dS extremal surfaces, replicas, pseudo-Renyis & time entanglement

K. Narayan
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- de Sitter space, dS/CFT and de Sitter entropy;
- dS extremal surfaces: future-past, no-boundary, time-entanglement/pseudo-entropy, analytic cont'ns, LM replica geometries, pseudo-Renyi
- QM: pseudo-entropy, time-entanglement $\rightarrow$ cosmological transition matrix
- Slow-roll inflation, no-bndy surfaces

2210.12963, 2310.00320, KN; 2303.01307, KN, Hitesh Saini (also 1501.03019, 1711.01107, 2002.11950, ...)

2509.02775, Kanhu Nanda, KN, Somnath Porey, Gopal Yadav, 2409.14208, Kaberi Goswami, KN, G Yadav

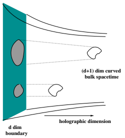
[collaborations: K. Goswami, D. Jatkar, K. Kolekar, K. Nanda, S. Porey, H. Saini, G. Yadav]

Holography and asymptotics

25+ yrs since AdS/CFT '97 Maldacena; '98 Gubser,Klebanov,Polyakov; Witten.

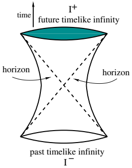
Holography: quantum gravity in $\mathcal{M} \leftrightarrow$ dual without gravity on $\partial\mathcal{M}$ ('t Hooft, Susskind).

(Witten@Strings'98, '01) Gauge/gravity duality and asymptotics —



$\Lambda < 0$: AdS $\rightarrow$ asymptotics at spatial infinity.
Dual: unitary Lorentzian CFT, includes time.

$\Lambda = 0$: flat space $\rightarrow$ null infinity.
S-matrix, symmetries, Carroll/celestial ...



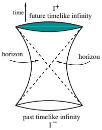
$\Lambda > 0$: de Sitter space
Boundary at future/past timelike infinity $\mathcal{I}^{\pm}$.
Dual $\rightarrow$ Euclidean CFT ... Time emergent.

[note: gravity dual of ordinary Euclidean CFT $\longrightarrow$ Euclidean AdS]

Might regard de Sitter as toy model for cosmology.

de Sitter space, entropy, dS/CFT

de Sitter entropy = area of cosmological horizon (Gibbons,Hawking).
Some sort of (holographic) entanglement? Ryu-Takayanagi generalizations?



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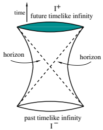

dS/CFT: ('01 Strominger; Witten) future timelike infinity $\mathcal{I}^+$ as a natural dS boundary. Euclidean non-unitary CFT dual. Time emergent.

[note: gravity dual of ordinary Eucl CFT → Eucl AdS]

(Maldacena '02) AdS, analytic continuation → $\boxed{Z_{CFT} = \Psi_{dS}}$ Hartle-Hawking Wavefunction of the Universe

Bulk expectation values $\langle \varphi_k \varphi_{k'} \rangle \sim \int D\varphi\ \varphi_k \varphi_{k'} |\Psi|^2 \rightarrow$ dual ≡ two CFT copies.

Dual energy-momentum $\langle TT \rangle$ 2-pt fn → $\mathcal{C}$ negative/imaginary, ghost-CFT.

Anninos,Hartman,Strominger: higher-spin dS_4 dual to $Sp(N)$ ghost CFT_3, ...

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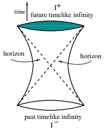

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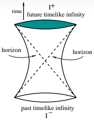
$$\left.\begin{array}{l} \underline{dS_4}, \text{ Poincare: } ds^2 = \frac{R_{dS}^2}{\tau^2}(-d\tau^2 + d\vec{x}^2) \\ r \rightarrow -i\tau, \quad R_{AdS} \rightarrow -iR_{dS}. \end{array}\right\} \quad \begin{array}{l} \Psi_{dS}[\varphi] \sim e^{iS_{cl}[\varphi]} \sim e^{-\int_k R_{dS}^2 k^3 \varphi^0_{-k} \varphi^0_k + \ldots} \\ \rightarrow \text{dual CFT: } \langle O_k O_{k'} \rangle \sim \frac{\delta^2 Z}{\delta\varphi^0_k \delta\varphi^0_{k'}} \rightarrow \mathcal{C}_3 \sim -\frac{R_{dS}^2}{G_4}\,. \end{array}$$

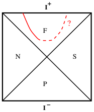
Global/static dS from global AdS: other analytic continuations.

de Sitter space, extremal surfaces

dS, future boundary, extremal surfaces

[KN '15-'25] A natural generalization of Ryu-Takayanagi to de Sitter $\equiv$ bulk analog of setting up entanglement entropy in dual Eucl CFT $\rightarrow$ define some boundary Eucl time slice $\rightarrow$ codim-2 RT/HRT surfaces anchored at I^+, dipping into holographic (time) direction.





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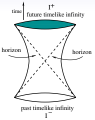

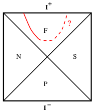

Extremization: surfaces anchored at future boundary $I^+ \rightarrow$ No real $I^+ \rightarrow I^+$ turning point (Lorentzian dS).
Surfaces do not return to I^+. *Interior boundary condns? Time contours?*

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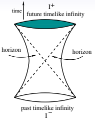

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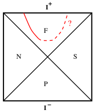

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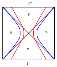
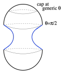
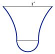


Future-past surfaces: stretch from I^+ to I^-, entirely Lorentzian dS.
Timelike, area $\propto$ overall $\pm i$ (relative to AdS spacelike surfaces).



No-boundary surfaces: Hartle-Hawking no-boundary dS. Complex area.
(top timelike f-p surface + real surface in hemisphere)



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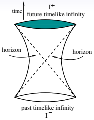

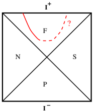

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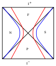

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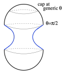

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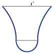

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Areas: new object $\rightarrow$ Pseudo entropy or "Time entanglement".
(EE-like structures, timelike separations) Doi,Harper,Mollabashi,Takayanagi,Taki; KN, '22

[LM replica on $\Psi_{dS} = Z_{CFT}$]

$\mathcal{T}^A_{F|I} = \mathrm{Tr}_B \left(\frac{|F\rangle\langle I|}{\mathrm{Tr}(|F\rangle\langle I|)} \right)$ [entropy of reduced transition matrix (Nakata,Takayanagi,Taki,Tamaoka,Wei,'20)]

“Entanglement” in ghost theories: ghost-spins

KN’16; Jatkar,KN’17; Jatkar,Kolekar,KN’18

- Replica arguments (Calabrese, Cardy) generalized to $c = -2$ ghost CFTs:
  twist operator 2-pt fn $\rightarrow$ $Re(S) < 0$. Subtleties. $[|\downarrow\rangle = |0\rangle;\ \langle -Q|T(z)|0\rangle = 0]$

- $\boxed{\begin{array}{c}\text{“Ghost-spin”} \rightarrow \text{2-state spin variable with indefinite norm.} \\ \langle\uparrow|\downarrow\rangle = \langle\downarrow|\uparrow\rangle = 1, \qquad \langle\uparrow|\uparrow\rangle = \langle\downarrow|\downarrow\rangle = 0\end{array}}$

  [ordinary spin: $\langle\uparrow|\uparrow\rangle = 1 = \langle\downarrow|\downarrow\rangle$]

  $|\pm\rangle \equiv \frac{1}{\sqrt{2}}(|\uparrow\rangle \pm |\downarrow\rangle)$; $\quad \langle\pm|\pm\rangle = \gamma_{\pm\pm} = \pm 1, \quad \langle +|-\rangle = \langle -|+\rangle = 0$

  Infinite ghost-spin chains, $\langle nn\rangle$-intns $\rightarrow$ continuum limit $\rightarrow$ bc-ghost CFT.

