(\mathcal{N}_t^{|I\rangle})^{-1} = \sum\ |c_{i,i'}|^2 e^{-iE_{i,i'}t}\big]$$

Time Evol'n op as pseudo-entropy transition matrix (doubled Hilbert sp) $\Big\}$ $|\psi_I\rangle = \sum_i e^{iE_it/2}\,|i\rangle_1|i\rangle_2,\quad |\psi_F\rangle = \sum_i e^{-iE_it/2}\,|i\rangle_1|i\rangle_2;$ $\mathcal{T}_{F|I} = |\psi_F\rangle\langle\psi_I| \ \rightarrow\ \mathrm{Tr}_2\,\mathcal{T}_{F|I} = \sum_i e^{-iE_it}\,|i\rangle_1\langle i|_1 = \mathcal{U}(t)$

"Time-Entanglement", examples: 2-qubits etc

2-state system: $H|k\rangle = E_k|k\rangle$, $(k = 1, 2;\ \langle 1|2\rangle = 0)$; $|k\rangle_F \equiv |k(t)\rangle = e^{-iE_k t}|k\rangle_P$.

$\rho_t = \frac{1}{1+e^{i\theta}}\left(|1\rangle\langle 1| + e^{i\theta}|2\rangle\langle 2|\right)$, $\theta = -(E_2 - E_1)t$; 2-spin analogy: $|1\rangle \equiv |++\rangle$, $|2\rangle \equiv |--\rangle$

$\xrightarrow{\mathrm{Tr}_B}$ ρ_t^A $\rightarrow$ entropy $S_A^\theta = -\mathrm{tr}(\rho_t^A \log \rho_t^A) = -\frac{1}{1+e^{i\theta}} \log \frac{1}{1+e^{i\theta}} - \frac{1}{1+e^{-i\theta}} \log \frac{1}{1+e^{-i\theta}}$

Real-valued, oscillating in time, periodicity $\sim \frac{1}{\Delta E}$; unbounded at $t = \frac{(2n+1)\pi}{\Delta E}$; min $S_A^\theta = \log 2$ at $t = \frac{2n\pi}{\Delta E}$.

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General 2-qubit Hamiltonian $H = E_{11}|11\rangle\langle 11| + E_{22}|22\rangle\langle 22| + E_{12}(|12\rangle\langle 12| + |21\rangle\langle 21|)$

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Generically complex-valued von Neumann entropy. (mixed EE, imaginary temp $\beta = it$)

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Generically complex-valued von Neumann entropy. (mixed EE, imaginary temp $\beta = it$)

$\frac{\rho_t|I\rangle\langle I|}{\mathrm{Tr}(\rho_t|I\rangle\langle I|)}$: Projection onto Thermofield-double initial states $|I\rangle = \sum_{1,2} c_{ii}|ii\rangle$

$$\xrightarrow{Tr_2} \boxed{\rho_t^{|I\rangle,A} = \frac{1}{|c_{11}|^2+|c_{22}|^2 e^{i\theta}}\Big(|c_{11}|^2|1\rangle\langle 1| + |c_{22}|^2 e^{i\theta}|2\rangle\langle 2|\Big)} \qquad [\theta = -(E_{22}-E_{11})t]$$

$\equiv$ reduced transition matrix for $|I\rangle$ and $|F\rangle = \sum c_{ii} e^{-iE_{ii}t}|ii\rangle$ ($\longrightarrow$ pseudo-entropy).

Max. entangled (Bell-pair) states $|c_{11}|^2 = |c_{22}|^2 = \frac{1}{2} \longrightarrow S_A^\theta$ (2-state above). Min $S_A^\theta = \log 2 = \mathrm{EE}(|I\rangle)$.

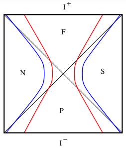
[Also 2-qutrits, qubit chains, uncoupled & coupled oscillators, CFT_2 ...; time-dep intns $\to$ no imag temp cont'n.]

dS future-past surfaces, time entanglement

dS future-past surfaces connecting I^+ to I^-.

(akin to Hartman-Maldacena surfaces (AdS bh) rotated)

Suggests future-past entanglement (betw $I^\pm$). Meaning?



Recall eternal AdS bh dual to $CFT_L \times CFT_R$ in TFD state (Maldacena)

Speculation: (Lorentzian) dS_4 approximately dual to $CFT_F \times CFT_P$ in thermofield-double-like entangled state $|\psi^{tfd}_{fp}\rangle = \sum \psi^{i^F_n, i^P_n} |i^F_n\rangle |i^P_n\rangle$?

Tracing over past copy gives mixed state at I^+.

KN '17; see also Arias,Diaz,Sundell,'19

(Witten, Strominger '01) bulk time evolution maps I^- to I^+, *i.e.* $|i^P_n\rangle \rightarrow |i^F_n\rangle \Rightarrow$ $|\psi^{tfd}_{fp}\rangle$ unitarily equivalent to $\sum \psi^{i^F_n, i^F_n} |i^F_n\rangle |i^F_n\rangle$ *i.e.* TFD-like state in two CFT copies at I^+.

These future-past TFD states give entirely positive structures (also ghost CFTs ▸ gs).

Connectedness of fp-TFD states & timelike entanglement $\leftrightarrow$ emergence of time?

van Raamsdonk: space emerges from entanglement. For factorized future-past states, $\mathrm{Tr}_P \rightarrow$ pure.

f-p TFD states: reduced transition matrix $\equiv$ time evolution operator; time evol'n $\equiv$ f-p EE. Timelike ER=EPR?

"Time-Entanglement": future-past EE

dS extremal surfaces at I^+ & areas $\equiv$ space-time rotations from AdS.
e.g. dS future-past surfaces $\leftrightarrow$ rotated Hartman-Maldacena surfaces (AdS bh).

