Lifshitz spacetimes from AdS null and cosmological solutions

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arXiv:1005.3291, Koushik Balasubramanian, KN.

[Also: hep-th/0602107, 0610053, Das, Michelson, KN, Trivedi;
arXiv:0711.2994, Awad, Das, KN, Trivedi;
arXiv:0807.1517, Awad, Das, Nampuri, KN, Trivedi.]

- Basic setup: Lifshitz spacetimes
- AdS/CFT with null and cosmological deformations
- Null: dim'nal reduction, holographic correlators, field theory
- Spacelike and anisotropic Lifshitz, AdS_4 and M-theory ...

AdS/condmat and Lifshitz systems

Interesting to explore the space of physical systems accessible within string theory, incorporating key qualitative physical features.

Generalizations of AdS/CFT to nonrelativistic systems $\rightarrow$ holographic condensed matter. Son; Balasubramanian,McGreevy; Adams et al; Herzog et al; Maldacena et al; ...

Lifshitz fixed points: arise in magnetic systems with antiferromagnetic interactions, dimer models, liquid crystals, ...

Symmetries: t, x_i-translations, x_i-rotations, anisotropic scaling $t \rightarrow \lambda^z t,\ x_i \rightarrow \lambda x_i$ [z: dynamical exponent].

Lifshitz spacetime: $ds_4^2 = -\frac{dt^2}{r^{2z}} + \frac{dx_i^2+dr^2}{r^2}$. Kachru,Liu,Mulligan

$[z = 1 :\ AdS]$ Scaling: $t \rightarrow \lambda^z t,\ x_i \rightarrow \lambda x_i,\ r \rightarrow \lambda r$.

Solution to 4-dim gravity with $\Lambda < 0$ and massive gauge field $A \sim \frac{dt}{r^z}$ (or alternatively gauge field + 2-form: dualize to get A-mass)

Lifshitz from string constructions?

Many holographic condmat investigations are at the level of effective gravity actions with appropriate matter content.

If basic qualitative features of a certain physical system are realizable holographically: string constructions? Expect to learn interesting things from string embedding, more microsopic description.

Naive expectations: $Li_d \times M^{10-d}$ or $Li_d \times M^{11-d}$ solution in string or M-theory. Supported by extra fluxes in the compact space M^{10-d}.

(Takayanagi et al) These ansatze (+warping) do not work. Basic violations — flux positivity, existence of certain forms assumed etc.

$z = \frac{3}{2}$ Lifshitz solutions from D3/D7-constructions (Takayanagi et al): necessarily anistropic, require scalar that breaks scaling symmetry. Similar solutions exist, where geometry is Lifshitz but scalar or other field breaks symmetry.

We'll describe alternative ways to construct $z = 2$ Lifshitz.

AdS null/cosmological deformations

$AdS_5 \times S^5 :$ $\quad ds^2 = \frac{1}{r^2}(\eta_{\mu\nu}dx^\mu dx^\nu + dr^2) + ds^2_{S^5}$ (Poincare),
with 5-form field strength and dilaton (scalar) $\Phi = const$.
Turn on non-normalizable deformations for metric, dilaton (DMNT):

$$ds^2 = \frac{1}{r^2}(\tilde{g}_{\mu\nu}dx^\mu dx^\nu + dr^2) + ds^2_{S^5} , \qquad \Phi = \Phi(x^\mu) .$$

This is a solution in string theory if

$$\tilde{R}_{\mu\nu} = \frac{1}{2}\partial_\mu\Phi\partial_\nu\Phi , \qquad \frac{1}{\sqrt{-\tilde{g}}}\,\partial_\mu(\sqrt{-\tilde{g}}\;\tilde{g}^{\mu\nu}\partial_\nu\Phi) = 0 ,$$

$i.e.$ if it is a solution to a 4-dim Einstein-dilaton system.
We'll be most interested in: $\Phi = \Phi(x^+)$ or $\Phi = \Phi(t)$.