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- $\rho = |\psi\rangle\langle\psi| \xrightarrow{\text{tr}_B} \text{RDM}_A$, remaining ghost-spin $\rightarrow$ von Neumann entropy.

  2 g.s., $\sum \psi^{ij}|ij\rangle$: $\langle\psi|\psi\rangle = \gamma_{ik}\gamma_{jl}\psi^{ij}\psi^{kl*} = |\psi^{++}|^2 + |\psi^{--}|^2 - |\psi^{+-}|^2 - |\psi^{-+}|^2 = \pm 1$

  RDM: $(\rho_A)^{ik} = \gamma_{jl}\psi^{ij}\psi^{kl*}$; EE: $S_A = -\gamma_{ij}(\rho_A \log \rho_A)^{ij}$ [new patterns]

  - $\boxed{-ve \text{ norm} \leftrightarrow \text{Im}(S_A)}$
  - $+ve$ norm $|\psi\rangle \not\Rightarrow +ve$ RDM, EE.
  - Nontrivial adjoint $\Rightarrow \rho \sim$ transition matrix $\rightarrow$ pseudo-entropies.

- 2 copies: entangle identical ghost-spins from each copy $\rightarrow$ $+ve$ norm, RDM, EE

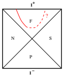
  $|\psi\rangle = \psi^{++}|+\rangle|+\rangle + \psi^{--}|-\rangle|-\rangle \Rightarrow$ Positivity $\longrightarrow$ correlated ghost-spins

  Also true for 2 copies of general ghost-spin ensembles: $|\psi\rangle = \sum_{|\sigma_n\rangle} \psi^{\sigma_n,\sigma_n}|\sigma_n\rangle|\sigma_n\rangle \rightarrow$ Positivity.

dS future-past extremal surfaces

$\underline{dS \text{ (Poincare)}}$: $\quad ds^2_{d+1} = \frac{R^2_{dS}}{\tau^2}(-d\tau^2 + dw^2 + dx_i^2)$

$S_{dS} \propto \int \frac{d\tau}{\tau^{d-1}}\sqrt{1-(\partial_\tau x)^2} \longrightarrow (\partial_\tau x)^2 = \frac{B^2\tau^{2d-2}}{1+B^2\tau^{2d-2}} \quad [B^2 > 0]$



Bndry Eucl time $w=const$ strip @ I^+ $\rightarrow$ codim-2.

Sign diff. from AdS $\Rightarrow$ $\boxed{\text{No } \underline{\text{real}}\ I^+ \rightarrow I^+ \text{ “turning point”.}}$

KN ’15; Sato ’15

[Analytic cont’ns from AdS RT $\rightarrow$ complex areas]

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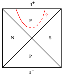

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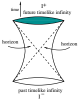
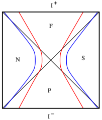


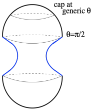
dS (static) $ds^2 = -\frac{dr^2}{r^2/l^2-1} + (\frac{r^2}{l^2}-1)dt^2 + r^2 d\Omega^2_{d-1}$. KN '17

Bndry Eucl time slice, any S^{d-1} equatorial plane (OR $t = const$ slice).

$\boxed{\text{Future-past (timelike) surfaces connecting } I^+ \text{ to } I^-}$



Hartman-Maldacena (AdS bh) rotated. [area div $-i\frac{\pi l^2}{G_4}\frac{R_c}{l}$]



dS (global): $ds^2_{d+1} = -d\tau^2 + l^2\cosh^2\frac{\tau}{l}\,d\Omega^2_d$.

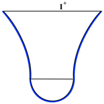
$\boxed{ds^2_{global}\Big|_{\theta_d=const} \;\equiv\; ds^2_{static}\Big|_{t=const} \quad [r = l\cosh\frac{\tau}{l}]}$

de Sitter no-boundary surfaces

[KN'22]; [dS_3: Hikida,Nishioka,Takayanagi,Taki]

Hartle-Hawking no-boundary proposal: Lorentzian dS evolves in time from a no-boundary Euclidean initial configuration. Cut global dS in middle ($\tau = 0$ slice), join top half with hemisphere in bottom half given by Euclidean continuation

$$ds^2 = l^2 d\tau_E^2 + l^2 \cos^2 \tau_E \, d\Omega_d^2 \,; \quad \tau = il\tau_E \,, \quad 0 \leq \tau_E \leq \frac{\pi}{2} \,.$$



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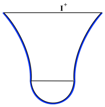

S^d equatorial plane (*i.e.* S^{d-1}) $\rightarrow$ (vertical) timelike f-p surface, $\theta = \frac{\pi}{2}$.
Join ($\tau = 0$) spatial surface going around hemisphere.

IR bottom surface: $ds^2 = l^2 d\tau_E^2 + l^2 \cos^2 \tau_E (d\theta^2 + \sin^2\theta \, d\Omega_{d-2}^2)\Big|_{\theta=\frac{\pi}{2}}$

$$\boxed{\text{Area} = \frac{l^{d-1}}{4G_{d+1}} V_{S^{d-2}} \int_0^{\pi/2} d\tau_E \ (\cos\tau_E)^{d-2} = \frac{1}{2}\, \frac{l^{d-1}\, V_{S^{d-1}}}{4G_{d+1}}}$$

Precisely half dS entropy: emerges differently from area of cosmological horizon (static patch observers). [One hemisphere direction here is Euclidean continuation of time in future universe]

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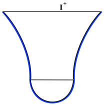

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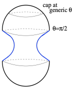
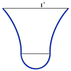


$S_{fp} = S_{nb} - S_{nb}^*$, $\quad$ $\mathrm{Re}(S_{nb}) = \frac{1}{2}\cdot dS$ entropy.
Suggests $\ S_{nb} \equiv \Psi_{dS}$, $\quad S_{fp} \equiv \Psi_{dS}, \Psi_{dS}^*\ (I^+ \cup I^-)$.