Recall f-p surfaces suggest future-past entanglement $|\psi\rangle_{fp} = \sum \psi^{i_n^F, i_n^P} |i_n\rangle_F |i_n\rangle_P$.

f-p density matrix $|\psi\rangle_{fp}\langle\psi|_{fp} \xrightarrow{Tr_p}$ red. d.m., nontrivial EE.

Example, 2-state QM: $H|k\rangle = E_k|k\rangle, \quad k = 1,2; \quad |k\rangle_F \equiv |k(t)\rangle = e^{-iE_k t}|k\rangle_P. \quad [\langle 1|2\rangle = 0]$

$$|\psi\rangle_{fp} = \frac{1}{\sqrt{2}}|1\rangle_F|1\rangle_P + \frac{1}{\sqrt{2}}|2\rangle_F|2\rangle_P = \frac{1}{\sqrt{2}}e^{-iE_1 t}|1\rangle_P|1\rangle_P + \frac{1}{\sqrt{2}}e^{-iE_2 t}|2\rangle_P|2\rangle_P$$

fp-density matrix $\rho = |\psi\rangle_{fp}\langle\psi|_{fp} \xrightarrow{\mathrm{Tr}_P} \delta_{ij}\psi_{fp}^{ki}(\psi_{fp}^*)^{lj} \to$ time-evol'n phases cancel $\to$

$$\rho_{fp} = \mathrm{Tr}_P|\psi\rangle_{fp}\langle\psi|_{fp} = \frac{1}{2}|1\rangle_F\langle 1|_F + \frac{1}{2}|2\rangle_F\langle 2|_F$$

Now imagine 2-spin analogy with $|1\rangle = |++\rangle$, $|2\rangle = |--\rangle$: partial trace over second component $\to \quad Tr_2\rho_{fp} = \frac{1}{2}|+\rangle_F\langle +|_F + \frac{1}{2}|-\rangle_F\langle -|_F \to$ positive entropy $\log 2$.

[Similar positive structures with ghost-spins]

Future-past TFD state with timelike separation quite different in principle from usual TFD. Positive structures in f-p d.m. despite timelike separation.

de Sitter extremal surfaces,
analytic continuations, space-time rotations,
subregion duality, entropy relns, Lewkowycz-Maldacena

dS no-boundary surfaces, analytic cont'n

$$ds^2 = -\frac{dr^2}{\frac{r^2}{l^2}-1} + \left(\frac{r^2}{l^2}-1\right)dt^2 + r^2 d\Omega_{d-1}^2 \quad \xrightarrow{l\to iL} \quad ds^2 = -\left(1+\frac{r^2}{L^2}\right)dt^2 + \frac{dr^2}{1+\frac{r^2}{L^2}} + r^2 d\Omega_{d-1}^2$$

$r > l$, $dS \leftrightarrow AdS$ $\quad$ [$r < l$: $t_E \to -it = [0, \frac{\pi}{2}] \to dS$ bottom Eucl hemisphere $\leftrightarrow EAdS$]

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Analytic cont'n ≡ space↔time rotation: AdS RT surface from $r \to \infty$ (boundary) to $r = 0$ (and back) $\longrightarrow$ IR dS RT/HRT surface from $r \to \infty$ (future boundary) to $r = l$ (Lorentzian dS) going around Eucl hemisphere ($r = l$ to $r = 0$) (& back to $\mathcal{I}^+$).

dS RT/HRT surfaces, $t = const$ slice (natural metaobservers?)

[$r > l$] $ds^2 = -\frac{dr^2}{\frac{r^2}{l^2}-1} + r^2 d\Omega_{d-1}^2 \xrightarrow{l\to iL} ds^2 = \frac{dr^2}{1+\frac{r^2}{L^2}} + r^2 d\Omega_{d-1}^2$

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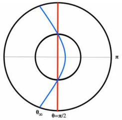
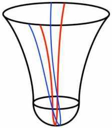




dS no-boundary surfaces, analytic cont'n

$ds^2 = -\frac{dr^2}{\frac{r^2}{l^2}-1} + \left(\frac{r^2}{l^2} - 1\right)dt^2 + r^2 d\Omega_{d-1}^2 \quad \xrightarrow{l\to iL} \quad ds^2 = -\left(1 + \frac{r^2}{L^2}\right)dt^2 + \frac{dr^2}{1+\frac{r^2}{L^2}} + r^2 d\Omega_{d-1}^2$

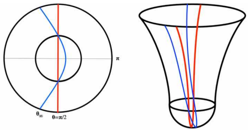
$r > l,\ dS \leftrightarrow AdS \quad [r < l\text{: } t_E \to -it = [0, \frac{\pi}{2}] \to dS$ bottom Eucl hemisphere $\leftrightarrow EAdS]$

Analytic cont'n $\equiv$ space$\leftrightarrow$time rotation: AdS RT surface from $r \to \infty$ (boundary) to $r = 0$ (and back) $\longrightarrow$ IR dS RT/HRT surface from $r \to \infty$ (future boundary) to $r = l$ (Lorentzian dS) going around Eucl hemisphere ($r = l$ to $r = 0$) (& back to $\mathcal{I}^+$).

dS RT/HRT surfaces, $t = const$ slice (natural metaobservers?)