General family of solutions: ($Z(x^m)$ harmonic function)

$$ds^2 = Z^{-1/2}\tilde{g}_{\mu\nu}dx^\mu dx^\nu + Z^{1/2}g_{mn}dx^m dx^n , \quad \Phi = \Phi(x^\mu),$$

$g_{mn}(x^m)$ is Ricci flat, and $\tilde{g}_{\mu\nu} = \tilde{g}_{\mu\nu}(x^\mu)$. $[\mu = 0123, m = 4\ldots 9.]$
AdS_4 solutions and M-theory: later.

AdS null deformations

$$ds^2 = \frac{1}{r^2}[e^{f(x^+)}(-2dx^+dy^- + dx_i^2) + dr^2] + ds^2_{S^5}\ , \qquad \Phi(x^+).$$

Constraint on these solutions: $\quad R_{++} = \frac{1}{2}(f')^2 - f'' = \frac{1}{2}(\Phi')^2\ .$

Function-worth of solutions: generically can choose $\Phi(x^+)$.

The coord. transfmn. $w = re^{-f/2}, \quad x^- = y^- - \frac{w^2 f'}{4}\ ,$ gives

$$ds^2 = \frac{1}{w^2}[-2dx^+dx^- + dx_i^2 + \frac{1}{4}w^2(\Phi')^2(dx^+)^2] + \frac{dw^2}{w^2}\ .$$

Now boundary at $w = 0$ manifestly flat 4D Minkowski spacetime.

This is a general feature: often possible to find new coordinates such that boundary metric $ds_4^2 = \lim_{r\to 0} r^2 ds_5^2$ is flat, at least as an expansion about the boundary ($r = 0$) if not exactly.

These are Penrose-Brown-Henneaux (PBH) transformations: subset of bulk diffeomorphisms leaving metric invariant (in Fefferman-Graham form), acting as Weyl transformation on boundary.

AdS null deformations

Conformal: $ds^2 = \frac{1}{r^2}[e^{f(x^+)}(-2dx^+dy^- + dx_i^2) + dr^2]$

Lightlike $\Rightarrow$ no nonzero contraction in curvature invariants. Potential singularities at $r = \infty$ likely absent. Tidal forces?

Earlier cosmological interpretation: $x^+ \equiv$ $lightcone\ time$. Bulk cosmological singularities (Bang/Crunch): choose $e^f \rightarrow 0$. Time-dependent cplng $g_{YM}^2 = e^{\Phi(x^+)}$ in dual gauge theory.

PBH: $ds^2 = \frac{1}{w^2}[-2dx^+dx^- + dx_i^2 + \frac{1}{4}w^2(\Phi')^2(dx^+)^2] + \frac{dw^2}{w^2}$

Regard x^+ as a compact coordinate with fixed size: $x^- \equiv t$ (time).

Consistent: $g_{++} \sim (\Phi')^2 > 0$ suggests ∂_+ is spacelike vector.

Strictly speaking: $x^+ = const$ surfaces are null surfaces, with null normal dx^+ ($g^{++} = 0$). $x^- = const$ surfaces spacelike ($g^{--} < 0$).

Corroborates that in PBH system, natural to take $x^- \sim$ $time$.

Choose e^f, Φ periodic in x^+, regular. Spacetime likely regular.

Tidal forces? (recall tidal forces in Lifshitz spacetimes)

AdS null deformations: symmetries

PBH: $ds^2 = \frac{1}{w^2}[-2dx^+dx^- + dx_i^2 + \frac{1}{4}w^2(\Phi')^2(dx^+)^2] + \frac{dw^2}{w^2}$

Symmetries: x^-, x_i-translations, x_i-rotations, $z = 2$ scaling

$x^- \equiv t \to \lambda^2 t,\ \ x_i \to \lambda x_i,\ \ w \to \lambda w$ (x^+ compact, no scaling).

x^+ noncompact: lightlike boosts $x^+ \to \lambda x^+,\ x^- \to \frac{1}{\lambda}x^-$.

x^+ compact $\Rightarrow$ lightlike boosts broken: they change physical parameters in compactified nonrelativistic theory which we hold fixed.

$g_{++} \sim (\Phi')^2 = 0$: AdS in lightcone coordinates with x^+ compactified — nonrelativistic, Schrodinger (Galilean) symmetries

[Goldberger, Barbon et al, Maldacena et al].