$[dS_4]$ $\ S_{fp} = -i\,\frac{\pi l^2}{G_4}\,\frac{R_c}{l}\,; \quad S_{nb} = -i\,\frac{\pi l^2}{2G_4}\,\frac{R_c}{l} + \frac{\pi l^2}{2G_4}$

dS_3, 2-dim CFT; timelike intervals

Future-past surfaces, entirely Lorentzian global dS_3. [some S^2 equatorial plane]

Area $S_{fp} = -i\frac{l}{G_3}\log\frac{l}{\varepsilon} \equiv 2(\frac{c}{3}\log\frac{l}{\varepsilon})$ with $c_{dS_3} = -i\frac{3l_{dS}}{2G}$ [2 copies].

No-boundary dS_3 surface: area $S_{nb} = -i\frac{l}{2G_3}\log\frac{l}{\varepsilon} + \frac{\pi l}{4G_3} \equiv \frac{c}{3}\log\frac{l}{\varepsilon} + \frac{c}{6}(i\pi)$.

$\mathrm{Im}(S_{nb}) \equiv \frac{c}{3}\log\frac{l}{\varepsilon}$ for maximal (IR) interval in Eucl CFT on circle.

$\mathrm{Re}(S_{nb})$: deep interior Euclideanization $\leftrightarrow$ "interior regularity", Eucl CFT

$\big[S_{nb}$ is overall $-i$ times EE for timelike interval in AdS_3 with $c = \frac{3l_{AdS}}{2G_3}.\big]$

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Ordinary unitary 2-dim CFTs: EE is $S = \frac{c}{6} \log \frac{\Delta^2}{\epsilon^2} = \frac{c}{6} \log \frac{-(\Delta t)^2 + (\Delta x)^2}{\epsilon^2}$.

Ordinary spacelike intervals $\Delta^2 > 0 \ \rightarrow \ S = \frac{c}{3} \log \frac{\Delta x}{\epsilon}$.

Entirely timelike interval, width Δt so $\Delta^2 < 0$: $S = \frac{c}{3} \log \frac{\Delta t}{\epsilon} + \frac{c}{6}(i\pi)$.

[Usual Eucl CFT replica $\rightarrow$ Wick-rotate to timelike interval]

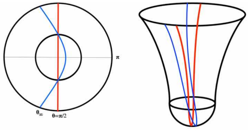
dS no-boundary surfaces, analytic cont'n

Analytic cont'n $\equiv$ space$\leftrightarrow$time rotation: AdS RT surface from $r \to \infty$ (boundary) to $r = 0$ (and back) $\longrightarrow$ IR dS RT/HRT surface from $r \to \infty$ (future boundary) to $r = l$ (Lorentzian dS) going around Eucl hemisphere ($r = l$ to $r = 0$) (& back to $\mathcal{I}^+$).

dS RT/HRT surfaces on bndry Eucl time slice $\quad [r = l \cosh \frac{\tau}{l}]$

$[r > l]\quad ds^2 = -\frac{dr^2}{\frac{r^2}{l^2} - 1} + r^2 d\Omega_{d-1}^2 \xrightarrow{l \to iL} ds^2 = \frac{dr^2}{1 + \frac{r^2}{L^2}} + r^2 d\Omega_{d-1}^2$

$[r < l]\quad ds^2 = \frac{dr^2}{1 - \frac{r^2}{l^2}} + r^2 d\Omega_{d-1}^2 \xrightarrow{l \to iL} ds^2 = \frac{dr^2}{1 + \frac{r^2}{L^2}} + r^2 d\Omega_{d-1}^2$



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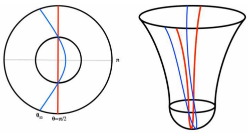


IR: max subregion

$$\frac{V_{S^{d-2}}}{4G_{d+1}}\int_0^{R_c}\frac{r^{d-2}dr}{\sqrt{1+\frac{r^2}{L^2}}} \xrightarrow{L\to -il} \frac{V_{S^{d-2}}}{4G_{d+1}}\left(\int_0^l\frac{r^{d-2}dr}{\sqrt{1-\frac{r^2}{l^2}}}+\int_l^{R_c}r^{d-2}\sqrt{\frac{dr^2}{-(\frac{r^2}{l^2}-1)}}\right)$$

(blue: generic θ_∞)

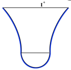
$$=\frac{1}{2}\frac{l^{d-1}V_{S^{d-1}}}{4G_{d+1}}-i\#\frac{l^{d-1}}{4G_{d+1}}\frac{R_c^{d-2}}{l^{d-2}}+\ldots$$

$\left[dS_4:\ \frac{\pi L^2}{2G_4}\left(\frac{R_c}{L}-1\right)\ \to\ -i\frac{\pi l^2}{2G_4}\frac{R_c}{l}+\frac{\pi l^2}{2G_4}\right]$ $\left[dS_3:\ \frac{2L}{4G_3}\log\frac{R_c}{L}\ \to\ -i\frac{l}{2G_3}\log\frac{R_c}{l}+\frac{\pi l}{4G_3}\right]$

dS no-bndry surfaces, Lewkowycz-Maldacena

[KN'23]

Hartle-Hawking Wavefunction of the Universe: amplitude (transition matrix) for creating universe (final bndry condns) from "nothing" (satisfying HH no-boundary condn).



Semiclassically $\Psi_{dS} \sim e^{iS^{(r>l)}}\, e^{S_E^{(r<l)}}$. Top Lorentzian (real $S^{(r>l)}$), pure phase.

Hemisphere: $iS_{cl} \xrightarrow{Eucl\ grav} S_E^{(r<l)} = -\int_{nbp} \sqrt{g}\,(R-2\Lambda) \xrightarrow{dS_4} \frac{\pi l^2}{2G_4}$ (nbp: $\tau_E = \frac{\pi}{2}$).

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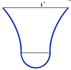

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Lewkowycz-Maldacena: bulk AdS replica dual to boundary replica EE argument.

$Z_{CFT} = Z_{bulk} \Rightarrow$ boundary entanglement entropy = bulk entanglement entropy.

Quotient: conical singularities smoothed by codim-2 cosmic brane (Dong). $\left[I_n = nI_1 + I_{brane}\ (\equiv \frac{n-1}{n}\frac{A}{4G})\right]$

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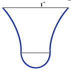

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dS/CFT: $Z_{CFT} = \Psi_{dS} \Rightarrow$ boundary replica via $Z_{CFT} \longrightarrow$ bulk replica on Ψ_{dS}

(single ket, not d.m.; non-hermitian) $\longrightarrow$ Pseudo-Entropy (entropy of transition matrix; complex).

Analytic cont'n (semicl.): $Z_n^{AdS} \sim e^{-I_n} \longrightarrow \Psi_n^{dS} \sim e^{iS_n}$; $S_t = \lim_{n\to 1}(1-n\partial_n)\log\Psi_n = \frac{S_t^{dS}}{4G}$

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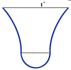

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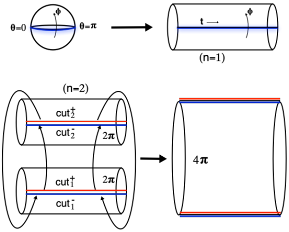
Cosmic brane not spacelike $\leftrightarrow$ Euclidean + timelike no-bndry dS extremal surface.
LM replica formulation: entropy = area of cosmic brane created from "nothing".
Amplitude divergent if Lorentzian part (till I^+) real. Here timelike part = pure phase cancels in probability (finite: bounded real part from hemisphere, set by dS entropy).