$[r > l]\quad ds^2 = -\frac{dr^2}{\frac{r^2}{l^2}-1} + r^2 d\Omega_{d-1}^2 \quad \xrightarrow{l\to iL} \quad ds^2 = \frac{dr^2}{1+\frac{r^2}{L^2}} + r^2 d\Omega_{d-1}^2$

$[r < l]\quad ds^2 = \frac{dr^2}{1-\frac{r^2}{l^2}} + r^2 d\Omega_{d-1}^2 \quad \xrightarrow{l\to iL} \quad ds^2 = \frac{dr^2}{1+\frac{r^2}{L^2}} + r^2 d\Omega_{d-1}^2$



IR: max subregion

$\frac{V_{S^{d-2}}}{4G_{d+1}} \int_0^{R_c} \frac{r^{d-2}dr}{\sqrt{1+\frac{r^2}{L^2}}} \quad \xrightarrow{L\to -il} \quad \frac{V_{S^{d-2}}}{4G_{d+1}} \int_0^{l} \frac{r^{d-2}dr}{\sqrt{1-\frac{r^2}{l^2}}} + \frac{V_{S^{d-2}}}{4G_{d+1}} \int_l^{R_c} r^{d-2} \sqrt{\frac{-dr^2}{\frac{r^2}{l^2}-1}}$

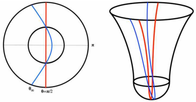
(blue: generic θ_∞)

$= \frac{1}{2}\frac{l^{d-1}V_{S^{d-1}}}{4G_{d+1}} - i\# \frac{l^{d-1}}{4G_{d+1}} \frac{R_c^{d-2}}{l^{d-2}} + \ \ldots$

$[dS_4\text{: } \frac{\pi L^2}{2G_4}\left(\frac{R_c}{L} - 1\right) \to -i\frac{\pi l^2}{2G_4}\frac{R_c}{l} + \frac{\pi l^2}{2G_4}] \qquad [dS_3\text{: } \frac{2L}{4G_3}\log\frac{R_c}{L} \to -i\frac{l}{2G_3}\log\frac{R_c}{l} + \frac{\pi l}{4G_3}]$

dS surfaces, subregion duality, geometrically

IR surface, $t = const$ slice, maximal subregion $\to$ red surface;
Generic subregion, blue: tilted "great circle" in hemisphere,
joining with tilted timelike surface in Lorentzian top half.
dS_3 explicitly solvable; dS_{d+1}, perturbatively analysed.



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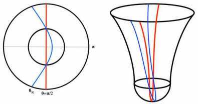
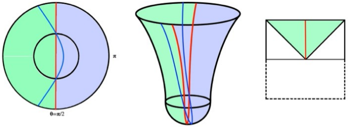


Time-entanglement/Pseudo-entanglement wedge: Max subregion, $t = const$ slice: green bulk region bounded by (red) IR surface and boundary subregion. (Violet complement region)



Including t-direction $\rightarrow$ top wedge (containing future of IR surface on vertical $t = const$ slice), bounded by I^+ subregion $\equiv$ analytic continuation from AdS. Space-time rotation from AdS EE wedge. (dS/CFT via relative entropy, modular flow etc?)

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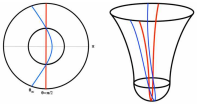

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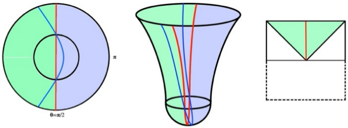

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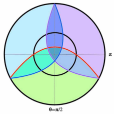
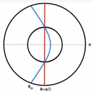
Space-time rotation from AdS EE wedge. (dS/CFT via relative entropy, modular flow etc?)

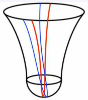
Multiple disjoint boundary subregions:
red, violet, blue no-boundary dS extremal surfaces.
Complex areas so entropy inequalities behave differently from AdS entanglement. Bulk subregions not disjoint.



[Note that $t = const$ slice geometric subregion duality is somewhat different from equatorial plane dS surfaces.]

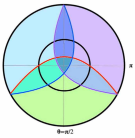
dS surfaces, entropy relations/inequalities





dS_3 : $S_t^{\theta_\infty} = -i\,\frac{l}{2G_3}\log\frac{R_c}{l} - i\,\frac{l}{4G_3}\log(\sin^2\theta_\infty) + \frac{\pi l}{4G_3}$

IR, $\theta_\infty = \frac{\pi}{2}$: $S_t^{IR} = -i\,\frac{l}{2G_3}\log\frac{R_c}{l} + \frac{\pi l}{4G_3}$



Two adjacent disjoint subregions A, B $(2\theta_\infty = \frac{\pi}{2})$; $A \cup B \equiv (2\theta_\infty = \pi)$.

"Mutual time-information" or "mutual pseudo-information":

$I_t[A,B] = S[A] + S[B] - S[A \cup B] = -i\,\frac{l}{2G_3}\log\frac{R_c}{l} + i\,\frac{l}{2G_3}\log 2 + \frac{\pi l}{4G_3}$

$\Rightarrow$ $\mathrm{Re}\, I_t \geq 0\,,$ $\mathrm{Im}\, I_t \leq 0.$ (antipodal subregions, $I_t = 0$)

dS surfaces, entropy relations/inequalities

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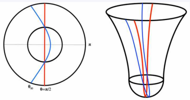

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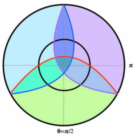

Tripartite time-information: 3 disjoint adjacent quadrant subregions A, B, C $(2\theta_\infty = \frac{\pi}{2})$.

$A\cup B$, $B \cup C$ maximal (IR) subregions. $A \cup C$, antipodal quadrants (extr. surf. = "inner" $(\equiv B)$ + "outer").

$S_A = S_B = S_C = S_t^{\pi/4}\,, \quad S_{AB} = S_{BC} = S_t^{\pi/2}\,, \quad S_{AC} = S_t^{\pi/4} + S_t^{\pi/4}\,, \quad S_{ABC} = S_t^{\pi/4};$

$I_3^t[A,B,C] = S_A + S_B + S_C - S_{AB} - S_{BC} - S_{AC} + S_{ABC} = i\frac{l}{2G_3}\log 2 \quad \Rightarrow \quad \mathrm{Im}\, I_3^t \geq 0.$

Strong subadditivity:
$$\left.\begin{array}{l} S_{AB} + S_{BC} - S_{ABC} - S_B = -i\frac{l}{2G_3}\log 2, \\ S_{AB} + S_{BC} - S_A - S_C = -i\frac{l}{2G_3}\log 2. \end{array}\right\} \qquad \begin{array}{l} \mathrm{Re}\, SSB_{1,2}^t \geq 0, \\ \mathrm{Im}\, SSB_{1,2}^t \leq 0 \end{array}$$

dS area/entropy relations special (relative to qubit system pseudo-entropies).