Galilean boosts $x_i \to x_i - v_i x^-,\ x^+ \to x^+ - \frac{1}{2}(2v_i x_i - v_i^2 x^-)$: broken here. $z = 2$ special conformal symmetry also broken.

Nontrivial x^+-dependence $\Rightarrow z = 2$ Galilean broken to Lifshitz.

Symmetries can also be seen in conformal coordinates earlier.

Dimensional reduction

PBH: $ds^2 = \frac{1}{w^2}[-2dx^+dx^- + dx_i^2 + \frac{1}{4}w^2(\Phi')^2(dx^+)^2] + \frac{dw^2}{w^2}$.

Naive, standard Kaluza-Klein reduction along x^+ as:

$ds^2 = g_{mn}dx^m dx^n = G_{\mu\nu}dx^\mu dx^\nu + G_{dd}(x^d + A_\mu dx^\mu)^2$.

Then $\{g_{++}, g_{+-}\}$-terms: $\frac{1}{w^2}[\frac{1}{4}w^2(\Phi')^2(dx^+)^2 - 2dx^+dx^-]$

$$= \frac{1}{4}(\Phi')^2[dx^+ - \frac{4dx^-}{w^2(\Phi')^2}]^2 - \frac{4(dx^-)^2}{w^4(\Phi')^2} .$$

Long-wavelength bulk 4-dim metric: $ds^2 = -\frac{4(dx^-)^2}{w^4(\Phi')^2} + \frac{dx_i^2}{w^2} + \frac{dw^2}{w^2}$.

$z = 2$ scaling: $x^- \equiv t \to \lambda^2 t, \quad x_i \to \lambda x_i, \quad w \to \lambda w$.

However, annoying remnant factor $(\Phi')^2$ with x^+-dependence.

Naive dim'nal reduction not valid here: nontrivial dependence of metric, scalar on compact x^+-dimension.

Must examine dim'nal reduction more carefully.

Dim'nal reduction, rigorously

Construct "minimal" off-shell metric ansatz containing PBH metric

$$[ds^2 = \frac{1}{w^2}[-2dx^+dx^- + dx_i^2 + \frac{1}{4}w^2(\Phi')^2(dx^+)^2] + \frac{dw^2}{w^2}]\ .$$

$$ds^2 = -N^2(x^+)K^2(s^i)dt^2 + \frac{1}{N^2(x^+)}(dx^+ + N^2(x^+)A)^2 + \frac{1}{w^2}(ds^i)^2$$

$N(x^+)$ controls g_{++} component, $x^- \equiv t$ and $s^i = x^i, w$.

Kaluza-Klein gauge field $A \equiv A_0 K dt$ (purely electric, $A_i = 0$).

On-shell solution: $g_{tt} = -N^2K^2(1 - A_0^2) = 0,\ \ N = \frac{2}{\Phi'}\ ,\ \ K = \frac{1}{w^2}\ .$

Want off-shell lower dim'nal effective action: expand out 5-dim Ricci scalar and action for scalar Φ, retaining $K(s^i)$ and $A_0(s^i)$ independently to separate gauge field parts from lower dim metric. From effective action, read off effective lower dim metric.

No nontrivial x^+-dependence $\Rightarrow$ standard KK-reduction. Lower dim effective action contains metric, $massless$ gauge field and scalar.

Dim'nal reduction, rigorously

$$ds^2 = -N^2(x^+)K^2(s^i)dt^2 + \frac{1}{N^2(x^+)}(dx^+ + N^2(x^+)A)^2 + \frac{1}{w^2}(ds^i)^2$$

$$\Rightarrow \ R^{(5)} = -2(NN'' + (N')^2) - [\tfrac{2}{K}(w^2\partial_i^2 K - w\partial_w K + 3K)] \\ + \tfrac{1}{8}F_{0i}^2 + 2(NN'' + (N')^2)A_0^2$$

$R^{(5)}$ contains $\quad R^{(4)} = -\frac{2}{K}(w^2\partial_i^2 K - w\partial_w K + 3K)$.

Suggests lower dim spacetime is: $\quad ds^2 = -K^2(s^i)dt^2 + \frac{1}{w^2}ds^{i\,2}$.