LM Replica geometries: dS_3

Nanda,KN,Porey,Yadav

Extend boundary I^+ replica $\rightarrow$ explicit dS-like smooth bulk replica$-n$ geometry

$$\boxed{\text{Boundary Renyi entropy } S_n = \frac{1}{1-n}\log(\frac{Z_n}{Z_1^n}) = \frac{1}{1-n}\log(\frac{\Psi_n}{\Psi_1^n}) \sim i\,\frac{I_n - nI_1}{1-n}}$$



Maximal subregion: $S^2 \rightarrow$ cylinder conformal transfm $\rightarrow$ smooth bulk replica geometry

$$ds^2 = -\frac{dr^2}{\frac{r^2}{l^2} - \frac{1}{n^2}} + \left(\frac{r^2}{l^2} - \frac{1}{n^2}\right)dt^2 + r^2 d\phi^2$$

Replica boundary condns $\phi \equiv \phi + 2\pi n$ at I^+.

[Euclidean part $r < \frac{l}{n}$: $t \rightarrow i\tau_E$ and $\tau_E = [0, \frac{\pi}{2}]$]

Renyi entropy $S_n^{nb} = \frac{1+n}{2n}\left(\frac{\pi l}{4G_3} + \frac{il}{2G_3}\log\frac{2R_c}{l}\right)$

$n \rightarrow 1$ limit $\longrightarrow$ extremal surface area earlier.

Cosmic brane: consider above metric in boundary-quotiented variables $\phi \equiv \phi + 2\pi$.
Now $r = 0$ is singularity (North+South poles). This is Schwarzschild dS_3 :

$$ds^2 = -\left(1 - 8G_3E - \frac{r^2}{l^2}\right)dt^2 + \frac{dr^2}{1 - 8G_3E - \frac{r^2}{l^2}} + r^2 d\tilde{\phi}^2\,, \qquad 8G_3E = 1 - \frac{1}{n^2}\,.$$

Cosmic brane ($r = 0$) timelike in Lorentzian region.

Replica geometries: dS_4

Nanda,KN,Porey,Yadav

Convenient to construct dS_4 replica in hyperbolic coords ($dH_2^2 = d\chi^2 + \sinh^2\chi\, d\psi^2$): recall AdS hyperbolic black holes, replicas, Renyi entropies, Hung,Myers,Smolkin,Yale,'11.

Pure dS_4: $ds^2 = -\frac{dr^2}{f(r)} + l^2 f(r) d\phi^2 + r^2 dH_2^2\,, \quad f(r) = \frac{r^2}{l^2} + 1\,, \qquad r = [il, 0] \cup [0, R_c]\,.$

Eucl: $r = i\rho\,, \quad \chi = i\varphi\,:$ $\qquad ds^2 = \frac{d\rho^2}{1 - \frac{\rho^2}{l^2}} + l^2\Big(1 - \frac{\rho^2}{l^2}\Big) d\phi^2 + \rho^2 (d\varphi^2 + \sin^2\varphi\, d\psi^2)$

Replica: $ds^2 = -\frac{dr^2}{f(r)} + l^2 f(r) d\phi^2 + r^2 dH_2^2\,, \qquad f(r) = \frac{r^2}{l^2} + 1 + \frac{c_1}{r}\,, \qquad \phi \equiv \phi + 2\pi n\,.$

$f(r_h) = 0$: $\qquad c_1 = r_h \frac{l^2 + r_h^2}{l^2}\,, \quad r_h = i\frac{l}{3n}(\sqrt{1+3n^2} + 1) \qquad [n = 1:\ r_h = il \text{ and } c_1 = 0]$

Regularity criterion: smooth Euclidean continuation $r = i\rho$ (dS_4 brane: Das,Das,KN,'13).
$\rightarrow\ -AdS$ type space. In dS variables $\rightarrow$ complex geometry, "ends" at complex horizon.

Boundary Renyi entropy $S_n = \frac{i}{1-n} \frac{2\pi A n}{16\pi G_4 l} \Big(r_h(-r_h^2 + l^2) - 2il^3 \Big)\,.$

$n \rightarrow 1$ limit, extremal surface area: $S_1 = -\frac{l^2}{4G_4} A \ \equiv\ \frac{\pi l^2}{2G_4}\big(\frac{iR_c}{l} + 1\big).$

Note: the hyperboloid area A is complex, since dS_4 in hyperbolic foliation is a nontrivial embedding in global dS_4 (even for $n = 1$) [$\cosh \chi_{max} \sim \frac{R_c}{il}$]. [Explicit dS_4 embedding coords for hyperboloid surface in covering space]
Can be generalized to higher dim dS. Overall this is equivalent to analytic continuation from AdS.

“Time-Entanglement”/Pseudo-entropy: QM entanglement with timelike separations

"Time-Entanglement": reduced time evol^n op

KN'22

dS surfaces anchored at I^+ do not return: extra data reqd in far past.

$\sim$ scattering amplitudes: final states from initial, $\equiv$ time evolution. [Witten '01, $dS \equiv$ past-future amplitudes]

$\rightarrow$ Entanglement-like structures from time evolution operator $\mathcal{U}(t)$ after partial trace over environment: *i.e.* "reduced transition amplitudes" and entropy.

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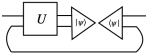
$$\mathcal{U}(t) = e^{-iHt} \;\rightarrow\; \rho_t(t) \equiv \frac{\mathcal{U}(t)}{\mathrm{Tr}\,\mathcal{U}(t)} \;\rightarrow\; \rho_t^A = tr_B\,\rho_t \;\rightarrow\; S_A = -tr(\rho_t^A \log \rho_t^A)$$

[Resemble finite temp EE, but imaginary temp $\beta = it$]

$\Leftrightarrow$ Pseudo-entropy [entropy of reduced transition matrix (Nakata,Takayanagi,Taki,Tamaoka,Wei,'20)].

[KN, Saini, '23]

Time Evol'n operator ρ_t	$\rightarrow$	"Components" of time Evol'n op $\rho_t^{\vert I\rangle} = \rho_t \vert I\rangle\langle I\vert$: ρ_t + projection onto initial state $\vert I\rangle \xrightarrow{tr_B} \rho_t^{\vert I\rangle, A}$	$\leftarrow$	Pseudo-entropy from reduced transition matrix $\mathcal{T}^A_{F\vert I} = \mathrm{Tr}_B\left(\frac{\vert F\rangle\langle I\vert}{\mathrm{Tr}(\vert F\rangle\langle I\vert)}\right)$



Ex. $\rho_t^{|I\rangle,A} = \frac{x|1\rangle\langle 1| + (1-x)e^{i\theta}|2\rangle\langle 2|}{x+(1-x)e^{i\theta}}$

$\left[\; |I\rangle = \sum_{1,2} c_{ii}|ii\rangle,\;\; |c_{11}|^2 \equiv x,\;\; \theta = -(E_{22}-E_{11})t \;\right]$

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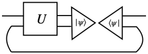

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- $|\psi_{fp}\rangle \;\rightarrow$ f-p density matrix $\rho_{fp} \equiv |\psi\rangle_{fp}\langle\psi|_{fp} \xrightarrow{Tr_p}$ positive structures.