Note: AdS analytic continuation $il \to -L \quad \Rightarrow \quad MI \geq 0, \quad I_3 \leq 0, \quad SSB^{1,2} \geq 0.$

Consistent with AdS RT/HRT areas which are also special Hayden,Headrick,Maloney,'11.

Qubits, pseudo-entropy inequalities

Pseudo-entropy $\rho_t = \frac{|F\rangle\langle I|}{\mathrm{Tr}(|F\rangle\langle I|)} = \frac{\mathcal{U}(t)|I\rangle\langle I|}{\mathrm{Tr}(\mathcal{U}(t)|I\rangle\langle I|)}$ [= time evoln op $\mathcal{U}(t) = e^{-iHt}$ with projection]

for TFD-type initial state $|I\rangle$ and its time-evolved final state $|F\rangle = \mathcal{U}(t)|I\rangle$.

2-qubits: $|I\rangle = c_{11}|11\rangle + c_{22}|22\rangle\ ,\quad |F\rangle = c_{11}e^{-iE_{11}t}|11\rangle + c_{22}e^{-iE_{22}t}|22\rangle$

$\left[|c_{11}|^2 + |c_{22}|^2 = 1;\ \ |c_{11}|^2 \equiv x;\quad \theta = -(E_{22}-E_{11})t\right]$

$$\rho_t^1 = \mathrm{Tr}_2\,\rho_t\,,\quad \rho_t^2 = \mathrm{Tr}_1\,\rho_t\,,\quad \rho_t^2 = \rho_t^1 = \frac{1}{x+(1-x)e^{i\theta}}\Big(x|1\rangle\langle 1| + (1-x)e^{i\theta}|2\rangle\langle 2|\Big)\,,$$

$$S_t^2 = S_t^1 = -\frac{x}{x+(1-x)e^{i\theta}}\log\frac{x}{x+(1-x)e^{i\theta}} - \frac{(1-x)e^{i\theta}}{x+(1-x)e^{i\theta}}\log\frac{(1-x)e^{i\theta}}{x+(1-x)e^{i\theta}}$$

Near $t=0$: $S_t^1(t) \sim S_t^1(0) + \frac{d}{dt}S_t^1(0)\,t \equiv S_0\,,$

$$S_t^1(0) = -x\log x - (1-x)\log(1-x)\,,\qquad \frac{d}{dt}S_t^1(0) = -i\,\Delta E\,x(1-x)\log\frac{x}{1-x}$$

Mutual pseudo-information: $I_t[1,2] = S_t^1 + S_t^2 - S_t \sim 2S_0;\quad \mathrm{Re}I_t > 0\,,\quad \mathrm{Im}I_t \gtrless 0$

Qubits, pseudo-entropy inequalities

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3-qubits: $|I\rangle = c_{111}|111\rangle + c_{222}|222\rangle\ , \quad |F\rangle = c_{111}e^{-iE_{111}t}|111\rangle + c_{222}e^{-iE_{222}t}|222\rangle$

$$\rho_t^{123} = \frac{|F\rangle\langle I|}{\mathrm{Tr}(|F\rangle\langle I|)}\,, \quad \rho_t^1 = \mathrm{Tr}_{23}\,\rho_t^{123} = \frac{1}{x+(1-x)e^{i\theta}}\left(x|1\rangle\langle 1| + (1-x)e^{i\theta}|2\rangle\langle 2|\right), \quad \rho_t^2 = \rho_t^3 = \rho_t^1\,,$$

$$\rho_t^{12} = \mathrm{Tr}_3\,\rho_t^{123} = \frac{1}{x+(1-x)e^{i\theta}}\left(x|11\rangle\langle 11| + (1-x)e^{i\theta}|22\rangle\langle 22|\right), \quad \rho_t^{23} = \rho_t^{13} = \rho_t^{12}$$

Tripartite pseudo-information: $I_3^t[1,2,3] = S_t^1 + S_t^2 + S_t^3 - S_t^{23} - S_t^{13} - S_t^{12} + S_t^{123} = 0$

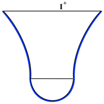
SSB: $SSB_1^t = S_t^{12} + S_t^{23} - S_t^{123} - S_t^2 = S_t^1; \quad SSB_2^t = S_t^{12} + S_t^{23} - S_t^1 - S_t^3 = 0$

$\mathrm{Re}\, SSB_1^t > 0\,, \quad \mathrm{Im}\, SSB_1^t \gtrless 0 \quad (x \neq \frac{1}{2})$

[Specific TFD states above; more general states?]

dS no-bndry surfaces, Lewkowycz-Maldacena

Hartle-Hawking Wavefunction of the Universe $\Psi_{dS}[h_{ij}, \phi_0] = \int_{nb}^{I^+} Dg\, D\phi\ e^{iS[g,\phi]}$. Best regarded as amplitude (transition matrix) for creating universe $M[h_{ij}]$ with final bndry condns (h_{ij} at I^+) from "nothing", *i.e.* satisfying Hartle-Hawking no-boundary condition.



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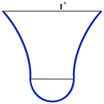

Semiclassically $\Psi_{dS} \sim e^{iS^{(r>l)}}\, e^{S_E^{(r<l)}}$.

Top Lorentzian part (real $S^{(r>l)}$) = pure phase.