Possible A-mass term actually vanishes: $\int dx^+\partial_+(NN')$.

However, nontrivial gauge field mass from scalar kinetic terms:

$$-\tfrac{1}{2}g^{++}(\partial_+\Phi)^2 \rightarrow \ -\tfrac{1}{2}N^2(1 - A_0^2)(\Phi')^2 + \ldots \ \rightarrow \ \tfrac{1}{2}N^2(\Phi')^2 A_0^2 \ .$$

On-shell: $\quad ds^2 = \frac{1}{w^2}[-2dx^+dx^- + dx_i^2 + \frac{1}{4}w^2(\Phi')^2(dx^+)^2] + \frac{dw^2}{w^2}$

$$\Rightarrow \ \ g_{tt} = -N^2K^2(1 - A_0^2) = 0, \ \ N = \frac{2}{\Phi'} \ , \quad K = \frac{1}{w^2} \ .$$

Fixes A-mass, and gives Li_4: $\quad ds^2 = -\frac{dt^2}{w^4} + \frac{dx^{i\,2}}{w^2} + \frac{dw^2}{w^2}$.

Dimensional reduction

$$ds^2 = -N^2(x^+)K^2(s^i)dt^2 + \frac{1}{N^2(x^+)}(dx^+ + N^2(x^+)A)^2 + \frac{1}{w^2}(ds^i)^2 \,.$$

Consistency: 5-dim system is an on-shell solution to Einstein equations with scalar depending only on x^+ if (from $[00]$-component) $(\partial_i A_0 + A_0 \frac{\partial_i K}{K})^2 = \frac{4}{w^2}$, admitting solution $K = \frac{1}{w^2},\ A = -\frac{dt}{w^2}$.
Also consistent with scalar EOM.

Perhaps surprising that naive dim'nal reduction involves $\Phi' \sim \frac{1}{N(x^+)}$ which however disappears in the effective metric Li_4 implied by $R^{(5)}$.
Nontrivial x^+-dependence might appear to complicate Wilsonian separation-of-scales argument: mixing with other modes?
$e.g.$ turn on KK vector potential $A_i dx^i$. No conclusive result here for consistent dim'nal reduction: then $R^{(5)}$ has extraneous factors of $N(x^+)$ appearing in analogous calculation.
Harder to interpret lower dim system. Explore further.

Scalar probes of Lifshitz

Long wavelength geometry seen by bulk supergravity scalar?

Scalar action $S = \frac{1}{G_5}\int d^5x\,\sqrt{-g}\,g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi$: restrict to modes with no x^+-dependence ($i.e.$ $\partial_+\varphi = 0$) $\Rightarrow$

$$S \rightarrow \frac{1}{G_5}\int \frac{d^4x}{w^5}\left[-w^4\left(\frac{\int dx^+(\Phi')^2}{4}\right)(\partial_-\varphi)^2 + w^2L(\partial_i\varphi)^2 + w^2L(\partial_w\varphi)^2\right]$$
$$= \frac{1}{G_4}\int \frac{d^4x}{w^5}\left[-w^4(\partial'_-\varphi)^2 + w^2(\partial_i\varphi)^2 + w^2(\partial_w\varphi)^2\right],$$

L: compactification size. $G_4 = \frac{G_5}{L}$, 4-dim Newton const.

Rescale $x^- \rightarrow x^{-\prime} = \frac{L}{\int dx^+(\Phi')^2}x^-$ $\rightarrow$ scalar action in 4-dim $z = 2$

Lifshitz background $ds^2 = -\frac{dt^2}{w^4} + \frac{{dx^i}^2}{w^2} + \frac{dw^2}{w^2}$.

Eqn of motion: 4-dim Lifshitz geometry arises on scales large compared with the typical scale of variation (compactification size), $i.e.$ effectively setting $\Phi' \sim const.$

Dual field theory

PBH: $ds^2 = \frac{1}{w^2}[-2dx^+dx^- + dx_i^2 + \frac{1}{4}w^2(\Phi')^2(dx^+)^2] + \frac{dw^2}{w^2}$.