- Transition matrix operator $\mathbb{T} \equiv \frac{|\psi(t)\rangle\langle\phi(t)|}{\mathrm{Tr}(|\psi\rangle\langle\phi|)} = \frac{\mathcal{U}(t)\,|\psi\rangle\langle\phi|\,\mathcal{U}(t)^\dagger}{\mathrm{Tr}(|\psi\rangle\langle\phi|)}$: (Nanda,KN,Porey,Yadav) useful for operators localized to subregions, autocorrn fns etc (also Milekhin,Adamska,Preskill).

“Cosmological transition matrix”

Nanda,KN,Porey,Yadav

Drawing experience from QM, revert to cosmology → consider

$$\boxed{\text{“Cosmological transition matrix” from } I^- \text{ to } I^+: \qquad \mathbb{T}_{dS} \equiv \Psi[I^+]\,\Psi^*[I^-]}$$

Can now construct time-entanglement or pseudo-entropy quantities in various cosmological contexts.

“Boundedness” since de Sitter entropy is finite. Consider $\Psi[I^-] = \Psi[I^+] = \Psi_{dS}^{HH}$. This amounts to gluing the past and future Euclidean hemispheres.

Replica $\Rightarrow \quad S_{\mathbb{T}_{dS}} = \left(1 - \frac{R_c}{3}\partial_{R_c}\right)\log \mathbb{T}_{dS} = S_{nb} + S_{nb}^* = \frac{\pi l^2}{G_4}$

using dS_4 : $\quad \log\Psi[I^+, R_c] = \frac{\pi l^2}{2G_4} + i\frac{\pi R_c^3}{8G_4}\left(-\frac{4}{l} + \frac{6l}{R_c^2}\right), \quad S_{nb} = \frac{\pi l^2}{2G_4} + i\frac{\pi l^2}{2G_4}\frac{R_c}{l}$

Entirely Lorentzian dS_4 suggests $\Psi[I^-] = \Psi[I^+]^*$

$$\rightarrow \quad S_{\mathbb{T}_{dS}} = \left(1 - \frac{R_c}{3}\partial_{R_c}\right)\log \mathbb{T}_{dS} = 2S_{nb} = \frac{\pi l^2}{G_4} + i\frac{\pi l^2}{G_4}\frac{R_c}{l}$$

$dS/CFT \Rightarrow Z_{CFT} = \Psi_{dS} \quad \longrightarrow \quad \boxed{\mathbb{T}_{dS} \equiv Z_{CFT}[I^+]\,Z_{CFT}^*[I^-]}$.

For $Z_{CFT}[I^+] = Z_{CFT}[I^-]$, this is positive: encodes dS entropy.

More generally, a single Z_{CFT} copy encodes boundary Renyi entropies.

Further aspects:
Slow-roll inflation, no-boundary surfaces

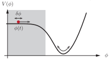
No-boundary slow-roll extremal surfaces

Goswami,KN,Yadav,'24

Inflation: brief period of exponential expansion phase in early universe.

[Standard Big-Bang Cosmology, horizon/flatness (antipodal points in causal contact in past; universe flattens out)]

Nearly de Sitter: dS small wiggles (slow-roll parameters $\epsilon,\ \eta$)
driven by inflaton scalar field slowly rolling down its potential.



Baumann,McAllister'14

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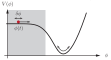
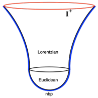
driven by inflaton scalar field slowly rolling down its potential.



Baumann,McAllister'14

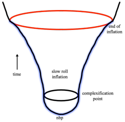
Inflationary perturbations to <u>no-boundary global dS</u>: preserve spherical symmetry.
No-boundary HH regularity at nbp for both inflaton and metric; no singularities.



$$ds^2 = -dt^2 + a(t)^2 d\Omega_d^2 \;=\; g_{aa}\, da^2 + a^2 d\Omega_d^2\,.$$

Slow-roll inflation: $g_{aa} = \frac{1}{1-r^2}\big(1 + 2\epsilon\, \beta_>(r)\big)$

$$\epsilon \equiv \frac{{V_*'}^2}{2V_*^2}\,, \qquad 3H^2 \sim V(\phi), \quad 3H\dot{\phi} \sim -V'(\phi)\,.$$



$$\beta_>(r) = \frac{8-9r^4+4ir^2\sqrt{r^2-1}+8i\sqrt{r^2-1}-6ir^4\sqrt{r^2-1}+r^6\left(6\log\left(1-i\sqrt{r^2-1}\right)-1+3i\pi\right)}{6r^4(r^2-1)}$$ [Lorentzian]

No-boundary slow-roll extremal surfaces

Goswami,KN,Yadav,'24

Inflation: brief period of exponential expansion phase in early universe.

[Standard Big-Bang Cosmology, horizon/flatness (antipodal points in causal contact in past; universe flattens out)]

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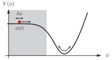

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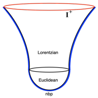

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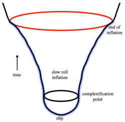


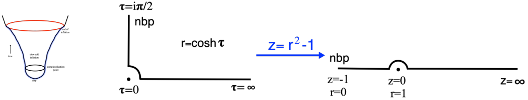
$$\beta_>(r) = \frac{8-9r^4+4ir^2\sqrt{r^2-1}+8i\sqrt{r^2-1}-6ir^4\sqrt{r^2-1}+r^6\left(6\log\left(1-i\sqrt{r^2-1}\right)-1+3i\pi\right)}{6r^4(r^2-1)} \qquad \text{[Lorentzian]}$$

Area $S_{sr_4}^{r<1}+S_{sr_4}^{r>1} \xrightarrow{O(\epsilon)} \quad S_{sr_4} \simeq \frac{\pi l^2}{2G_4}\left(-i\int_1^{R_c/l}\frac{1+\epsilon\,\beta_>(r)}{\sqrt{r^2-1}}\;r\,dr + \int_0^1\frac{1+\epsilon\,\beta_<(r)}{\sqrt{1-r^2}}\;r\,dr\right)$

Extra singular terms at complexification point $r=1$ (poles in $\beta(r)$ terms) (like Wavefn)

Define as complex-time-plane integral + time-contour (avoid $r=1$). Normalize to leading dS.