Bottom hemisphere: Lorentzian iS_{cl} continues to Eucl gravity action $S_E^{(r<l)} = -\int_{nbp} \sqrt{g}\,(R - 2\Lambda) \longrightarrow \frac{1}{2}\,\frac{l^4 V_{S^4}}{16\pi G_4}\,\frac{6}{l^2} = \frac{\pi l^2}{2G_4}$ for dS_4 (nbp is $\tau_E = \frac{\pi}{2}$).

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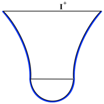

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Lewkowycz-Maldacena: bulk AdS replica dual to boundary replica EE argument.
$Z_{CFT} = Z_{bulk} \Rightarrow$ boundary entanglement entropy = bulk entanglement entropy.

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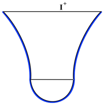

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dS/CFT: $Z_{CFT} = \Psi_{dS} \Rightarrow$ boundary replica via $Z_{CFT} \longrightarrow$ bulk replica on Wavefunction Ψ_{dS} (single ket, not density matrix).

This is Pseudo-Entropy (entropy of transition matrix).

Intrinsic object = $\Psi_{dS} \Rightarrow$ non-Hermitian. Complex areas expected (imaginary parts from Lorentzian part of Ψ_{dS}).

dS no-bndry surfaces, Lewkowycz-Maldacena

LM AdS review

Subregion, const time slice $\rightarrow$ bndry, codim-2 $\rightarrow$ codim-2 bulk extremal surface.

Replica space: n copies permuted by $\mathbb{Z}_n$ replica symmetry.

Bndry replica space M_n extends into a smooth bulk replica covering space $\mathcal{B}_n$.

(smooth covering space, replica bndry condns for gluing n copies cyclically)

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(bndry $= \partial\tilde{\mathcal{B}}_n = M_n/\mathbb{Z}_n = M_1$, original bndry space)

$\rightarrow$ conical (orbifold) singularities $\equiv$ bulk $\mathbb{Z}_n$ fixed points ($n \neq 1$).

Like 4d cosmic string geometry: locally each of n copies in $\mathcal{B}_n$ same as single copy.

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Like 4d cosmic string geometry: locally each of n copies in $\mathcal{B}_n$ same as single copy.

$\Rightarrow$ adding codim-2 cosmic brane source (wrapping transverse directions, area A) will desingularize conical singularities. [deficit angle $2\pi - \frac{2\pi}{n}$]

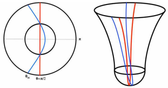
Smooth action $I_n = nI_1 + I_{brane} = nI_1 + \frac{n-1}{n}\frac{A}{4G} \xrightarrow{n=1+\epsilon} (1+\epsilon)I_1 + \epsilon\,\frac{A}{4G}$.

Semiclassically $Z_n = Z[\tilde{\mathcal{B}}_n] \sim e^{-I_n}$ so entropy via replica is

$$S = -\lim_{n\rightarrow 1} n\partial_n(\log Z_n - n\log Z_1) = \lim_{n\rightarrow 1}(1-n\partial_n)\log Z_n = -\lim_{n\rightarrow 1}(1-n\partial_n)I_n$$

$$= \frac{A}{4G} = \text{area of RT/HRT entangling surface.}$$

dS no-bndry surfaces, Lewkowycz-Maldacena



LM dS: consider IR surface on $t = const$ slice. Twist operators at $\theta = \pm\frac{\pi}{2}$ endpoints of hemispherical maximal subregion $\in I^+$ $\rightarrow n$ copies of Wavefunction Ψ_{dS} glued appropriately.

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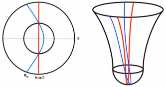

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Analytic cont'n: $Z^{AdS}_{bulk} \sim e^{-I_{bulk}} \longrightarrow \Psi_{dS} \sim e^{iS_{cl}} = e^{iS^{(r>l)}}\, e^{S_E^{(r<l)}}$ (single copy).

Replica, n copies: $\frac{Z_n}{Z_1^n} \longrightarrow \frac{\Psi_n}{\Psi_1^n}$ (Ψ_n, quotient bulk replica space, replica bndry condns @I^+)

Semiclassically: $Z_n \sim e^{-I_n} \longrightarrow \Psi_n \sim e^{iS_n}$; $\quad -I_n \to iS_n = iS_n^{(r>l)} + S_E^{(r<l)}$

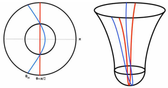
Orbifold singularities in Ψ_n ($n \neq 1$) smoothed out by codim-2 cosmic brane.

$n \to 1$ limit: satisfies HH no-boundary condn, wraps will-be no-bndry dS surface.

$$-I_{brane} \to \frac{1-n}{n}\frac{A_{brane}}{4G} \qquad (I_n = nI_1 + I_{brane})$$

Pseudo-entropy: $S_t = \lim_{n\to 1}(1 - n\partial_n)\log\Psi_n = \lim_{n\to 1}(1-n\partial_n)\,\frac{1-n}{n}\,\frac{A_{brane}}{4G} = \frac{A^{dS}_{brane}}{4G}$

dS no-bndry surfaces, Lewkowycz-Maldacena



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Cosmic brane not spacelike $\leftrightarrow$ Euclidean + Lorentzian (timelike) no-bndry dS extremal surface. Complex area (pure imaginary top timelike part) reasonable.

LM replica formulation: entropy = area of codim-2 brane created from "nothing".

Amplitude for this process divergent if Lorentzian part (which goes all the way to late times) were real. Here timelike part = pure phase cancels in probability (finite: bounded real part from hemisphere, set by dS entropy).

Conclusions, questions

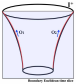
- Future boundary: no $I^+ \to I^+$ turning point. Surfaces do not return to I^+.

(a) Future-past surfaces, end at past boundary I^-. Pure imaginary area.

Suggest a $CFT_F \times CFT_P$ dual in f-p TFD-like entangled state.