(DMNT) $d = 4$ $\mathcal{N}{=}4$ SYM theory with gauge coupling lightlike-deformed as $g_{YM}^2(x^+) = e^{\Phi(x^+)} \rightarrow$ dim'nally reduce.

boundary metric $ds_4^2 = \lim_{w\rightarrow 0} w^2 ds_5^2$ manifestly flat. Lightlike deformation $\Rightarrow$ no nonzero contraction exists involving metric and coupling alone (only $\partial_+\Phi$ nonvanishing, $g^{++} = 0$) $\Rightarrow$ various physical observables (in particular trace anomaly) unaffected.

After dim'nal reduction along x^+: free gauge theory exhibits $z = 2$ Lifshitz scaling. Varying coupling breaks nonrelativistic $z = 2$ Schrodinger symmetry to $z = 2$ Lifshitz, perhaps not surprising.
On general grounds, this is DLCQ of $\mathcal{N}{=}4$ SYM with coupling having nontrivial variation $g_{YM}^2(x^+)$: better defined?

Dual theory: holographic 2-pt fn

PBH: $ds^2 = \frac{1}{w^2}[-2dx^+dx^- + dx_i^2 + \frac{1}{4}w^2(\Phi')^2(dx^+)^2] + \frac{dw^2}{w^2}$.

Strong coupling gauge theory $\equiv$ weak coupling Lifshitz geometry.

Holographic 2-point function: operators $\mathcal{O}$ dual to massive scalars φ.

Exact PBH coord calculation hindered: scalar wave eqn not exactly solvable. Near boundary, mode fns asymptote to AdS lightcone mode fns. Thus 2-pt fn asymptotes to AdS_5 2-pt fn (lightcone coords)

$\langle O(x)O(x')\rangle \sim \frac{1}{[(\Delta\vec{x})^2]^\Delta}$, with $\Delta = 2 + \sqrt{4+m^2}$,

$(\Delta\vec{x})^2 = -2(\Delta x^+)(\Delta x^-) + \sum_{i=1,2}(\Delta x^i)^2$.

Compactified x^+-dimension: $\Delta x^+ \ll \Delta x^-, \Delta x^i$ [$\equiv$ equal time]

$$\Rightarrow \quad (\Delta\vec{x})^2 = -2(\Delta x^+)(\Delta x^-) + \sum_{i=1,2}(\Delta x^i)^2 \ \sim\ \sum_i(\Delta x^i)^2$$

$$\Rightarrow \quad \langle O(x)O(x')\rangle_{PBH} \ \sim\ \frac{1}{[\sum_i(\Delta x^i)^2]^\Delta} .$$

Agrees with equal-time 2-pt fn of $2+1$-dim Lifshitz theory (Kachru,Liu,Mulligan, $\Delta = 4$, massless bulk scalar).

Equal time correlators: $2+1$-Lifshitz $\equiv$ 2D Eucl. CFT (Fradkin et al)

Dual theory: holographic 2-pt fn

Conformal coords: $ds^2 = \frac{1}{r^2}[e^{f(x^+)}(-2dx^+dy^- + dx_i^2) + dr^2]$.

Boundary metric $\tilde{g}_{\mu\nu} = e^{f(x^+)}\eta_{\mu\nu}$, coupling $g_{YM}^2(x^+) = e^{\Phi(x^+)}$. Lightlike $e^f \Rightarrow$ various observables unaffected. However, conformal factor provides dressing factors for operators, correlators: specifically, conformally dressed operators in conformally flat background behave like undressed operators in flat space.

Exact holographic 2-pt fn for conformally dressed operators:

$$\langle e^{\frac{f(x)\Delta}{2}}O(x)e^{\frac{f(x')\Delta}{2}}O(x')\rangle = e^{\frac{f(x)(\Delta-1)}{2}}e^{\frac{f(x')(\Delta-1)}{2}}\left(\frac{\Delta\lambda}{\Delta x^+}\right)^{1-\Delta}\frac{1}{[(\Delta\vec{x})^2]^\Delta} ,$$

where $\Delta = 2 + \sqrt{4+m^2}$, $\lambda = \int e^{f(x^+)}dx^+$.