Slow-roll no-boundary extremal surfaces



$$\boxed{\begin{aligned} S \;&=\; \frac{\pi\, l^2}{2G_4} \int_C \frac{1+\epsilon\,\beta_>(r)}{2\sqrt{1-r^2}}\; r\, dr \\ &\equiv\; \frac{\pi\, l^2}{2G_4} \int_C \frac{1+\epsilon\,\beta_>(z)}{2\sqrt{-z}}\; dz\,. \end{aligned}}$$

$$\longrightarrow \quad \begin{aligned} S \;=\; \frac{\pi\, l^2}{2G_4} \Bigg[1 - \sqrt{\delta} \;+\; \epsilon \left[\left(\log 4 - \frac{7}{6} + i\pi \right) - \left(-\frac{2-3i\pi}{6\sqrt{\delta}} + \frac{5}{3} \right) \right] \;+\; \sqrt{\delta}(-i+1) \\ +\; \epsilon \left[\frac{2-3i\pi}{6i\sqrt{\delta}} - \frac{2-3i\pi}{6\sqrt{\delta}} \right] \;+\; \frac{1}{i}\left(\sqrt{z_c} - \sqrt{\delta}\right) + \epsilon \left[\left(1 + i\frac{7}{6}\sqrt{z_c} - i\sqrt{z_c}\log\sqrt{z_c} \right) - \left(\frac{2-3i\pi}{6i\sqrt{\delta}} + \frac{5}{3} \right) \right] \Bigg] \end{aligned}$$

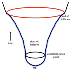
Various cancellations as expected. Details of regulating semicircle contour unimportant.

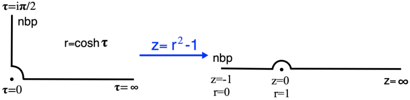
$$\boxed{S_{sr_4} \;=\; \frac{\pi\, l^2}{2G_4} \left(-i\,\frac{R_c}{l} + 1 \right) \;+\; \epsilon\, \frac{\pi\, l^2}{2G_4} \left(-i\,\frac{R_c}{l} \log \frac{R_c}{l} \;+ i\,\frac{7}{6}\,\frac{R_c}{l} \;+\; \log 4 - \frac{7}{2} + i\pi \right)}$$

- Divergent parts pure imaginary. Vindicates finite cosmic brane creation probability, set by size of maximal hemisphere ($\equiv dS$ entropy $+$ slow-roll corrections $[<0]$).
- No clean separation betw real/imaginary parts of area, slow-roll corrections mix all.

Finite terms in particular arise from entire surface, both timelike and hemisphere parts.

Slow-roll no-boundary extremal surfaces





$$\boxed{\begin{aligned} S &= \frac{\pi\, l^2}{2G_4} \int_C \frac{1+\epsilon\,\beta_>(r)}{2\sqrt{1-r^2}}\; r\, dr \\ &\equiv \frac{\pi\, l^2}{2G_4} \int_C \frac{1+\epsilon\,\beta_>(z)}{2\sqrt{-z}}\; dz\,. \end{aligned}}$$

$$\longrightarrow \quad \begin{aligned} S = \frac{\pi\, l^2}{2G_4}\Bigg[& 1-\sqrt{\delta} \;+\; \epsilon\left[\left(\log 4-\frac{7}{6}+i\pi\right)-\left(-\frac{2-3i\pi}{6\sqrt{\delta}}+\frac{5}{3}\right)\right] \;+\; \sqrt{\delta}(-i+1) \\ & +\;\epsilon\left[\frac{2-3i\pi}{6i\sqrt{\delta}}-\frac{2-3i\pi}{6\sqrt{\delta}}\right]+\frac{1}{i}(\sqrt{z_c}-\sqrt{\delta})+\;\epsilon\left[\left(1+i\frac{7}{6}\sqrt{z_c}-i\sqrt{z_c}\log\sqrt{z_c}\right)-\left(\frac{2-3i\pi}{6i\sqrt{\delta}}+\frac{5}{3}\right)\right]\Bigg] \end{aligned}$$

Various cancellations as expected. Details of regulating semicircle contour unimportant.

$$\boxed{S_{sr_4} = \frac{\pi\, l^2}{2G_4}\left(-i\frac{R_c}{l}+1\right) \;+\; \epsilon\,\frac{\pi\, l^2}{2G_4}\left(-i\frac{R_c}{l}\log\frac{R_c}{l}\;+i\frac{7}{6}\frac{R_c}{l}\;+\;\log 4-\frac{7}{2}+i\pi\right)}$$

• Divergent parts pure imaginary. Vindicates finite cosmic brane creation probability, set by size of maximal hemisphere ($\equiv dS$ entropy $+$ slow-roll corrections $[<0]$).

• No clean separation betw real/imaginary parts of area, slow-roll corrections mix all. Finite terms in particular arise from entire surface, both timelike and hemisphere parts.

Finite parts above (cosmic brane probability) match those in dS_4 Wavefunction:

$$iI_{sr4} = \frac{\pi\, l^2}{2G_4}\left[1+\epsilon\left(\log 4-\frac{7}{2}+i\pi\right)-i\left(r_c^3-\frac{3}{2}r_c\right)+i\epsilon\left(r_c^3\left(\log r_c-\frac{1}{6}\right)+\frac{r_c}{4}\,(6\log r_c-11)\right)\right]$$

(Maldacena '24) $\Psi \sim e^{iI_{sr4}}$, obtained by evaluating on-shell action via ADM formulation.

• AdS BH: IR RT surface wraps horizon, $S^{fin} \sim$ BH entropy $\leftarrow$ action $\equiv$ partition fn

Conclusions, questions

- dS future boundary: no $I^+ \to I^+$ turning point. Surfaces do not return to I^+.

(a) Future-past surfaces, $I^+ \leftrightarrow I^-$. Timelike. (b) No-boundary surfaces, Real part (hemisphere) $= \frac{1}{2}\, dS$ entropy.

Pseudo-entropy: AdS analytic cont'n $\equiv$ space $\leftrightarrow$ time rotation, Lewkowycz-Maldacena.

Explicit dS replica geometries for maximal subregions $\to$ boundary Renyi entropies.

? *Complex dS-like geometries:* rules for replicas, time-contours?
multiple extremal surfaces? KSW?

QM & Time-entanglement/Pseudo-entropy: EE-like structures, timelike separations:
(i) reduced time evolution operator $\leftrightarrow$ red. transition amplitudes,
(ii) positivity in future-past entangled states & density matrices.

"Cosmological transition matrix" using dS Wavefunction $\to$ dS entropy, future-past areas...

? Dual using two copies of ghost-like CFTs?

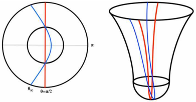
- Slow-roll inflation: no-boundary areas must be defined carefully via complex time plane integrals with appropriate time contours avoiding potential poles.

Maximal cosmic brane creation probability ($= dS_4$ entropy + slow-roll corrections $[< 0]$)
matches $|\text{Wavefunction}|^2$. Dual CFT understanding? Time-contours, more general cosmologies?