(b) No-boundary surfaces, top timelike f-p joined with Eucl surface

in bottom hemisphere. Real finite part of area is half dS entropy.



codim-2 surfaces $\leftrightarrow$ antipodal metaobservers?

$\to$ *Pseudo-entropy*. AdS, analytic cont'n $\equiv$ space $\leftrightarrow$ time rotations.

pseudo-entanglement wedge, entropy inequalities, Lewkowycz-Maldacena.

Various new features. Deeper understanding? More generally, extremal surfaces, cosmology, past boundary condns?

- Spatial infinity boundary (AdS) rotated to timelike infinity boundary (dS): spacelike RT/HRT surface (real area) rotated, includes timelike components (complex area).

Time-entanglement/Pseudo-entropy: entanglement-like structures, timelike separations:

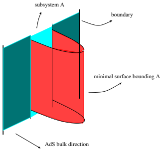
(i) reduced time evolution operator, mixed state EE + imaginary temperature
$\leftrightarrow$ reduced transition amplitudes, pseudo-entropy, ...

(ii) positivity in future-past entangled states & density matrices.

Holographic entanglement entropy

Entanglement entropy: entropy of reduced density matrix of subsystem.

EE for spatial subsystem A, $S_A = -tr\rho_A \log \rho_A$, with partial trace $\rho_A = tr_B \rho$.



Ryu-Takayanagi: $EE = \frac{A_{min.surf.}}{4G}$

[$\sim$ black hole entropy] Area of codim-2 minimal surface in gravity dual.

Non-static situations: extremal surfaces (Hubeny, Rangamani, Takayanagi).

Operationally: Const time slice, boundary subsystem $\rightarrow$ bulk slice, codim-2 extremal surface.

Ex.: CFT$_d$ ground state = empty AdS_{d+1}, $ds^2 = \frac{R^2}{r^2}(dr^2 - dt^2 + dx_i^2)$. Strip, width $\Delta x = l$, infinitely long.

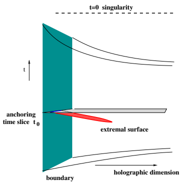
Bulk surface $x(r)$. Turning point r_*. $S_A = \frac{V_{d-2} R^{d-1}}{4G_{d+1}} \int \frac{dr}{r^{d-1}} \sqrt{1 + (\partial_r x)^2}$ $\rightarrow$ extremize $\rightarrow$

$$(\partial_r x)^2 = \frac{(r/r_*)^{2d-2}}{1-(r/r_*)^{2d-2}}\,, \qquad \frac{l}{2} = \int_0^{r_*} dr \partial_r x.$$

$$S_A = \frac{V_{d-2} R^{d-1}}{4G_{d+1}} \int_\epsilon^{r_*} \frac{dr}{r^{d-1}} \frac{2}{\sqrt{1-(r/r_*)^{2d-2}}}$$

[2d] $\quad S_A = \frac{R}{2G_3} \log \frac{l}{\epsilon}\,, \qquad \frac{3R}{2G_3} = c.$

CFT thermal state (AdS black brane): minimal surface wraps horizon. $S^{fin} \sim \frac{R^{d-1}}{G_{d+1}} T^{d-1} V_{d-2} l$



AdS Kasner:

spacelike extremal surfaces in reliable semiclassical region far from singularity.

Surface lies almost on $t = const$ slice and bends away from singularity.

(Manu,KN,Paul,'20)

Back

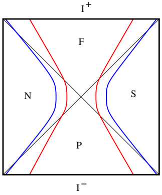
de Sitter future-past surfaces

Entirely timelike surface so overall $-i$ in area S $\quad\rightarrow\quad$ $S = -i\frac{l^{d-1}V_{S^{d-2}}}{4G_{d+1}}\int\frac{d\tau}{\tau^{d-1}}\sqrt{\frac{1}{f}-f(w')^2}$

Boundary Eucl time slice: $S^{d-2}\in S^{d-1}$; codim-2 surfaces wrap S^{d-2} [all S^{d-1} equatorial planes equivalent]

Extremize $\rightarrow$ $\quad \dot{w}^2 \equiv (1-\tau^2)^2(w')^2 = \frac{B^2\tau^{2d-2}}{1-\tau^2+B^2\tau^{2d-2}}$

B=const, $\quad S = -i\frac{2l^{d-1}V_{S^{d-2}}}{4G_{d+1}}\int_\epsilon^{\tau_*}\frac{d\tau}{\tau^{d-1}}\frac{1}{\sqrt{1-\tau^2+B^2\tau^{2d-2}}}$



Future-past surfaces stretching from I^+ to I^- [KN'17]

Hartman-Maldacena surfaces (AdS bh) rotated.

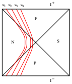
[real turning point τ_* at $|\dot{w}|\rightarrow\infty$: $1-\tau_*^2+B^2\tau_*^{2d-2}=0$]

Limiting surface as $\Delta w\rightarrow\infty$, whole space at $I^\pm$ $\quad$ (dS_4 : $B\rightarrow\frac{1}{2}$: $\tau_*\rightarrow\sqrt{2}$.)

Area law divergence $S^{div}\sim -i\frac{\pi l^2}{G_4}\frac{l}{\epsilon_c}$; $\quad$ Finite part $S^{fin}\sim -i\frac{\pi l^2}{G_4}\Delta w$

Scaling: de Sitter entropy $\rightarrow$ akin to number of degrees of freedom in dual CFT.

Suggest TFD-like entangled dual of two CFT copies at I^+. $\qquad$ [AdS_4 BH RT-EE $\sim\frac{R^2}{G_4}(\frac{V}{\epsilon}+\#T^2Vl)$]



Vanishing mutual information, SSB saturated, “entanglement wedge”, subregion duality, . . .