Long wavelength: approximate $e^{f(x^+)} \sim 1$, $\frac{\Delta\lambda}{\Delta x^+} \sim \frac{d\lambda}{dx^+} = e^f \sim 1$:

$$\Rightarrow \quad \langle e^{\frac{f(x)\Delta}{2}}O(x)e^{\frac{f(x')\Delta}{2}}O(x')\rangle \sim \frac{1}{[(\Delta\vec{x})^2]^\Delta} \sim \frac{1}{[\sum_i(\Delta x^i)^2]^\Delta} .$$

Agrees with equal-time 2-pt fn of $2+1$-dim Lifshitz theory (Kachru,Liu,Mulligan, $\Delta = 4$, massless bulk scalar).

Time-dep: AdS cosmologies

Recall: $ds^2 = \frac{1}{r^2}(\tilde{g}_{\mu\nu}dx^\mu dx^\nu + dr^2) + ds^2_{S^5}\ , \ \ \Phi = \Phi(x^\mu)\ .$

Solution if: $\tilde{R}_{\mu\nu} = \frac{1}{2}\partial_\mu\Phi\partial_\nu\Phi\ , \quad \frac{1}{\sqrt{-\tilde{g}}}\,\partial_\mu(\sqrt{-\tilde{g}}\,\tilde{g}^{\mu\nu}\partial_\nu\Phi) = 0\ .$

AdS_5-Kasner cosmologies: $e^\Phi = |t|^{\sqrt{2(1-\sum_i p_i^2)}}, \qquad \sum_i p_i = 1\ ,$

$$ds^2 = \frac{1}{r^2}[dr^2 - dt^2 + \sum_{i=1}^3 t^{2p_i}(dx^i)^2]\ .$$

Anisotropic Lifshitz-like scaling symmetries:

$$r \to \lambda r, \quad t \to \lambda t, \quad x^i \to \lambda^{1-p_i} x^i\ .$$

Dilaton, 5-form also respect symmetries.

Const dilaton $\Rightarrow$ no symmetric AdS-Kasner solution ($p_i = \frac{1}{3}$).

Qualitatively different from null: time-dep, cosmological singularity.

Also singularity at $r \to \infty$ ($R_{ABCD}R^{ABCD}$, R diverge).

Initial conditions evolving to these cosmological singularities?

[Static radial Kasner-like solutions exist: require nontrivial dilaton.]

AdS-BKL cosmologies

In fact, larger family of cosmological solutions where spatial metric is one of the homogenous spaces in the Bianchi classification (ADNNT):

$$ds^2 = \frac{1}{r^2}\left[dr^2 - dt^2 + \eta_{ab}(t)(e^a_\alpha dx^\alpha)(e^b_\beta dx^\beta)\right] , \quad e^\Phi = e^{\Phi(t)} .$$

$e^a_\alpha dx^\alpha$ are a triad of 1-forms defining symmetry directions. Spatially homogenous dilaton means spatial $R^a_{(a)}$ vanish, and $R^0{}_0 = \frac{1}{2}(\partial_0\Phi)^2$.

Bianchi-IX: $ds^2 = \frac{1}{r^2}\left[dr^2 - dt^2 + \eta_i^2(t)e^i_\alpha e^i_\beta dx^\alpha dx^\beta\right],\ e^\Phi = |t|^\alpha$.

Approximate Kasner-like solution $\eta_i(t) \simeq t^{p_i}$ with

$\sum_i p_i = 1,\ \sum_i p_i^2 = 1 - \frac{\alpha^2}{2}$, approximate Lifshitz scaling regime.

If all $p_i > 0$, cosmology "stable". Else, spatial curvatures force BKL bounces between distinct Kasner-Lifshitz regimes. Dilaton-driven attractor-like behaviour: α increases along bounce. Attractor basin: generic Kasner-like solution with all $p_i > 0$.

Const dilaton ($\alpha = 0$) $\Rightarrow$ bounces continue indefinitely.

AdS cosmologies

Qualitatively different from earlier null AdS solutions:
time-dependence, bulk cosmological singularity.
Gauge theory duals: conjectured as $\mathcal{N}=4$ SYM theory living on time-dep base space $\tilde{g}_{\mu\nu}$, time-dep cplng $g_{YM}^2 = e^{\Phi}$ (dilatonic cases).
Time-dep of boundary metric $\tilde{g}_{\mu\nu}$ imparts BKL-bouncing behaviour to gauge theory too. Bounces betw distinct Lifshitz regimes.