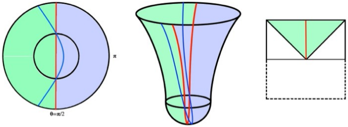
? Pseudo-entropy meta-observables $\leftrightarrow$ standard Big-Bang cosmology observers/observables?

dS surfaces, subregion duality, geometrically

IR surface, $t = const$ slice, maximal subregion $\rightarrow$ red surface;
Generic subregion, blue: tilted "great circle" in hemisphere, joining with tilted timelike surface in Lorentzian top half.
dS_3 explicitly solvable; dS_{d+1}, perturbatively analysed.

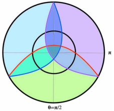


Time-entanglement/Pseudo-entanglement wedge: Max subregion, $t = const$ slice: green bulk region bounded by (red) IR surface and boundary subregion. (Violet complement region)



Including t-direction $\rightarrow$ top wedge (containing future of IR surface on vertical $t = const$ slice), bounded by I^+ subregion $\equiv$ analytic continuation from AdS.
Space-time rotation from AdS EE wedge. (dS/CFT via relative entropy, modular flow etc?)

Multiple disjoint boundary subregions: red, violet, blue no-bndry dS extremal surfaces. Complex areas so quite different from AdS EE.
Bulk subregions not disjoint: except for IR (maximal) subregions.

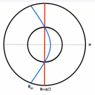


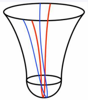
[Maybe other possibilities: subregion duality in Lorentzian dS via future-past surfaces..]

dS surfaces, entropy relations/inequalities

dS_3 : $S_t^{\theta_\infty} = -i\frac{l}{2G_3}\log\frac{R_c}{l} - i\frac{l}{4G_3}\log(\sin^2\theta_\infty) + \frac{\pi l}{4G_3}$

IR, $\theta_\infty = \frac{\pi}{2}$: $S_t^{IR} = -i\frac{l}{2G_3}\log\frac{R_c}{l} + \frac{\pi l}{4G_3}$



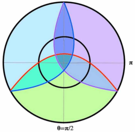


Two adjacent disjoint subregions A, B $(2\theta_\infty = \frac{\pi}{2})$; $A \cup B \equiv (2\theta_\infty = \pi)$.

"Mutual time-information" or "mutual pseudo-information":

$I_t[A,B] = S[A] + S[B] - S[A \cup B] = -i\frac{l}{2G_3}\log\frac{R_c}{l} + i\frac{l}{2G_3}\log 2 + \frac{\pi l}{4G_3}$

$\Rightarrow$ $\mathrm{Re}\, I_t \geq 0$, $\mathrm{Im}\, I_t \leq 0$. (antipodal subregions, $I_t = 0$)



Tripartite time-information: 3 disjoint adjacent quadrant subregions A, B, C $(2\theta_\infty = \frac{\pi}{2})$.

$A \cup B$, $B \cup C$ maximal (IR) subregions. $A \cup C$, antipodal quadrants (extr. surf. = "inner" $(\equiv B)$ + "outer").

$S_A = S_B = S_C = S_t^{\pi/4}$, $S_{AB} = S_{BC} = S_t^{\pi/2}$, $S_{AC} = S_t^{\pi/4} + S_t^{\pi/4}$, $S_{ABC} = S_t^{\pi/4}$;

$I_3^t[A,B,C] = S_A + S_B + S_C - S_{AB} - S_{BC} - S_{AC} + S_{ABC} = i\frac{l}{2G_3}\log 2 \quad \Rightarrow \quad \mathrm{Im}\, I_3^t \geq 0.$

Strong subadditivity: $\left.\begin{array}{l} S_{AB} + S_{BC} - S_{ABC} - S_B = -i\frac{l}{2G_3}\log 2, \\ S_{AB} + S_{BC} - S_A - S_C = -i\frac{l}{2G_3}\log 2. \end{array}\right\}$ $\begin{array}{l} \mathrm{Re}\, SSB_{1,2}^t \geq 0, \\ \mathrm{Im}\, SSB_{1,2}^t \leq 0 \end{array}$

dS area/entropy relations special (relative to qubit system pseudo-entropies).

Note: AdS analytic continuation $il \to -L$ $\Rightarrow$ $MI \geq 0$, $I_3 \leq 0$, $SSB^{1,2} \geq 0$.

Consistent with AdS RT/HRT areas which are also special Hayden,Headrick,Maloney,'11.

Qubits, pseudo-entropy inequalities

Pseudo-entropy $\rho_t = \frac{|F\rangle\langle I|}{\mathrm{Tr}(|F\rangle\langle I|)} = \frac{\mathcal{U}(t)|I\rangle\langle I|}{\mathrm{Tr}(\mathcal{U}(t)|I\rangle\langle I|)}$ [= time evoln op $\mathcal{U}(t) = e^{-iHt}$ with projection]

for TFD-type initial state $|I\rangle$ and its time-evolved final state $|F\rangle = \mathcal{U}(t)|I\rangle$.

2-qubits: $|I\rangle = c_{11}|11\rangle + c_{22}|22\rangle\,, \quad |F\rangle = c_{11}e^{-iE_{11}t}|11\rangle + c_{22}e^{-iE_{22}t}|22\rangle$

$\left[|c_{11}|^2 + |c_{22}|^2 = 1; \quad |c_{11}|^2 \equiv x; \quad \theta = -(E_{22} - E_{11})t\right]$

$$\rho_t^1 = \mathrm{Tr}_2\,\rho_t\,, \quad \rho_t^2 = \mathrm{Tr}_1\,\rho_t\,, \quad \rho_t^2 = \rho_t^1 = \frac{1}{x+(1-x)e^{i\theta}}\Big(x|1\rangle\langle 1| + (1-x)e^{i\theta}|2\rangle\langle 2|\Big)\,,$$

$$S_t^2 = S_t^1 = -\frac{x}{x+(1-x)e^{i\theta}}\log\frac{x}{x+(1-x)e^{i\theta}} - \frac{(1-x)e^{i\theta}}{x+(1-x)e^{i\theta}}\log\frac{(1-x)e^{i\theta}}{x+(1-x)e^{i\theta}}$$

Near $t = 0$: $\quad S_t^1(t) \sim S_t^1(0) + \frac{d}{dt}S_t^1(0)\,t \equiv S_0\,,$

$$S_t^1(0) = -x\log x - (1-x)\log(1-x)\,, \qquad \frac{d}{dt}S_t^1(0) = -i\,\Delta E\,x(1-x)\log\frac{x}{1-x}$$

Mutual pseudo-information: $I_t[1,2] = S_t^1 + S_t^2 - S_t \sim 2S_0; \quad \mathrm{Re}I_t > 0\,, \quad \mathrm{Im}I_t \gtrless 0$