$A\equiv(w_1,w_2)$, $B\equiv(w_3,w_4)\rightarrow S[\mathcal{A}\cup\mathcal{B}] = S[w_1]+S[w_2]+S[w_3]+S[w_4] = S[\mathcal{A}]+S[\mathcal{B}]$

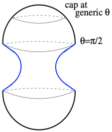
Back

de Sitter future-past surfaces

[Future-past surfaces: entirely timelike surface, overall $-i$ in area.]

dS global: sphere foliations. $ds^2_{d+1} = -d\tau^2 + l^2 \cosh^2 \frac{\tau}{l} d\Omega^2_d$.

Bndry Eucl time: any S^d equatorial plane. Cap-like subregion ($I^{\pm}$): $\theta = const$ latitude on S^{d-1}.



Future-past surfaces stretching betw caps at $I^{\pm}$, wrapping S^{d-2}.

$$S = -i \frac{2l^{d-2} V_{S^{d-2}}}{4G_{d+1}} \int d\tau \ (\cosh \tau)^{d-2} \ (\sin \theta)^{d-2} \sqrt{1 - \cosh^2 \tau \, (\partial_\tau \theta)^2}$$

IR $\rightarrow \theta = \frac{\pi}{2}$: $S = -i \frac{\pi l^2}{G_4} \int_0^{\tau_c/l} d\tau \ \cosh \tau \sim -i \frac{\pi l^2}{2G_4} \frac{l}{T_c}$. [$dS_4$]

Area law divergence, no finite part. [cutoff $T_c = l e^{-\tau_c/l} \sim 0$ near $\tau_c \rightarrow \infty$]

$$ds^2_{global}\big|_{\theta_d = const} = -d\tau^2 + l^2 \cosh^2 \frac{\tau}{l} \, d\Omega^2_{d-1} \equiv ds^2_{static}\big|_{t=const} \quad [r = l \cosh \frac{\tau}{l}]$$

dS static: Bndry Eucl time slice: $t = const$ slice [S^{d-1} eq.planes earlier]

[t is Killing time in AdS BH analogy before rotating to dS; also Killing time t in static patch.]

Cap-like subregion ($I^{\pm}$): $\theta = const$ latitude on S^{d-1}. [generic θ difficult to analyse explicitly.]

$\theta = \frac{\pi}{2}$ $\rightarrow$ simplifications $\rightarrow$ extremal future-past surface $\longrightarrow$ Area $S \xrightarrow{dS_4} -i \frac{\pi l^2}{G_4} \frac{1}{\epsilon}$

Back

Entanglement in ghost theories: "ghost-spins"

KN; Jatkar,KN; Jatkar,Kolekar,KN

- Replica arguments (Calabrese, Cardy) can be generalized to $c = -2$ ghost CFTs: twist operator 2-pt fn $\rightarrow$ $c < 0 \Rightarrow S < 0$. $[|\downarrow\rangle = |0\rangle; \ \langle -Q|T(z)|0\rangle = 0]$

- "Ghost-spin" $\rightarrow$ 2-state spin variable with indefinite norm.
  $\langle\uparrow|\downarrow\rangle = \langle\downarrow|\uparrow\rangle = 1, \qquad \langle\uparrow|\uparrow\rangle = \langle\downarrow|\downarrow\rangle = 0$

  [ordinary spin: $\langle\uparrow|\uparrow\rangle = 1 = \langle\downarrow|\downarrow\rangle$]

  $|\pm\rangle \equiv \frac{1}{\sqrt{2}}(|\uparrow\rangle \pm |\downarrow\rangle); \quad \langle\pm|\pm\rangle = \gamma_{\pm\pm} = \pm 1, \quad \langle +|-\rangle = \langle -|+\rangle = 0$

  Infinite ghost-spin chains, $\langle nn\rangle$-intns $\rightarrow$ continuum limit $\rightarrow$ bc-ghost CFT.

  Back

- Two ghost-spins: $|\psi\rangle = \sum \psi^{ij}|i\rangle|j\rangle \ \rightarrow \rho = |\psi\rangle\langle\psi| \ \rightarrow$ partial trace $\rightarrow$ RDM for remaining ghost-spin $\rightarrow$ von Neumann entropy.

$$\langle\psi|\psi\rangle = \gamma_{ik}\gamma_{jl}\psi^{ij}\psi^{kl*} = |\psi^{++}|^2 + |\psi^{--}|^2 - |\psi^{+-}|^2 - |\psi^{-+}|^2 = \pm 1$$

$$\text{RDM: } (\rho_A)^{ik} = \gamma_{jl}\psi^{ij}\psi^{kl*}; \qquad \text{EE: } S_A = -\gamma_{ij}(\rho_A \log \rho_A)^{ij}$$

In general: (i) $+ve$ norm $|\psi\rangle \not\Rightarrow +ve$ RDM, EE. (ii) new entanglement patterns.

e.g. $(\rho_A)_i^k e_k = \lambda e_i$ *i.e.* $(\rho_A)^{ij} e_j = \gamma^{ij}\lambda e_j$. -ve norm $\Rightarrow$ eigenvalues λ in general complex.

- Entangle identical ghost-spins from each copy $\rightarrow$ $+ve$ norm, RDM, EE

  $|\psi\rangle = \psi^{++}|+\rangle|+\rangle + \psi^{--}|-\rangle|-\rangle \Rightarrow \langle\psi|\psi\rangle > 0 \quad \longrightarrow$ correlated ghost-spins

  Also true for 2 copies of ghost-spin ensembles: $|\psi\rangle = \sum_{|\sigma_n\rangle} \psi^{\sigma_n,\sigma_n}|\sigma_n\rangle|\sigma_n\rangle, \ \langle\psi|\psi\rangle > 0$

Quantum extremal surfaces

Exciting recent developments in black hole information paradox

Penington; Almheiri,Engelhardt,Marolf,Maxfield; Almheiri,Mahajan,Maldacena,Zhao; ... .

New insights from generalized entropy and quantum extremal surfaces.