Symmetric Kasner-like AdS-BKL-dilaton cosmology family: near singularity, spatial curvatures unimportant. Leading singular behaviour dilaton-driven, symmetric AdS-Kasner spacetime.
Gauge theory response: near-singularity wavefunction singular, divergent energy. If time-dep coupling $g_{YM}^2 = e^{\Phi}$ does not strictly vanish, likely nonsingular.

AdS_4 Lifshitz and M-theory

$ds^2 = \frac{1}{r^2}(\tilde{g}_{\mu\nu}dx^\mu dx^\nu + dr^2) + 4d\Omega^2_{X^7}, \quad G_4 = 6\mathrm{vol}(M_4) + C d\Phi \wedge \Omega_3.$

C normalization const, X^7 Sasaki-Einstein 7-manifold admitting harmonic 3-form Ω_3. Scalar $\Phi(x^\mu)$: no natural interpretation in 11-dim, arises from 4-form flux after compactification.

$\Phi = const, \;\; \tilde{g}_{\mu\nu} = \eta_{\mu\nu} \;\; \Rightarrow$ usual $AdS_4 \times X^7$ solution.

Effective 3-dim system: $\tilde{R}_{\mu\nu} = \frac{1}{2}\partial_\mu\Phi\partial_\nu\Phi, \quad \Box\Phi = 0\,.$

3-dim gravity $\tilde{g}_{MN}$ trivial: dynamics driven by scalar.

Null: $ds^2 = \frac{1}{w^2}[-2dx^+dx^- + dx_i^2 + \frac{1}{2}w^2(\Phi')^2(dx^+)^2] + \frac{dw^2}{w^2}\,.$

Dim'nal reduction $\Rightarrow$ $2+1$-dim bulk $z = 2$ Lifshitz spacetime

$$ds^2 = -\frac{dt^2}{w^4} + \frac{dx^2}{w^2} + \frac{dw^2}{w^2}\,.$$

Field theory dual: $1+1$-dim strongly coupled theory, dim'nal redux of null deformation of Chern-Simons theories on M2-branes at conical (CY 4-fold) singularities. ABJM generalizations … Explore further.

AdS Lifshitz and M-theory

AdS_5 null deformation: lifts to (11-dim) M-theory. Similar to earlier solution: $ds^2 = \frac{-2dx^+dx^- + d\vec{x}^2 + dw^2}{w^2} + (dx^+)^2, \ \ \varphi(x^+) = \sqrt{\frac{2}{\ell^2}}\frac{e^{i\ell x^+}}{R}$.

Note: no x^+-dep in metric, KK-reduction standard, gives $z = 2\ Li_4$ and gauge field.

M-theory lift: $AdS_5^{null} \times \mathbb{CP}^2 \times S^1 \times S^1$ supported by G-flux.

Boundary theory: unclear, possibly M5-brane dual.

Time-dep solutions: AdS_4 cosmologies. 3-dim gravity driven purely by scalar. AdS_4-Kasner-Lifshitz: $e^{\Phi} = |t|^{\sqrt{2(1-\sum_i p_i^2)}}, \quad \sum_i p_i = 1$,

$$ds^2 = \frac{1}{r^2}[dr^2 - dt^2 + \sum_{i=1}^{2} t^{2p_i}(dx^i)^2]\ .$$

AdS_4-BKL-dilaton cosmologies: dilaton driven.

No nontrivial AdS_4-Kasner-Lifshitz or AdS_4-BKL cosmologies with const scalar (2 Kasner exponents with 2 constraints).

Conclusions, open questions

AdS null solutions $\rightarrow$ dim'nal redux $\rightarrow z = 2$ Lifshitz spacetimes: dual to DLCQ of $\mathcal{N}=4$ SYM with varying coupling.

• This is on-shell solution in string theory: fluctuations about Lifshitz (or null AdS)? Finite temperature?

• More general Lifshitz-z from string theory?

• Interesting radial dependence of scalar/metric: holographic RG flows between AdS-lightcone (Schrodinger), Lifshitz, or $Li_{z_1} \rightarrow Li_{z_2}$? Holographic superfluids?