3-qubits: $|I\rangle = c_{111}|111\rangle + c_{222}|222\rangle\,, \quad |F\rangle = c_{111}e^{-iE_{111}t}|111\rangle + c_{222}e^{-iE_{222}t}|222\rangle$

$$\rho_t^{123} = \frac{|F\rangle\langle I|}{\mathrm{Tr}(|F\rangle\langle I|)}\,, \quad \rho_t^1 = \mathrm{Tr}_{23}\,\rho_t^{123} = \frac{1}{x+(1-x)e^{i\theta}}\Big(x|1\rangle\langle 1| + (1-x)e^{i\theta}|2\rangle\langle 2|\Big), \quad \rho_t^2 = \rho_t^3 = \rho_t^1\,,$$

$$\rho_t^{12} = \mathrm{Tr}_3\,\rho_t^{123} = \frac{1}{x+(1-x)e^{i\theta}}\Big(x|11\rangle\langle 11| + (1-x)e^{i\theta}|22\rangle\langle 22|\Big), \quad \rho_t^{23} = \rho_t^{13} = \rho_t^{12}$$

Tripartite pseudo-information: $I_3^t[1,2,3] = S_t^1 + S_t^2 + S_t^3 - S_t^{23} - S_t^{13} - S_t^{12} + S_t^{123} = 0$

SSB: $SSB_1^t = S_t^{12} + S_t^{23} - S_t^{123} - S_t^2 = S_t^1; \quad SSB_2^t = S_t^{12} + S_t^{23} - S_t^1 - S_t^3 = 0$

$\mathrm{Re}\,SSB_1^t > 0\,, \quad \mathrm{Im}\,SSB_1^t \gtrless 0 \quad (x \neq \frac{1}{2})$

[Specific TFD states above; more general states?]

“Time-Entanglement”, examples: 2-qubits etc

2-state system: $H|k\rangle = E_k|k\rangle$, $(k=1,2;\ \langle 1|2\rangle = 0)$; $|k\rangle_F \equiv |k(t)\rangle = e^{-iE_k t}|k\rangle_P$.

$\rho_t = \frac{1}{1+e^{i\theta}}\left(|1\rangle\langle 1| + e^{i\theta}|2\rangle\langle 2|\right)$, $\theta = -(E_2-E_1)t$; 2-spin analogy: $|1\rangle \equiv |{+}{+}\rangle$, $|2\rangle \equiv |{-}{-}\rangle$

$$\xrightarrow{\mathrm{Tr}_B} \rho_t^A \to \boxed{\text{entropy } S_A^\theta = -\mathrm{tr}(\rho_t^A \log \rho_t^A) = -\frac{1}{1+e^{i\theta}}\log\frac{1}{1+e^{i\theta}} - \frac{1}{1+e^{-i\theta}}\log\frac{1}{1+e^{-i\theta}}}$$

Real-valued, oscillating in time, periodicity $\sim \frac{1}{\Delta E}$; unbounded at $t = \frac{(2n+1)\pi}{\Delta E}$; min $S_A^\theta = \log 2$ at $t = \frac{2n\pi}{\Delta E}$.

General 2-qubit Hamiltonian $H = E_{11}|11\rangle\langle 11| + E_{22}|22\rangle\langle 22| + E_{12}(|12\rangle\langle 12| + |21\rangle\langle 21|)$

$$\rho_t = \mathcal{N}_t \sum_{i,j} e^{-iE_{ij}t}|ij\rangle\langle ij| = \frac{(|11\rangle\langle 11| + e^{i\theta_1}|22\rangle\langle 22| + e^{i\theta_2}(|12\rangle\langle 12| + |21\rangle\langle 21|))}{1+e^{i\theta_1}+2e^{i\theta_2}} \qquad \left[\xrightarrow{t=0} \tfrac{1}{4}\hat{\mathbf{1}}\right]$$

$$\xrightarrow{Tr_2} \boxed{\rho_t^A = \frac{1}{1+e^{i\theta_1}+2e^{i\theta_2}}\Big((1+e^{i\theta_2})|1\rangle\langle 1| + (e^{i\theta_1}+e^{i\theta_2})|2\rangle\langle 2|\Big)} \qquad \begin{array}{l}\theta_1 = -(E_{22}-E_{11})t,\\ \theta_2 = -(E_{12}-E_{11})t.\end{array}$$

Generically complex-valued von Neumann entropy. (mixed EE, imaginary temp $\beta = it$)

$\frac{\rho_t|I\rangle\langle I|}{\mathrm{Tr}(\rho_t|I\rangle\langle I|)}$: Projection onto Thermofield-double initial states $|I\rangle = \sum_{1,2} c_{ii}|ii\rangle$

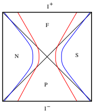
$$\xrightarrow{Tr_2} \boxed{\rho_t^{|I\rangle,A} = \frac{1}{|c_{11}|^2+|c_{22}|^2e^{i\theta}}\Big(|c_{11}|^2|1\rangle\langle 1| + |c_{22}|^2e^{i\theta}|2\rangle\langle 2|\Big)} \qquad [\theta = -(E_{22}-E_{11})t]$$

$\equiv$ reduced transition matrix for $|I\rangle$ and $|F\rangle = \sum c_{ii}e^{-iE_{ii}t}|ii\rangle$ ($\longrightarrow$ pseudo-entropy).

Max. entangled (Bell-pair) states $|c_{11}|^2 = |c_{22}|^2 = \frac{1}{2} \longrightarrow S_A^\theta$ (2-state above). Min $S_A^\theta = \log 2 = \mathrm{EE}(|I\rangle)$.

Future-past surfaces, f-p “entanglement”

dS future-past surfaces connecting I^+ to I^-.
(Hartman-Maldacena (AdS bh) rotated)
Suggests future-past entanglement (betw $I^\pm$).



Recall eternal AdS bh dual to $CFT_L \times CFT_R$ in TFD state (Maldacena)

Speculation: (Lorentzian) dS_4 dual to $CFT_F \times CFT_P$ in thermofield-double entangled state $|\psi^{tfd}_{fp}\rangle = \sum \psi^{i^F_n, i^P_n} |i^F_n\rangle |i^P_n\rangle$?

[KN ’17; also Arias,Diaz,Sundell,’19]

Tracing fp-dm over past copy gives mixed state at I^+.

2 copies of future-past entangled states & density matrices: positive entropy $EE > 0$.

$|\psi_{fp}\rangle \rightarrow$ f-p density matrix $\rho_{fp} \equiv |\psi\rangle_{fp}\langle\psi|_{fp} \xrightarrow{Tr_p}$ positive structures.
Connectedness of fp-entangled states & timelike entanglement $\leftrightarrow$ emergence of time?

van Raamsdonk: space emerges from entanglement.
Factorized fp-states $|\psi^{(1)}_f\rangle|\psi^{(2)}_P\rangle$: $\mathrm{Tr}_P \rho_{fp} \rightarrow$ pure.
Entangled fp-states: reduced transition matrix $\equiv$ time evolution operator.

$[\mathcal{U}(t) = Tr_2(|\psi_{fp}\rangle\langle\psi_I|)]$ Time evol’n $\equiv$ f-p EE. Timelike ER=EPR?