2-dim CFT techniques → adapt Calabrese,Cardy formula for subleading bulk matter contribution to entanglement entropy.

$$S_{gen} = S_{cl} + S_{bulk} = \frac{\phi}{4G_2} + \frac{c}{12} \log\left(\Delta^2\ e^f\big|_{(t,r)}\right) + \ \ldots \qquad (\Delta^2 = (\Delta r)^2 - (\Delta t)^2)$$

→ extremize → QES. (retaining only terms relevant for extremization)

- S_{gen} pertains to 2-dim theory from dim. red'n of higher dim space.
- subsystems here are full space (higher dim transverse space compactified): QES is point in 2d space → IR limit of higher dim RT/HRT.
- $1 \ll c \ll \frac{1}{G}$: classical area term S_{cl} dominant but S_{bulk} appreciable.
- If S_{bulk} overpowers S_{cl}, Bekenstein bound violated → islands.
  Hartman,Jiang,Shagoulian

Quantum extremal surfaces

Manu,KN,Paul
[2d redux, Bhattacharya,KN,Paul]

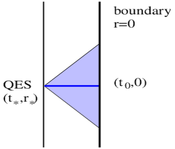
[holographic boundary observer at $(t_0, 0)$]

$$S_{gen} = \frac{\phi}{4G_2} + \frac{c}{12}\log\left(\Delta^2\, e^f\big|_{(t,r)}\right) + \ \ldots \qquad (\Delta^2 = r^2 - (t-t_0)^2)$$

Time-independent examples: everything is on const time slice ($t = t_0$ above).

AdS_{d_i+2} reduction (earlier): $\phi = \frac{R^{d_i}}{r^{d_i}}, \quad ds^2 = \frac{R^{d_i+1}}{r^{d_i+1}}(-dt^2 + dr^2)$

$$S_{gen} = \frac{\phi_r}{4G}\,\frac{R^{d_i}}{r^{d_i}} + \frac{c}{12}\log\left(\frac{r^2/\epsilon_{UV}^2}{(r/R)^{d_i+1}}\right) \ \Rightarrow\ \partial_r S_{gen} = -\frac{\phi_r}{4G}\,\frac{d_i R^{d_i}}{r^{d_i+1}} - \frac{c}{6}\left(\frac{d_i-1}{2}\right)\frac{1}{r} = 0$$



Both terms negative: $c > 0$ and $d_i > 1$

$\Rightarrow\ \ r_* \to \infty\ \ \to$ entire Poincare wedge

$\equiv$ usual AdS_{d_i+2} entanglement wedge. (maximin)

[$S^{o.s.} \sim 0$: AdS ground state]

Can be recast as $S_{gen} \sim \frac{\phi}{4G} + \frac{c}{12}\,\frac{d_i-1}{d_i}\,\log\phi$

$\Rightarrow$ S_{bulk} is subleading to classical area [ϕ not too small]

$\to$ Bekenstein bound not violated $\to$ no islands.

Quantum extremal surfaces: de Sitter (Poincare)

Goswami,KN,Saini

dS_{d_i+2}: $ds^2 = \frac{R^2}{\tau^2}(-d\tau^2 + dx^2 + dy_i^2) \ \rightarrow\ \phi = \frac{R^{d_i}}{(-\tau)^{d_i}}\,, \quad ds^2 = \frac{R^{d_i+1}}{(-\tau)^{d_i+1}}(-d\tau^2 + dx^2)$

Generalized entropy: $S_{gen} = \frac{\phi_r}{4G}\frac{R^{d_i}}{(-\tau)^{d_i}} + \frac{c}{6}\log\left(\Delta^2 \frac{R^{(d_i+1)/2}}{(-\tau)^{(d_i+1)/2}}\right), \quad \Delta^2 = (\Delta x)^2 - (\tau-\tau_0)^2$

Extremization: $\frac{c}{3}\frac{\Delta x}{\Delta^2} = 0\,, \quad \frac{d_i\phi_r}{4G}\frac{R^{d_i}}{(-\tau)^{d_i+1}} + \frac{c}{12}\frac{d_i+1}{(-\tau)} - \frac{c}{3}\frac{\tau-\tau_0}{\Delta^2} = 0$

- Timelike-separated QES: ($d_i = 1 \leftrightarrow dS_2$, Chen,Gorbenko,Maldacena)

  $$\Delta x = 0\,, \quad \Delta^2 = -(\tau-\tau_0)^2\,; \qquad \frac{d_i\phi_r}{4G}\frac{R^{d_i}}{(-\tau)^{d_i+1}} + \frac{c}{12}\frac{d_i+1}{(-\tau)} + \frac{c}{3}\frac{1}{\tau-\tau_0} = 0$$

  Late-time observer $\tau_0 \sim 0$: $\Delta x = 0\,, \quad \tau_* = -R\left(\frac{d_i}{3-d_i}\frac{3\phi_r}{Gc}\right)^{1/d_i}$

  Timelike-separated $\Rightarrow \Delta^2 < 0 \rightarrow$ generalized entropy acquires imaginary part.

- Spacelike-separated QES: exist in certain regimes with spatial regulator.

  $$\Delta^2 \sim R_c^2\,, \quad \frac{d_i\phi_r}{4G}\frac{R^{d_i}}{(-\tau)^{d_i+1}} + \frac{c}{12}\frac{d_i+1}{(-\tau)} \sim \frac{c}{3}\frac{\tau-\tau_0}{R_c^2}.$$

  * $R_c \rightarrow \infty \ \Rightarrow \tau \rightarrow -\infty$. * Late-times $\rightarrow$ no real solution.

Back

FRW, scalar source $p = w\rho$: $ds^2 = -dt^2 + a(t)^2 dx_i^2 \ \rightarrow\ \phi = a^{d_i}\,, \quad ds^2 = a^{d_i+1}(-d\tau^2 + dx^2)$