• Further exploration of $2+1$-dim field theory dual to Lifshitz: DLCQ of $\mathcal{N}=4$ SYM with varying coupling.

• Further exploration of AdS_4 solutions: $1+1$-dim field theory dual possibly deformation of Chern-Simons theories on M2-branes at singularities (generalizations of ABJM $\ldots$).

* More on AdS BKL-cosmologies

Bianchi IX: symmetry algebra of $X_a = e_a^\alpha \partial_\alpha$ is $SU(2)$.

Spatial Ricci, decomposing along triad $R^a_{(a)} = R^a_\alpha e^\alpha_a$:

$$R^1_{(1)} = \frac{\partial_t(\eta_2\eta_3\partial_t\eta_1)}{\eta_1\eta_2\eta_3} - \frac{1}{2(\eta_1\eta_2\eta_3)^2}[(\eta_2^2 - \eta_3^2)^2 - \eta_1^4] = 0\,, \qquad \ldots.$$

Say $p_1 < 0$: then $\eta_1^4 \sim t^{-4|p_1|}$ non-negligible at some time. This forces metric to transit from one Kasner regime to another. As long as some $p_i < 0$, these bounces continue as:

$$p_i^{(n+1)} = \frac{-p_-^{(n)}}{1+2p_-^{(n)}}\,, \qquad p_j^{(n+1)} = \frac{p_+^{(n)}+2p_-^{(n)}}{1+2p_-^{(n)}}\,, \qquad \alpha_{(n+1)} = \frac{\alpha_n}{1+2p_-^{(n)}}\,,$$

for the bounce from the (n)-th to the $(n+1)$-th Kasner regime.

If $p_- < 0$, then $\alpha_{n+1} > \alpha_n$. Also $\alpha_{n+1} - \alpha_n = \alpha_n(\frac{-2p_-}{1+2p_-})$, i.e., α increases slowly for small α: attractor-like behaviour. Finite number of bounces. If all $p_i > 0$, no bounce: cosmology “stable”.

For no dilaton ($\alpha = 0$), BKL bounces purely oscillatory.

* More on AdS BKL-cosmologies

Parametrization: $p_1 = x,\ p_{2,3} = \frac{1-x}{2} \pm \frac{\sqrt{1-\alpha^2+2x-3x^2}}{2}$.

Lower bound: $p_1 \geq \frac{1-\sqrt{4-3\alpha^2}}{3}$. Solution existence forces $\alpha^2 \leq \frac{4}{3}$.

Under bounces, α increases, window of allowed p_i shrinks. Lower bound hits $p_1 \geq 0 \Rightarrow \alpha^2 \geq 1$. Bounces stop, cosmology "stabilizes".

Attractor-like behaviour: e.g.: $\{p_1^0 = x_0 = 0.3,\ \alpha_0 = 0.001\}$, flows (initially slowly) to $\{p_i > 0\}$ after 15 oscillations ($\alpha_{15} = 1.0896$).

E.g.: $(-\frac{1}{5}, \frac{9}{35}, \frac{33}{35}) \ \rightarrow\ (-\frac{5}{21}, \frac{7}{21}, \frac{19}{21}) \ \rightarrow\ (-\frac{3}{11}, \frac{5}{11}, \frac{9}{11}) \ \rightarrow$ $(-\frac{1}{5}, \frac{3}{5}, \frac{3}{5}) \ \rightarrow\ (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. [multiple flows with same endpoint]

Chaotic behaviour: 7% change to smallest exponent $-\frac{1}{5}$ gives $(-\frac{13}{70}, \frac{9}{35}, \frac{65}{70}) \ \rightarrow\ (-\frac{2}{11}, \frac{13}{44}, \frac{39}{44}) \ \rightarrow\ (-\frac{3}{28}, \frac{2}{7}, \frac{23}{28}) \ \rightarrow\ (\frac{1}{11}, \frac{3}{22}, \frac{17}{22})$, drastically different endpoint.

Note also that dilatonic ($\alpha \neq 0$) [attractor-like] and non-dilatonic ($\alpha = 0$) [oscillatory] flows drastically different.